ANALYSIS OF A CLASS OF NONCONFORMING FINITE ELEMENTS FOR CRYSTALLINE MICROSTRUCTURES

By

Petr Klouček
Bo Li
and
Mitchell Luskin

IMA Preprint Series # 1196
January 1994
ANALYSIS OF A CLASS OF NONCONFORMING FINITE ELEMENTS FOR CRYSTALLINE MICROSTRUCTURES

PETR KLOUČEK, BO LI AND MITCHELL LUSKIN

University of Minnesota

Abstract. An analysis is given for two types of nonconforming Lagrange-type finite elements which have been successfully utilized to approximate the solution of a variational problem modeling the deformation of martensitic crystals with microstructure. Convergence is proven and error estimates, including estimates for superconvergence, are obtained. Finally, an analysis of the effect of numerical quadrature is given.

1. Introduction

Recent years have seen the development of a continuum theory for martensitic crystals based on the minimization of the Ericksen-James elastic energy [3–4, 10–11, 14, 16]. The elastic energy density attains a minimum value at several symmetry-related deformation gradients. Thus, the deformations of energy-minimizing sequences often exhibit a microstructure—the simplest of which are fine-scale layers in which the deformation gradient is nearly constant and across which the deformation gradient oscillates between the energy wells—to allow the effective energy of a deformation to be that of a macroscopic or relaxed energy. Further, the parallel planes defining the layering in the microstructure are constrained by the symmetry of the energy density to be a member of a finite family of parallel planes.

If the deformation is constrained on the boundary, then the deformation cannot generally attain a minimum energy by forming a microstructure with layers of nonzero thickness [4]. Rather, minimizing sequences of deformations must be constructed from layers with a thickness which converges to zero. Such minimizing sequences define the solution to the variational problems. They can be described physically by the notion of microstructure and mathematically by the Young measure [3–4, 14, 16].

When an energy minimizing deformation is sought in a finite element space, the fineness of the layers is limited not only by the mesh size, but also by the nature of the finite element used. The most accurate finite element spaces will be those which can...
approximate microstructures with the most fine-scale layers possible on meshes which are oriented arbitrarily with respect to the layers defining the microstructure.

Several approaches have been developed for the finite element approximation of microstructure. The most commonly used finite element spaces are the conforming spaces with continuous deformations which are either piecewise linear or multilinear with respect to some mesh. Although these spaces can approximate well microstructure with layers which are oriented with respect to the mesh, we have had difficulty approximating microstructure with these conforming spaces when the layers are not oriented with respect to the mesh. We have not generally been able to obtain solutions with conforming spaces which have a layer thickness of less than three elements if the grid is not oriented so that the planes across which the gradients of the deformations in the conforming finite element space are allowed to be discontinuous are not parallel to the layers.

Two alternative methods have been developed to allow microstructure to be approximated on meshes which are not aligned with the microstructure. The first method was that of reduced integration of the multilinear element [7–9]. This method has been effectively used to compute microstructure with fine-scale layers on meshes which are not oriented with respect to the microstructure. For Laplace’s equation on a uniform grid, the deformation computed with this method can be shown to converge strongly, but the deformation gradients do not converge strongly. This would not be an effective method for the minimization of a quasi-convex energy, however this method can be used effectively with the non-convex Ericksen-James energy since its energy-minimizing deformations converge strongly while its gradients do not converge strongly. Most importantly, numerical experiments indicate the convergence of the microstructure or Young measure for the piecewise constant projection of the gradients of the deformation.

The approach analyzed here is that given by a family of nonconforming finite elements. The use of nonconforming finite elements is intuitively appealing for problems with microstructure because the admissible deformations have more flexibility to approximate oscillatory functions. These elements were first proposed and analyzed in [20] for solving the Stokes problem. Recently, we have used these finite elements to simulate the deformation of martensitic crystals with microstructure [15], and we found that with a suitable numerical quadrature they produce a very robust approximation method. Our analysis demonstrates that the deformation, as well as the deformation gradient, converges strongly for second order elliptic problems.

The first version of the considered elements is a finite element defined on rectangles (respectively, rectangular parallelepipeds) with degrees of freedom given by the values at midpoints of edges of the rectangles (respectively, the centers of the faces of the rectangular parallelepipeds). The second version is a finite element defined on rectangles (respectively, rectangular parallelepipeds) with the degrees of freedom given by the averages over the four edges of the rectangles (respectively, the six faces of the rectangular parallelepipeds). A discussion of the construction of these elements in the setting of the mixed finite element method can be found in [2].

Unlike most other nonconforming finite elements, these elements do not have any conforming counterparts. Consequently, the error analysis is nontrivial. We prove error estimates for these finite element approximations in both the $H^1$ and the $L^2$ norms. Our analysis contributes to the understanding of these elements by emphasizing their relation to the conforming multilinear elements. We also give new superconvergence
estimates for the error of the deformation gradient. In view of practical computations, especially for the computation of material microstructures, we also give an analysis of the effect of the numerical integration.

Throughout this paper we will mostly focus on the three-dimensional approximations. Naturally all results hold true for the two-dimensional approximations as well. For simplicity, let \( \Omega = (0, L_1) \times (0, L_2) \times (0, L_3) \) be a rectangular parallelepiped with faces parallel to the coordinate planes. We consider the following divergence-type second order elliptic boundary value problem,

\[
\mathcal{L}u \equiv -\frac{\partial}{\partial x} \left( a_1 \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( a_2 \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial z} \left( a_3 \frac{\partial u}{\partial z} \right) + cu = f, \quad \text{in} \ \Omega, \\
u = 0, \quad \text{on} \ \partial \Omega,
\]

where \( a_1, a_2, a_3 \in W^{1,\infty}(\Omega) \), \( a_1, a_2, a_3 \geq a_0 = \text{constant} > 0 \), a.e. \( \Omega \), \( c \in L^{\infty}(\Omega) \), \( c \geq 0 \), a.e. \( \Omega \), \( f \in L^2(\Omega) \). We define \( a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} \) by

\[
a(v, w) \equiv \int_\Omega \left( a_1 \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} + a_2 \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} + a_3 \frac{\partial v}{\partial z} \frac{\partial w}{\partial z} + cvw \right) dxdydz.
\]

It is obvious that \( a(\cdot, \cdot) \) is symmetric, continuous and bilinear. Furthermore, by the Poincaré inequality, \( a(\cdot, \cdot) \) is \( H_0^1(\Omega) \)-elliptic. We denote by \((\cdot, \cdot)\) the \( L^2(\Omega) \) inner product. The existence and uniqueness of the solution to the problem (1.1) follow from the Lax-Milgram lemma. The following theorem gives the regularity of the solution [12, 13].

**Theorem 1.1.** For any \( f \in L^2(\Omega) \), there exits a unique \( u \in H_0^1(\Omega) \cap H^2(\Omega) \) such that

\[
a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega).
\]

Furthermore, there holds the a priori estimate

\[
\|u\|_{2, \Omega} \leq C \|f\|_{0, \Omega},
\]

where \( C = C(a_1, a_2, a_3, c, \Omega) > 0 \) is a constant.

The rest of this paper is organized as follows. In Section 2 we define the finite elements and their corresponding finite element spaces based on a rectangular partition of \( \Omega \). We then prove a Poincaré type inequality for the test functions. In Section 3 we give the error estimates in both the \( H^1 \) and the \( L^2 \) norms. In Section 4, we discuss the relation of the considered finite elements to multilinear finite elements. In section 5, we give a superconvergence estimate for the gradients based on cubic partitions. Finally, in Section 6, we apply numerical quadrature to the finite element approximations, and we study the rates of convergence for several resulting schemes.

## 2. The Finite Elements

The first finite element is defined by the triple \((Q, P_Q, \Sigma_Q^p)\) where \( Q \equiv [a-\tau, a+\tau] \times [b-s, b+s] \times [c-t, c+t] \) is a rectangular parallelepiped with its center at \((a, b, c)\) and the lengths of its edges \( 2\tau, 2s, 2t \), where \( r, s, t > 0 \),

\[
P_Q = \text{Span} \left\{ 1, x, y, z, \left(\frac{x}{\tau}\right)^2 - \left(\frac{y}{s}\right)^2, \left(\frac{x}{\tau}\right)^2 - \left(\frac{z}{t}\right)^2 \right\},
\]

\[
\Sigma_Q^p = \{ q(M_i) : i = 1, \ldots, 6 \},
\]
where $M_i, i = 1, \ldots, 6$, are the centers of the faces of $Q$. This Lagrange-type element is well-defined since it is easy to verify that $\Sigma_P^i$ is $P_Q$-unisolvent, i.e., for any given $\alpha_i \in \mathbb{R}, i = 1, \ldots, 6$, there exists a unique $q \in P_Q$ such that

$$q(M_i) = \alpha_i, \quad i = 1, \ldots, 6.$$ 

We define $\varphi_i = \varphi_i(x, y, z) \in P_Q, i = 1, \ldots, 6$, such that

$$\varphi_i(M_j) = \delta_{ij}, \quad i, j = 1, \ldots, 6,$$

by permuting the terms $(x - a)/r, (y - b)/s$ and $(z - c)/t$ in the polynomial

$$\varphi(x, y, z) = \frac{1}{6} \left( \frac{x - a}{r} \right)^2 - \frac{1}{6} \left( \frac{y - b}{s} \right)^2 + \frac{1}{3} \left( \frac{z - c}{t} \right)^2 + \epsilon \frac{z - c}{2t} + \frac{1}{6},$$

where $\epsilon = \pm 1$. Hence, $\{\varphi_i\}_{i=1}^6$ is the standard basis for the finite element $(Q, P_Q, \Sigma_P^i)$. We then define the affine family of finite elements $(R, P_R, \Sigma_R^i)$ where $R$ is a rectangular parallelepiped.

Next, we define the averaged version of the preceding finite element to be the triple $(Q, P_Q, \Sigma_Q^a)$. The polynomial space $P_Q$ is the same as defined in (2.1) and the set of degrees of freedom is defined by

$$\Sigma_Q^a = \left\{ \int_{F_i} q dS : i = 1, \ldots, 6 \right\},$$

where $F_i, i = 1, \ldots, 6$, are the faces of the rectangular parallelepiped $Q$ and

$$\int_F = \frac{1}{|F|} \int_{F}$$

for faces $F \subset \partial Q$. This finite element is well-defined since $\Sigma_Q^a$ is $P_Q$-unisolvent. This can be easily checked by considering the six polynomials $\psi_i = \psi_i(x, y, z), i = 1, \ldots, 6$, obtained by permuting the terms $(x - a)/r, (y - b)/s$ and $(z - c)/t$ in the polynomial

$$\psi(x, y, z) = \frac{1}{4} \left( \frac{x - a}{r} \right)^2 - \frac{1}{4} \left( \frac{y - b}{s} \right)^2 + \frac{1}{2} \left( \frac{z - c}{t} \right)^2 + \epsilon \frac{z - c}{2t} + \frac{1}{6},$$

where $\epsilon = \pm 1$. It is obvious that $\psi_i \in P_Q, i = 1, \ldots, 6$, and it is easily checked that with a suitable labeling of the indices

$$\int_{F_i} \psi_j dS = \delta_{ij}.$$ 

Thus, $\{\psi_i\}_{i=1}^6$ is the standard basis for the finite element $(Q, P_Q, \Sigma_Q^a)$. Again, we define the affine family of finite elements $(R, P_R, \Sigma_R^a)$ where $R$ is a rectangular parallelepiped.

To construct a rectangular partition $\tau$ of $\Omega$, we define one-dimensional partitions of $[0, L_k]$, for $k = 1, 2, 3$, by

$$0 = x_k^0 < x_k^1 < \cdots < x_k^{m_k} = L_k,$$
where \( m_k \) are positive integers. We then define the rectangular parallelepipeds
\[
R_{i_1,i_2,i_3} = [x_1^{i_1-1}, x_1^{i_1}] \times [x_2^{i_2-1}, x_2^{i_2}] \times [x_3^{i_3-1}, x_3^{i_3}], \quad 1 \leq i_1 \leq m_1, \ldots, 1 \leq i_3 \leq m_3,
\]
and the rectangular partition
\[
\tau_h = \{ R_{i_1,i_2,i_3} : 1 \leq i_1 \leq m_1, \ldots, 1 \leq i_3 \leq m_3 \}
\]
with the mesh size parameter \( h \) defined by \( h = \max\{h_k : 1 \leq k \leq 3\} \), where \( h_k = \max\{x_k^{-1} - x_k^{-1} : 1 \leq i \leq m_k\} \) is the maximal discretization size in the \( k \)-th coordinate direction for \( k = 1, 2, 3 \).

For the first finite element, we define the set of nodal points \( N_h \) to be the set of all the centers of faces of elements in \( \tau_h \). The finite element spaces over the partition \( \tau_h \) are then defined respectively to be
\[
V_h^p = \left\{ v_h \in L^2(\Omega) : v_h |_{R} \in P_R, \forall R \in \tau_h ; v_h \text{ is continuous on } N_h \right\},
\]
\[
V_h^a = \left\{ v_h \in L^2(\Omega) : v_h |_{R} \in P_R, \forall R \in \tau_h ; \int_F v_h \big|_{R'} dS = \int_F v_h \big|_{R''} dS, \forall \text{ faces } F = \partial R' \cap \partial R'' \neq \emptyset, R', R'' \in \tau_h \right\}.
\]

To solve the Dirichlet problem (1.1), we define
\[
V_{oh}^p = \left\{ v_h \in V_h^p : v_h = 0 \text{ on } N_h \cap \partial \Omega \right\},
\]
\[
V_{oh}^a = \left\{ v_h \in V_h^a : \int_F v_h dS = 0, \forall \text{ faces } F \subset \partial R \cap \partial \Omega \neq \emptyset, R \in \tau_h \right\}.
\]

It is obvious that all of the spaces \( V_h^p, V_{oh}^p, V_h^a \) and \( V_{oh}^a \) are finitely dimensional subspaces of \( L^2(\Omega) \). They are also affine finite element spaces [6]. For \( v_h \in V_h^p \) (or \( V_{oh}^p, V_h^a, V_{oh}^a \)), we have in general that \( v_h \not\in C(\Omega) \) since \( v_h \) is continuous necessarily only at centers (or at some other points in the case of the \( V_h^a \)-approximation) of faces of adjacent elements. Therefore, \( V_h^p \) (or \( V_{oh}^p, V_h^a, V_{oh}^a \)) \( \not\subset C(\Omega) \), and, hence, \( V_h^p \) (or \( V_{oh}^p, V_h^a, V_{oh}^a \)) \( \not\subset H^1(\Omega) \). Thus, in view of solving a second order elliptic boundary value problem, the finite elements are nonconforming.

For convenience, we define for an integer \( k \geq 0 \) and \( p \in [1, \infty] \) the space
\[
W^{k,p}(\Omega) = \left\{ v \in L^p(\Omega) : v |_{R} \in W^{k,p}(R), \forall R \in \tau_h \right\},
\]
and equip \( W^{k,p}(\Omega) \) with the following seminorm and norm
\[
| \cdot |_{k,p,h} = \begin{cases} \left( \sum_{R \in \tau_h} | \cdot |_{k,p,R}^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max_{R \in \tau_h} | \cdot |_{k,\infty,R}, & \text{if } p = \infty; \end{cases}
\]
\[
\| \cdot \|_{k,p,h} = \begin{cases} \left( \sum_{R \in \tau_h} \| \cdot \|_{k,p,R}^p \right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \max_{R \in \tau_h} \| \cdot \|_{k,\infty,R}, & \text{if } p = \infty, \end{cases}
\]
where, for \( R \in \tau_h, | \cdot |_{k,p,R} \) and \( \| \cdot \|_{k,p,R} \) are the usual seminorm and norm on the Sobolev space \( W^{k,p}(R) \) [1]. If \( p = 2 \) we write \( H^k_h(\Omega) \) for \( W^{k,p}(\Omega) \) and omit \( p \) in all the above seminorm and norm expressions.

Now it is obvious that \( | \cdot |_{1,h} \) defines a norm on \( V_{oh}^p \) and \( V_{oh}^a \). We next prove a Poincaré type inequality for functions in the finite element spaces \( V_{oh}^p \) and \( V_{oh}^a \). This inequality leads to the uniform \( V_{oh}^p \)- and \( V_{oh}^a \)-ellipticity which is required in deriving the second Strang lemma [6,21].
Theorem 2.1. For any $v_h \in V_{0h}^p \cup V_{0h}^q$, we have

$$
(2.8) \quad \|v_h\|_{0,\Omega} \leq \sqrt{6} h \|v_h\|_{1,h} + \sqrt{2} L_k \left\| \frac{\partial v_h}{\partial x_k} \right\|_{0,h}, \quad k = 1, 2, 3.
$$

Proof. Let us fix $v \equiv v_h \in V_{0h}^p \cup V_{0h}^q$. For any $\bar{x} = (\bar{x}_1, \bar{x}') \in \Omega$, where $\bar{x}' = (\bar{x}_2, \bar{x}_3)$, let $R = [x_{1}^{i-1}, x_{1}^{i}] \times R'$, where $R' = [x_{2}^{j}, x_{2}^{j+1}] \times [x_{3}^{j}, x_{3}^{j+1}]$, be such that $\bar{x} \in R$. Without loss of generality, we assume $i_1 \geq 2$. Denote by $v_j$ the restriction of $v$ on $[x_{1}^{j}, x_{1}^{j+1}] \times R'$, for $j = 1, \ldots, m_1$. If $v \in V_{0h}^q$, we have

$$
\int_{R'} v_1(x_1^0, x') dx' + \sum_{j=1}^{i_1-1} \left[ \int_{R'} v_{j+1}(x_1^j, x') dx' - \int_{R'} v_j(x_1^j, x') dx' \right] = 0.
$$

Consequently, there exists $z' = (z_2^j, z_3^j) \in R'$ so that

$$
(2.9) \quad v_1(x_1^0, z') + \sum_{j=1}^{i_1-1} \left[ v_{j+1}(x_1^j, z') - v_j(x_1^j, z') \right] = 0.
$$

Observe that on each element in $\tau_h$,

$$
(2.10) \quad \frac{\partial v}{\partial x_k} \in \text{Span}\{1, x_k\}, \quad k = 1, 2, 3.
$$

By (2.9) and (2.10), we have

$$
(2.11) \quad v(\bar{x}) = v_{i_1}(\bar{x}_1, z') - v_{i_1}(x_1^{i_1-1}, z') + \sum_{j=1}^{i_1-1} \left[ v_{j}(x_1^j, z') - v_{j}(x_1^{j-1}, z') \right]
$$

$$
= \sum_{k=1}^{3} \int_{z_k}^{x_k} \frac{\partial v_{i_1}(x)}{\partial x_k} dx_k + \sum_{j=1}^{i_1-1} \int_{x_1^{j-1}}^{x_1^j} \frac{\partial v_j(x)}{\partial x_1} dx_1,
$$

where $x_1 = x_1^{i_1-1}$. This is also true for $v \in V_{0h}^p$ if we choose $z' \in R'$ so that $(x_1^0, z')$ is the center of the face $\{x_1^0\} \times R'$ of the element $[x_1^0, x_1^1] \times R'$. It then follows from (2.11), (2.10) and the Cauchy-Schwarz inequality that

$$
(2.12) \quad |v(\bar{x})|^2 \leq 6h \sum_{k=1}^{3} \int_{z_k^{i_k-1}}^{x_k^{i_k-1}} \left| \frac{\partial v_{i_1}(x)}{\partial x_k} \right|^2 dx_k + 2L_1 \sum_{j=1}^{m_1} \int_{x_1^{j-1}}^{x_1^j} \left| \frac{\partial v_j(x)}{\partial x_1} \right|^2 dx_1.
$$

Integrating (2.12) over $R$ and summing up the integrals over $R \in \tau_h$, we thus obtain (2.8) for $k = 1$. The same argument applies to $k = 2, 3$. $\Box$
3. $H^1_h$ and $L^2$ Error Estimates

We define $a_h(\cdot, \cdot) : H^1_h(\Omega) \times H^1_h(\Omega) \rightarrow \mathbb{R}$ by

$$a_h(v, w) \equiv \sum_{R \in \tau_h} \int_R \left( a_1 \frac{\partial v}{\partial x} \frac{\partial w}{\partial x} + a_2 \frac{\partial v}{\partial y} \frac{\partial w}{\partial y} + a_3 \frac{\partial v}{\partial z} \frac{\partial w}{\partial z} + cvw \right) dx dy dz.$$ 

We also denote $V_h = V^p_h$ or $V^a_h$ and $V_{0h} = V^p_{0h}$ or $V^a_{0h}$, respectively. It is clear that $a_h(\cdot, \cdot)$ is symmetric, continuous and bilinear. By Theorem 2.1, it is also uniform $V_{0h}$-elliptic, i.e., there exits a constant $\alpha > 0$, independent of $h$, such that

$$a_h(v_h, v_h) \geq \alpha \|v_h\|_{1, h}^2, \quad \forall v_h \in V_{0h}. \tag{3.1}$$

Therefore, by the Lax-Milgram lemma, there exists a unique finite element approximation $u_h \in V_{0h}$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{0h}. \tag{3.2}$$

In the sequel, the rectangular partitions $\tau_h$ are always assumed to be quasi-uniform, i.e., there exists a constant $\sigma > 0$, independent of $h$, such that

$$\min \{ x^i_k - x^{i-1}_k : i = 1, \ldots, m_k, k = 1, 2, 3 \} \geq \sigma h.$$ 

We denote the Lagrange interpolation operator $I_h : C(\tilde{\Omega}) \rightarrow V_h$ to be either $I^p_h : C(\tilde{\Omega}) \rightarrow V^p_h$ or $I^a_h : C(\tilde{\Omega}) \rightarrow V^a_h$ which are defined respectively for $I^p_h v \in V^p_h$ and $I^a_h v \in V^a_h$ by

$$I^p_h v(M) = v(M), \quad \forall M \in N_h,$$

$$\int_F I^a_h v \, dS = \int_F v \, dS, \quad \forall \text{ faces } F \subset \partial R \text{ where } R \in \tau_h,$$

for any $v \in C(\tilde{\Omega})$. We also use the same notation $I_h, I^p_h$ and $I^a_h$ to denote the restrictions of these operators over an element of the partition $\tau_h$.

We use the symbol $C$ to denote a generic constant varying with the context. This constant is always assumed to be independent of all the trial and test functions, the solution $u$ to (1.2) and the mesh size parameter $h$ unless the dependence is otherwise stated.

Let us recall the following well-known results on the estimates for interpolation errors and the inverse estimates for later use [6].

**Theorem 3.1.** For $k = 0, 1, 2$, we have

$$\|I_h v - v\|_{k, R} \leq Ch^{2-k} |v|_{2, R}, \quad \forall R \in \tau_h, \forall v \in H^2(R),$$

$$\|I_h v - v\|_{k, h} \leq Ch^{2-k} |v|_{2, h}, \quad \forall v \in H^2_h(\Omega). \quad \Box$$
Theorem 3.2. Let $k$ and $l$ be two integers such that $0 \leq k \leq l \leq 2$. Then for any $R \in \tau_h$ and any $v_h \in V_h$ we have
\[
|v_h|_{l,R} \leq C h^{k-l} |v_h|_{k,R},
\]
\[
|v_h|_{l,h} \leq C h^{k-l} |v_h|_{k,h},
\]
\[
|v_h|_{l,\infty,R} \leq C h^{k-l-\frac{3}{2}} |v_h|_{k,R}
\]
\[
|v_h|_{l,\infty,h} \leq C h^{k-l-\frac{3}{2}} |v_h|_{k,h}.
\]

Our main results in this section are the error estimates for the finite element approximations in the $H^1_h(\Omega)$ and the $L^2(\Omega)$ norms.

Theorem 3.3. Let $u \in H^1_0(\Omega) \cap H^2(\Omega)$ and $u_h \in V_0$ be the solutions to (1.2) and (3.2), respectively. We have
\[
\|u - u_h\|_{m,h} \leq C h^{2-m} \|u\|_{2,\Omega}, \quad m = 0, 1.
\]

To prove the theorem we need some auxiliary lemmas.

Lemma 3.4. Let $R \in \tau_h$ and $F \subset \partial R$ be a face of $R$ and let $P_0 \in R$ be an arbitrary point. Then the following estimates hold
\[
|w(P_0) - \int_R w \, dx \, dy \, dz| \leq C h |w|_{1,\infty,R}, \quad \forall w \in W^{1,\infty}(R),
\]
\[
\int_F \left| w - \int_F w \, dS \right|^p dS \leq C h^{2-\frac{p}{2}} |w|_{1,\infty,R}^p, \quad \forall w \in H^1(R), \quad p = 1, 2,
\]
\[
\int_{\partial R} w^2 \, dS \leq \frac{C}{h} |w|_{0,R}^2 + C h |w|_{1,R}^2, \quad \forall w \in H^1(R).
\]

Proof. Let $R = [a-r, a+r] \times [b-s, b+s] \times [c-t, c+t]$. Denote $\hat{R} = [-1, 1] \times [-1, 1] \times [-1, 1]$. Define the affine mapping $K_R : \hat{R} \rightarrow R$ by $K(\xi, \eta, \zeta) = (x, y, z)$ where
\[
x = r\xi + a, \quad y = s\eta + b, \quad z = t\zeta + c.
\]

For any function $w = w(P), P \in R$, write $\hat{w} = w \circ K_R$. Thus, by (3.7) and the imbedding $W^{1,\infty}(\hat{R}) \hookrightarrow C(\hat{R})$, we get
\[
|w(P_0) - \int_R w \, dx \, dy \, dz| = \left| \hat{w}(\hat{P}_0) - \int_{\hat{R}} \hat{w} \, d\xi \, d\eta \, d\zeta \right| \leq C \|\hat{w}\|_{1,\infty,\hat{R}},
\]

where $\hat{P}_0 = K_R^{-1}(P_0)$. Replacing $w$ by $w + p$ in (3.8) with $p$ any constant, by a known result similar to the Bramble-Hilber lemma [6], we have
\[
|w(P_0) - \int_R w \, dx \, dy \, dz| \leq C \inf_{\hat{p} \text{ constant}} \|\hat{w} + \hat{p}\|_{1,\infty,\hat{R}} \leq C \|\hat{w}\|_{1,\infty,\hat{R}} \leq C h |w|_{1,\infty,R}.
\]
This proves (3.4). A similar argument together with an application of the trace theorem (see, e.g., [1]) leads to (3.5). Finally, by the transformation (3.7), the quasi-uniformity of \( \tau_h \) and the trace theorem, we have

\[
\int_{\partial R} w^2 \, dS \leq C h^2 \int_{\partial R} \tilde{w}^2 \, d\tilde{S} \leq C h^2 \left[ \int_R \tilde{w}^2 \, d\xi d\eta d\zeta + \int_R |\nabla \tilde{w}|^2 \, d\xi d\eta d\zeta \right]
\]

\[
\leq C h^2 \left[ h^{-3} \int_R w^4 \, dx dy dz + h^{-1} \int_R |\nabla w|^2 \, dx dy dz \right],
\]

leading to (3.6). \( \square \)

In what follows, for \( R \in \tau_h \) and a face \( F \subset \partial R \), we define the functional \( T_F \) by either \( T_F(w) = w(M_F) \) for \( w \in C(F) \), where \( M_F \) is the center of the face \( F \), when considering the \( V_h^p \)-approximation, or by \( T_F(w) = \int_F w \, dS \) for \( w \in L^2(F) \) when considering the \( V_h^a \)-approximation.

**Lemma 3.5.** For any \( R \in \tau_h \) and any face \( F \subset \partial R \), we have

\[
\int_F [v_h - T_F(v_h)]^2 \, dS \leq C h|v_h|_{1,R}^2, \quad \forall v_h \in V_h.
\]

**Proof.** Let \( R, \hat{R} \), the mapping \( K_R : \hat{R} \rightarrow R \) be the same as in the proof of Lemma 3.4. Without loss of generality, we assume that \( F = \{a + r\} \times [b - s, b + s] \times [c - t, c + t] \). Let \( P_F \equiv (a + r, \bar{y}, \bar{z}) \in F \) be such that \( T_F(v_h) = v_h(P_F) \). Now, by (2.10), the Cauchy-Schwarz inequality and the quasi-uniformity of the partitions \( \tau_h \), we have

\[
\int_F [v_h - T_F(v_h)]^2 \, dS = \int_{b-s}^{b+s} \int_{c-t}^{c+t} [v_h(a + r, y, z) - v_h(a + r, \bar{y}, \bar{z})]^2 \, dy dz
\]

\[
= \int_{b-s}^{b+s} \int_{c-t}^{c+t} \left[ \int_{\bar{y}}^{y} \frac{\partial v_h(a + r, y', z)}{\partial y'} \, dy' + \int_{\bar{z}}^{z} \frac{\partial v_h(a + r, \bar{y}, z')}{\partial z'} \, dz' \right]^2 \, dy dz
\]

\[
\leq 4st \left( 4s \int_{b-s}^{b+s} \left| \frac{\partial v_h}{\partial y} \right|^2 \, dy + 4t \int_{c-t}^{c+t} \left| \frac{\partial v_h}{\partial z} \right|^2 \, dz \right) \leq C h|v_h|_{1,R}^2,
\]

completing the proof. \( \square \)

**Lemma 3.6.** Let \( R \in \tau_h \) and let \( F \subset \partial R \) be a face of \( R \). Then, for any trilinear function \( w = w(x, y, z) \) on \( R \), we have

\[
(3.9) \quad \int_F [(w - I_h w) - T_F(w - I_h w)] \, dS = 0.
\]

**Proof.** Without loss of generality, let \( R \in \tau_h \) and \( F \subset \partial R \) be the same as in the proof of Lemma 3.5. If \( w = 1, x, y \) or \( z \), then, \( I_h w = w \). Hence, (3.9) holds trivially. Now if \( w \) is not linear but multilinear with respect to the variables \( x - a, y - b \) and \( z - c \), then, a simple calculation shows that \( I_h w = 0 \), \( T_F(w) = 0 \) and (3.9) is true as well. Our proof is complete since all the trilinear functions are linear combinations of those functions tested above. \( \square \)
Now we are in a position to prove our theorem.

**Proof of the $H_h^1$ error estimate.** By Theorem 2.1, $a_h(\cdot, \cdot)$ is uniformly $V_{0h}$-elliptic. Hence, by the second Strang lemma [6,21], we have

\[
\|u - u_h\|_{1,h} \leq C \left[ \inf_{v_h \in V_{0h}} \|u - v_h\|_{1,h} + \sup_{0 \neq v_h \in V_{0h}} \frac{|d_h(u, v_h)|}{\|v_h\|_{1,h}} \right],
\]

where $d_h : H^2(\Omega) \times H_h^1(\Omega) \to \mathbb{R}$ is the nonconforming error functional defined by

\[
d_h(\varphi, v) = a_h(\varphi, v) - (L\varphi, v), \quad \varphi \in H^2(\Omega), \quad v \in H_h^1(\Omega),
\]

and where $L$ is the differential operator defined in (1.1). Since $u = 0$ on $\partial \Omega$, and $u \in H^2(\Omega) \hookrightarrow C(\Omega)$, we have that $I_hu \in V_{0h}$. Thus, Theorem 3.1 leads to the estimate

\[
\inf_{v_h \in V_{0h}} \|u - v_h\|_{1,h} \leq \|u - I_hu\|_{1,h} \leq C h \|u\|_{2,\Omega}.
\]

To estimate the error functional $d_h(\cdot, \cdot)$, we fix an arbitrary function $v = v_h \in V_{0h}$. Since $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is the solution to (1.2), by integration by parts, we get

\[
d_h(u, v) = \sum_{R \in \tau_h} \int_R \left( a_1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + a_2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} + a_3 \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} + cuv - f v \right) \, dxdydz
\]

\[
= \sum_{R \in \tau_h} \int_{\partial R} a_1 \frac{\partial u}{\partial x} v_n \, dS + \sum_{R \in \tau_h} \int_{\partial R} a_2 \frac{\partial u}{\partial y} v_n \, dS + \sum_{R \in \tau_h} \int_{\partial R} a_3 \frac{\partial u}{\partial z} v_n \, dS + I_1 + I_2 + I_3,
\]

where $n = (n_1, n_2, n_3)$ is the unit outer normal to the boundary $\partial R$ of an element $R \in \tau_h$.

By virtue of the definition of the $V_h^p$- and $V_h^a$-approximations, we have

\[
I_1 = \sum_{R \in \tau_h} \int_{\partial R} a_1 \frac{\partial u}{\partial x} v_n \, dS
\]

\[
= \sum_{R \in \tau_h} \sum_{\text{face } F \subset \partial R} \int_F a_1 \frac{\partial u}{\partial x} [v - T_F(v)] n_1 \, dS
\]

\[
= \sum_{R \in \tau_h} \sum_{\text{face } F \subset \partial R} \int_F \left( a_1 - \int_R a_1 \, dxdydz \right) \frac{\partial u}{\partial x} [v - T_F(v)] n_1 \, dS
\]

\[
+ \sum_{R \in \tau_h} \sum_{\text{face } F \subset \partial R} \left( \int_R a_1 \, dxdydz \right) \int_F \frac{\partial u}{\partial x} [v - T_F(v)] n_1 \, dS
\]

\[
= I_4 + I_5.
\]

It follows from the Cauchy-Schwarz inequality, Lemma 3.4 and Lemma 3.5 that

\[
|I_4| \leq C h \|u\|_{2,\Omega} \|v\|_{1,h}.
\]

\[
(3.15)
\]
To estimate $I_5$, we first consider the $V_h^a$-approximation. In this case, since $T_F(v) = \int_F v \, dS$, by the Cauchy-Schwarz inequality and Lemma 3.4, we have

$$|I_5| = \left| \sum_{R \in \mathcal{F}_h} \sum_{F \in \partial R} \left( \int_R a_1 \, dxdydz \right) \int_F \frac{\partial u}{\partial x} \left[ v - T_F(v) \right] n_1 \, dS \right|$$

$$= \left| \sum_{R \in \mathcal{F}_h} \sum_{F \in \partial R} \left( \int_R a_1 \, dxdydz \right) \int_F \left[ \frac{\partial u}{\partial x} - T_F \left( \frac{\partial u}{\partial x} \right) \right] \left[ v - T_F(v) \right] n_1 \, dS \right|$$

$$\leq C h \| u \|_{2, \Omega} \| v \|_{1, h}.$$  \hspace{1cm} (3.16)

In the case of the $V_h^p$-approximation, we fix an element $R = [a - r, a + r] \times [b - s, b + s] \times \{ c - t, c + t \} \in \mathcal{F}_h$ and consider its two opposite faces $F_\pm = \{ a \pm r \} \times [b - s, b + s] \times \{ c - t, c + t \}$ with $n_1 = n_\pm = \pm 1$. We then have by (2.10), by the Cauchy-Schwarz inequality, and by the quasi-uniformity of the partitions $\mathcal{F}_h$, that

$$\int_{F_+} \frac{\partial u}{\partial x} \left[ v - T_{F_+}(v) \right] n_+ \, dS + \int_{F_-} \frac{\partial u}{\partial x} \left[ v - T_{F_-}(v) \right] n_- \, dS$$

$$= \left| \int_{b-s}^{b+s} \int_{c-t}^{c+t} \frac{\partial u(a + r, y, z)}{\partial x} \left[ v(a + r, y, z) - v(a + r, b, c) \right] \, dydz \right|$$

$$- \int_{b-s}^{b+s} \int_{c-t}^{c+t} \frac{\partial u(a - r, y, z)}{\partial x} \left[ v(a - r, y, z) - v(a - r, b, c) \right] \, dydz \right|$$

$$\leq C h^{-2} \| u \|_{2, 1, R} \| v \|_{1, 1, R} \leq C h \| u \|_{2, 2, R} \| v \|_{1, R}.$$  \hspace{1cm} (3.17)

Consequently, by rearranging the terms in the summation $I_5$, we obtain the estimate for the $V_h^p$-approximation

$$|I_5| \leq C h \| u \|_{2, \Omega} \| v \|_{1, h}.$$  \hspace{1cm} (3.18)

It follows from (3.14)–(3.18) that

$$|I_1| \leq C h \| u \|_{2, \Omega} \| v \|_{1, h}.$$  \hspace{1cm} (3.19)

Similar estimates hold for $I_2$ and $I_3$. We then obtain by (3.11) and (3.13) the following estimate

$$|d_h(u, v_h)| \equiv |a_h(u, v_h) - (L, v_h)|$$

$$\leq C h \| u \|_{2, \Omega} \| v_h \|_{1, h}, \hspace{1cm} \forall v_h \in V_{0h}.$$  \hspace{1cm} (3.20)

This, together with (3.10) and (3.12), leads to (3.3) with $m = 1$. The first part of the proof of the theorem is finished.
Proof of the $L^2$ error estimate. We follow the nonconforming version of the Aubin-Nitsche argument [18,19]. Let $g \in L^2(\Omega)$. By Theorem 1.1, there exists a unique $\varphi_g \in H^1_0(\Omega) \cap H^2(\Omega)$ such that

\begin{align}
L \varphi_g &= g, \quad \text{in } \Omega, \\
\quad u &= 0, \quad \text{on } \Omega,
\end{align}

which by Theorem 1.1 satisfies

\begin{equation}
\| \varphi_g \|_{2,\Omega} \leq C \| g \|_{0,\Omega}.
\end{equation}

It is easy to verify, for any $\psi_h \in V_{0h}$, that

\begin{equation}
(u - u_h, g) = a_h(u - u_h, \varphi_g - \psi_h) - d_h(u, \varphi_g - \psi_h) - d_h(\varphi_g, u - u_h).
\end{equation}

Consequently,

\begin{equation}
\begin{aligned}
\| u - u_h \|_{0,\Omega} &= \sup_{0 \neq g \in L^2(\Omega)} \frac{(u - u_h, g)}{\| g \|_{0,\Omega}} \\
&\leq \sup_{0 \neq g \in L^2(\Omega)} \frac{1}{\| g \|_{0,\Omega}} \inf_{\psi_h \in V_{0h}} \left[ |a_h(u - u_h, \varphi_g - \psi_h)| + |d_h(u, \varphi_g - \psi_h)| \\
&\quad + |d_h(\varphi_g, u - u_h)| \right].
\end{aligned}
\end{equation}

Fix $g \in L^2(\Omega)$. Let $W_{0h} \subset C(\Omega) \cap H^1_0(\Omega)$ be the trilinear finite element space over the partition $\tau_h$. Denote by $Q_h : C(\Omega) \rightarrow W_{0h}$ the corresponding trilinear interpolation operator. We choose $\psi_h = I_h(Q_h \varphi_g) \in V_{0h}$. By the $H^1$ error estimate, Theorem 3.1, the well-known interpolation properties of the operator $Q_h$ [6], and (3.21), we have

\begin{equation}
|a_h(u - u_h, \varphi_g - \psi_h)| \leq C \| u - u_h \|_{1,h} \| \varphi_g - \psi_h \|_{1,h} \\
\leq C h \| u \|_{2,\Omega} \left( \| \varphi_g - Q_h \varphi_g \|_{1,\Omega} + \| Q_h \varphi_g - I_h(Q_h \varphi_g) \|_{1,h} \right) \\
\leq C h^2 \| u \|_{2,\Omega} \left( \| \varphi_g \|_{2,\Omega} + \| Q_h \varphi_g \|_{2,h} \right) \| u \|_{2,\Omega} \leq C h^2 \| g \|_{0,\Omega}.
\end{equation}

Since $Q_h \varphi_g \in H^1_0(\Omega)$ and $u \in H^1_0(\Omega) \cap H^2(\Omega)$ is the solution to (1.2), by denoting $\chi_h = Q_h \varphi_g - \psi_h$, we have

\begin{equation}
\begin{aligned}
d_h(u, \varphi_g - \psi_h) &= d_h(u, Q_h \varphi_g - \psi_h) \\
&= \sum_{R \in \tau_h} \int_{\partial R} \left[ a_1 \frac{\partial u}{\partial x} \chi_h n_1 + a_2 \frac{\partial u}{\partial y} \chi_h n_2 + a_3 \frac{\partial u}{\partial z} \chi_h n_3 \right] dS \\
&\equiv J_1 + J_2 + J_3.
\end{aligned}
\end{equation}

On a face $F \subset \partial R$ for an element $R \in \tau_h$, we have by a simple calculation (cf. (3.9)) that $f_{F} \chi_h dS = 0$. Hence, since $Q_h \varphi_g \in C(\Omega)$, by the same argument as in the proof
of the $H^1_0$ error estimate (cf. (3.14), (3.16)) we have

$$J_1 \equiv \sum_{R \in T_h} \int_{\partial R} a_1 \frac{\partial u}{\partial x} \chi n_1 dS = \sum_{R \in T_h, \text{face}} \int_{F} a_1 \frac{\partial u}{\partial x} \chi n_1 dS$$

$$= \sum_{R \in T_h, \text{face}} \int_{F} \left( a_1 - a_1 dxdydz \right) \frac{\partial u}{\partial x} \chi n_1 dS$$

$$+ \sum_{R \in T_h, \text{face}} \int_{F} \left( a_1 dxdydz \right) \frac{\partial u}{\partial x} \chi n_1 dS$$

$$= \sum_{R \in T_h, \text{face}} \int_{F} \left( a_1 - a_1 dxdydz \right) \frac{\partial u}{\partial x} \chi n_1 dS$$

$$+ \sum_{R \in T_h, \text{face}} \int_{F} \left( a_1 dxdydz \right) \frac{\partial u}{\partial x} \chi n_1 dS.$$

Therefore, by Lemma 3.4, the Cauchy-Schwarz inequality, Theorem 3.1, the properties of the operator $Q_h$, and (3.21), we get

$$|J_1| \leq C \sum_{R \in T_h, \text{face}} \int_{F} \left( h \|\nabla u\|_{0,F} + \left\| \frac{\partial u}{\partial x} - \int_{F} \frac{\partial u}{\partial x} dS \right\|_{0,F} \right) \left\| \chi - \int_{F} \chi dS \right\|_{0,F}$$

$$\leq C h \|u\|_{2,\Omega} \|\chi\|_{1,h} \leq C h^2 \|u\|_{2,\Omega} \|\varphi\|_{2,\Omega} \leq C h^2 \|u\|_{2,\Omega} \|g\|_{0,\Omega}. \quad (3.25)$$

Similarly,

$$|J_2| + |J_3| \leq C h^2 \|u\|_{2,\Omega} \|g\|_{0,\Omega}. \quad (3.26)$$

Now it follows from the fact $Q_h u \in H^1_0(\Omega)$ that

$$d_h (\varphi_g, u - u_h) = d_h (\varphi_g, Q_h u - I_h Q_h u) + d_h (\varphi_g, I_h Q_h u - u_h) \equiv J_4 + J_5. \quad (3.27)$$

By the same argument used in estimating $d_h (u, \varphi_g - \psi_h)$ (cf. (3.24)-(3.26)), we obtain

$$|J_4| \equiv |d_h (\varphi_g, Q_h u - I_h Q_h u)|$$

$$\leq C h^2 \|\varphi_g\|_{2,\Omega} \|Q_h u\|_{2,h} \leq C h^2 \|u\|_{0,\Omega} \|u\|_{2,\Omega}. \quad (3.28)$$

Denote $v = I_h Q_h u - u_h \in V_{Vh}$. Replacing $u$ by $\varphi_g$ in (3.19), by the $H^1_0$ error estimate, Theorem 3.1, the known properties of the operator $Q_h$ and (3.21), we then have

$$|J_5| \equiv |d_h (\varphi_g, v)| \leq C h \|\varphi_g\|_{2,\Omega} \|v\|_{1,h}$$

$$\leq C h \|g\|_{0,\Omega} (\|I_h Q_h u - Q_h u\|_{1,h} + \|Q_h u - u\|_{1,\Omega} + \|u - u_h\|_{1,h})$$

$$\leq C h^2 \|u\|_{2,\Omega} \|g\|_{0,\Omega}. \quad (3.29)$$

Finally, the $L^2$ error estimate (3.3) with $m = 0$ is a direct consequence of (3.22)-(3.29). \qed
4. Connection with multilinear finite elements

In the previous proof, we made use of the piecewise linear function \( I_h Q_h u \) several times as an approximation function of \( u \). This makes some connection between the considered nonconforming elements and the conforming multilinear elements. Furthermore, it is in fact true that all the piecewise linear functions in \( V_{0h} \) approximate the solution \( u \) well enough.

To be more precise, let \( W_{0h} \subset C(\bar{\Omega}) \cap H^1_0(\Omega) \) be again the trilinear finite element space over the mesh \( \tau_h \). By the proof of Lemma 3.6, we know that all the functions in the subspace \( I_h W_{0h} \subset V_{0h} \) are piecewise linear functions. Furthermore, by taking the boundary condition into account, it is easy to see that the operator \( I_h : W_{0h} \rightarrow I_h W_{0h} \) is in fact one-to-one and onto. Thus, the subspace \( I_h W_{0h} \subset V_{0h} \) has the same number of degrees of freedom as the trilinear finite element space \( W_{0h} \) does. Now by the Lax-Milgram lemma, there exists a unique \( \bar{u}_h \in I_h W_{0h} \) such that

\[
(4.1) \quad a_h(\bar{u}_h, v_h) = (f, v_h), \quad \forall v_h \in I_h W_{0h}.
\]

We realize that \( \bar{u}_h \) is in fact the projection of the finite element solution \( u_h \in V_{0h} \) into the subspace \( I_h W_{0h} \), since by (3.2) and (4.1), we have

\[
(4.2) \quad a_h(u_h - \bar{u}_h, v_h) = 0, \quad \forall v_h \in I_h W_{0h}.
\]

Now let us write the error

\[
(4.3) \quad u - \bar{u}_h = (u - I_h u_h^t) + (I_h u_h^t - \bar{u}_h),
\]

where \( u_h^t \in W_{0h} \) is the trilinear finite element approximation of \( u \), i.e.,

\[
(4.4) \quad a(u_h^t, w_h) = (f, w_h), \quad \forall w_h \in W_{0h}.
\]

By the known results on this approximation \( u_h^t \) [6], and by Theorem 3.1, we get

\[
(4.5) \quad \|u - I_h u_h^t\|_{1,h} \leq \|u - u_h^t\|_{1,\Omega} + \|u_h^t - I_h u_h^t\|_{1,h} \leq C h \|u\|_{2,\Omega}.
\]

Notice that \( \lambda_h \equiv I_h u_h^t - \bar{u}_h \in I_h W_{0h} \subset V_{0h} \). By (3.1), (4.5), (3.19) and the known results on \( u_h^t \), we have

\[
(4.6) \quad a_h(\lambda_h, \lambda_h) = a_h(I_h u_h^t - u, \lambda_h) + a_h(u - \bar{u}_h, \lambda_h)
\]

\[
\leq C \|I_h u_h^t - u\|_{1,h} \|\lambda_h\|_{1,h} + C \|d_h(u, \lambda_h)\| \leq C h \|u\|_{2,\Omega} \|\lambda_h\|_{1,h}.
\]

It follows from (4.3)-(4.6) that

\[
(4.7) \quad \|u - \bar{u}_h\|_{1,h} \leq C h \|u\|_{2,\Omega}.
\]

Now, by the definition of \( d_h(\cdot, \cdot) \) (cf. (3.11)) and (4.2), we have for any \( g \in L^2(\Omega) \) and any \( \psi_h \in I_h W_{0h} \) that

\[
(4.8) \quad (u - \bar{u}_h, g) = (u - u_h, g) + (u_h - \bar{u}_h, g)
\]

\[
= (u - u_h, g) + a_h(u_h - \bar{u}_h, \varphi_g - \psi_h) - d_h(\varphi_g, u_h - \bar{u}_h).
\]
NONCONFORMING ELEMENTS FOR MICROSTRUCTURES

Setting \( \psi_h = I_h Q_h \varphi_g \in I_h W_{0h} \) in (4.8), we have

\[
|a_h(u_h - \bar{u}_h, \varphi_g - \psi_h) - C \|u_h - \bar{u}_h\|_{1,h} \|\varphi_g - I_h Q_h \varphi_g\|_{1,h} \\
\leq C \left( \|u_h - u\|_{1,h} + \|u - \bar{u}_h\|_{1,h} \right) \left( \|\varphi_g - Q_h \varphi_g\|_{1,h} + \|Q_h \varphi_g - I_h Q_h \varphi_g\|_{1,h} \right) \\
(4.9)
\leq C h^2 \|u\|_{2,\Omega} \left( \|\varphi_g\|_{2,\Omega} + \|Q_h \varphi_g\|_{2,\Omega} \right) \leq C h^2 \|u\|_{2,\Omega} \|g\|_{0,\Omega},
\]

where we used the known properties of \( Q_h \), Theorem 3.1, Theorem 3.3, (4.7) and (3.21). Replacing \( u \) by \( \varphi_g \) in (3.19), we then get by (3.21), Theorem 3.1 and (4.7) that

\[
|d_h(\varphi_g, u_h - \bar{u}_h)| \leq C h \|\varphi_g\|_{2,\Omega} \|u_h - \bar{u}_h\|_{1,h} \\
\leq C h \|g\|_{0,\Omega} \left( \|u_h - u\|_{1,h} + \|u - \bar{u}_h\|_{1,h} \right) \\
\leq C h^2 \|u\|_{2,\Omega} \|g\|_{0,\Omega}.
(4.10)
\]

We have in fact proved, by (4.7)–(4.10) and Theorem 3.3, the following

**Theorem 4.1.** Let \( u \in H^1_0(\Omega) \cap H^2(\Omega) \) and \( \bar{u}_h \in I_h W_{0h} \) be the solutions to (1.2) and (4.1), respectively. Then,

\[
\|u - \bar{u}_h\|_{0,\Omega} + h \|u - \bar{u}_h\|_{1,h} \leq C h^2 \|u\|_{2,\Omega}. \quad \square
\]

5. A SUPERCONVERGENCE ESTIMATE

We first give a superconvergence estimate for the interpolation error gradients. Denote by \( C_R \) the gravitational center of a rectangular element \( R \in \tau_h \).

**Lemma 5.1.** For any \( R \in \tau_h \), we have

\[
(5.1) \quad |\nabla (v - I_h v)(C_R)| \leq C h^2 \|v\|_{3,\infty,R}, \quad \forall v \in W^{3,\infty}(\hat{R}).
\]

**Proof.** Let \( \hat{R} = [-1,1] \times [-1,1] \times [-1,1] \). Define: \( \hat{F} : W^{3,\infty}(\hat{R}) \longrightarrow \mathbb{R} \) by

\[
\hat{F}(\hat{v}) = \partial_\xi (\hat{v} - \hat{I} \hat{v})(O), \quad \hat{v} \in W^{3,\infty}(\hat{R}),
\]

where \( O = (0,0,0) \) and \( \hat{I} \) is the interpolation operator over \( \hat{R} \) for the considered elements. By the imbedding \( W^{3,\infty} \hookrightarrow C^1(\hat{R}) \), we have

\[
(5.2) \quad |\hat{F}(\hat{v})| \leq C \|\hat{v}\|_{3,\infty,\hat{R}}, \quad \hat{v} \in W^{3,\infty}(\hat{R}).
\]

Now the basis functions for our elements over \( \hat{R} \) can be easily obtained by setting \( a = b = c = 0 \) and \( r = s = t = 1 \) in (2.4) and (2.6), respectively. By their properties (cf. (2.3), (2.7)) and by the Taylor expansion, a series of calculations then lead to

\[
(5.3) \quad \hat{F}(\hat{\rho}) = 0, \quad \forall \hat{\rho} \in P_2(\hat{R}),
\]
where $P_2(\hat{R})$ is the set of all polynomials over $\hat{R}$ with degrees at most 2. It follows from (5.2), (5.3) and the Bramble-Hilbert lemma that

$$\|\hat{F}(\hat{v})\| \leq C\|\hat{v}\|_{3,\infty,\hat{R}}, \quad \hat{v} \in W^{3,\infty}(\hat{R}).$$

This, together with the affine transformation from $\hat{R}$ to $R$ (cf. (3.7)), leads to

$$|\partial_x(v - I_hv)(C_R)| \leq Ch^{-1}\|\hat{F}(\hat{v})\| \leq Ch^2\|v\|_{3,\infty,R}.$$

Similar estimates hold for $\partial_y(v - I_hv)$ and $\partial_z(v - I_hv)$. □

In the rest of this section, we will only consider the $V_h^a$-approximation, i.e., the averaged element approximation, to the solution of the model problem

$$-\Delta u = f, \quad \text{in } \Omega,$$

$$u = 0, \quad \text{on } \partial\Omega. \quad (5.4)$$

The following result shows that the nonconforming error functional $d_h(\cdot,\cdot)$, as defined in (3.11), is of one order higher than usual. Hence the nonconformity, in this case, is weak.

**Lemma 5.2.** Let $u \in H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$ be the solution to (5.4). Then,

$$\|d_h(u,v_h)\| \leq Ch^2\|u\|_{3,\infty,\Omega}\|v_h\|_{1,h}, \quad \forall v_h \in V_{0h}^a. \quad (5.5)$$

**Proof.** Fix $v = v_h \in V_{0h}^a$. We have as before that

$$d_h(u,v) = \sum_{R \in \tau_h} \int_{\partial R} \left( \frac{\partial u}{\partial x} v_{n_1} + \frac{\partial u}{\partial y} v_{n_2} + \frac{\partial u}{\partial z} v_{n_3} \right) dS \equiv K_1 + K_2 + K_3. \quad (5.6)$$

By virtue of the $V_{0h}^a$-approximation, we can further write

$$K_1 = \sum_{R \in \tau_h} \int_{\partial R} \frac{\partial u}{\partial x} v_{n_1} dS$$

$$= \sum_{R \in \tau_h} \sum_{\text{face } F \in \partial R} \int_F \left[ \frac{\partial u}{\partial x} - \frac{\partial u(M_F)}{\partial x} \right] [v - T_F(v)] n_1 dS, \quad (5.7)$$

where $M_F$ is the center of a face $F$. Fix $R = [a-r, a+r] \times [b-s, b+s] \times [c-t, c+t] \in \tau_h$ and consider its two opposite faces $F_{\pm} = \{a \pm r\} \times [b - s, b + s] \times [c - t, c + t]$ with $n_1 = n_{\pm} = \pm 1$. An application of the Bramble-Hilbert lemma leads to the estimate

$$\int_R \left[ \frac{\partial^2 u(x, y, z)}{\partial x^2} - \frac{\partial^2 u(x, b, c)}{\partial x^2} \right]^2 dx dy dz \leq Ch^5\|u\|_{3,\infty,R}. \quad (5.8)$$
It then follows from (2.10), the Cauchy-Schwarz inequality, (5.8) and Lemma 3.5, that

\[
\begin{aligned}
&\left| \int_{F_+} \left[ \frac{\partial u}{\partial x} - \frac{\partial u(M_{F_+})}{\partial x} \right] \left[ v - T_{F_+}(v) \right] n_+ dS \\
+ &\int_{F_-} \left[ \frac{\partial u}{\partial x} - \frac{\partial u(M_{F_-})}{\partial x} \right] \left[ v - T_{F_-}(v) \right] n_- dS \right| \\
= &\left| \int_{b-s}^{b+s} \int_{c-t}^{c+t} \left\{ \left[ \frac{\partial u(a + r, y, z)}{\partial x} - \frac{\partial u(a + r, b, c)}{\partial x} \right] \right. \\
- &\left. \left[ \frac{\partial u(a - r, y, z)}{\partial x} - \frac{\partial u(a - r, b, c)}{\partial x} \right] \right\} \left( v(a + r, y, z) - T_{F_+}(v) \right) dydz \right| \\
= &\left| \int_{b-s}^{b+s} \int_{c-t}^{c+t} \left\{ \int_{a-r}^{a+r} \left[ \frac{\partial^2 u(x, y, z)}{\partial x^2} - \frac{\partial^2 u(x, b, c)}{\partial x^2} \right] dx \right. \\
&\left. \left( v(a + r, y, z) - T_{F_+}(v) \right) dydz \right| \\
\leq & C h^{\frac{3}{2}} \| u \|_{3, \infty, \Omega} \| v \|_{1, R}.
\end{aligned}
\]

Consequently, by an rearrangement of the terms in the summation \( K_1 \), we have

\[
|K_1| \leq C h^{\frac{3}{2}} \| u \|_{3, \infty, \Omega} \sum_{R \in \tau_h} \| v \|_{1, R} \leq C h^2 \| u \|_{3, \infty, \Omega} \| v \|_{1, \Omega},
\]

where we also used the Cauchy-Schwarz inequality and the fact that

\[
(5.9) \quad |\tau_h| \equiv \sum_{R \in \tau_h} 1 \leq C h^{-3}.
\]

In the two-dimensional case, \( h^{-3} \) should be replaced by \( h^{-2} \). Similar estimates hold for \( K_2 \) and \( K_3 \) as well. Hence, (5.5) follows. \( \square \)

**Lemma 5.3.** Let \( u \in H_0^1(\Omega) \cap W^{3, \infty}(\Omega) \) be the solution to (5.4) and \( u_h \in V_{0h}^a \) its \( V_{0h}^a \)-approximation. Assume that all the elements in \( \tau_h \) are cubes. Then

\[
(5.10) \quad \| I_h u - u_h \|_{1, h} \leq C h^2 \| u \|_{3, \infty, \Omega}.
\]

**Proof.** Denote \( \gamma_h \equiv I_h u - u_h \in V_{0h}^a \). Since each element in \( \tau_h \) is assumed to be a cube, it is easy to see that \( \gamma_h \) is piecewise harmonic. On the other hand, since \( I_h = I_h^a \) is the interpolation operator for the \( V_{0h}^a \)-approximation, we have

\[
\int_F (I_h u - u) dS = 0, \quad \forall \text{ faces } F \subset \partial R, \forall R \in \tau_h.
\]

It then follows from (2.10) that
\[ a_h(I_h u - u, \gamma_h) = \sum_{R \in \tau_h} \int_R \nabla (I_h u - u) \nabla \gamma_h dx dy dz \]

(5.11) \[ = \sum_{R \in \tau_h} \sum_{F \in \partial R} \left[ \int_F (I_h u - u) \frac{\partial \gamma_h}{\partial x} n_1 dS + \int_F (I_h u - u) \frac{\partial \gamma_h}{\partial y} n_2 dS + \int_F (I_h u - u) \frac{\partial \gamma_h}{\partial z} n_3 dS \right] = 0. \]

Now, by (3.1), (5.11) and Lemma 5.2, we have

\[ \alpha \| \gamma_h \|_{1,h}^2 \leq a_h(\gamma_h, \gamma_h) = a_h(I_h u - u_h, \gamma_h) = a_h(u - u_h, \gamma_h) \leq d_h(u, \gamma_h) \leq C h^2 \| u \|_{3,\infty,\Omega} \| \gamma_h \|_{1,h}, \]

leading to (5.10). \qed

Now we present the main result in this section.

**Theorem 5.4.** With the same assumption as in Lemma 5.3, we have

(5.12) \[ \left[ \sum_{R \in \tau_h} |\nabla (u - u_h)(C_R)|^2 h^3 \right]^{\frac{1}{2}} \leq C h^2 \| u \|_{3,\infty,\Omega}. \]

*Proof.* By Lemma 5.1, Theorem 3.2 and Lemma 5.3, we have

\[ \left[ \sum_{R \in \tau_h} |\nabla (u - u_h)(C_R)|^2 h^3 \right]^{\frac{1}{2}} \leq C \left[ \sum_{R \in \tau_h} |\nabla (u - I_h u)(C_R)|^2 h^3 + \sum_{R \in \tau_h} |\nabla (I_h u - u_h)(C_R)|^2 h^3 \right]^{\frac{1}{2}} \]

\[ \leq C \left[ h^4 \| u \|_{3,\infty,\Omega}^2 + h^3 \sum_{R \in \tau_h} \left( h^{-\frac{3}{2}} \| I_h u - u_h \|_{1,R} \right)^2 \right]^{\frac{1}{2}} \]

\[ \leq C h^2 \| u \|_{3,\infty,\Omega}, \]

completing the proof. \qed

Now let us turn back to both the \( V_h^P \) and \( V_h^a \)-approximations with general partitions of the solution to the general problem (1.1). Recall that \( Q_h : C(\breve{\Omega}) \rightarrow W_{0h} \) is the interpolation operator for the trilinear finite element. We can easily obtain, if the solution \( u \) is smooth, that

(5.13) \[ |\nabla (u - Q_h u)(C_R)| + |\nabla (u - I_h Q_h u)(C_R)| = O(h^2). \]

As discussed in Section 4, the considered elements are connected with the conforming multilinear finite elements through the subspace \( I_h W_{0h} \). Thus, the estimate (5.13),
Lemma 5.1 and the known superconvergence results (cf. [17]) on the multilinear elements naturally lead to a conjecture on the pointwise superconvergence estimates for the error of the gradient: if the solution $u$ is smooth enough and the partitions $\tau_h$ are suitably regular, then

$$\max_{R \in \tau_h} \left| \nabla (u - u_h) (C_R) \right| = O(h^2).$$

In its discrete average form, the superconvergence estimate (5.12) for the simplest case makes the estimate (5.14) believable somewhat. However, compared with a recent work on higher order error estimates on the nonconforming Wilson finite element [5], the proof or disproof of (5.14) will be more difficult since our elements do not have any conforming counterparts, though there is some connection between our elements and the conforming multilinear elements.

6. EFFECT OF NUMERICAL INTEGRATION

We define on the reference element $\hat{R} = [-1,1] \times [-1,1] \times [-1,1]$ the numerical integration scheme

$$\int_{\hat{R}} \hat{g}(\xi, \eta, \zeta) d\xi d\eta d\zeta = \sum_{i=1}^{I} \hat{\omega}_i \hat{g}(\hat{Q}_i), \quad \hat{g} \in C(\hat{R}),$$

where $\hat{\omega}_i > 0$, $\hat{Q}_i \equiv (\xi_i, \eta_i, \zeta_i) \in \hat{R}$, $i = 1, \ldots, I$, and $I$ is a positive integer. Let us denote

$$\hat{P} = \text{Span}\{1, \xi, \eta, \zeta, \xi^2 - \eta^2, \xi^2 - \zeta^2\}.$$

We shall assume that the quadrature scheme is exact on $\hat{P}$, i.e.,

$$\int_{\hat{R}} \hat{p}(\xi, \eta, \zeta) d\xi d\eta d\zeta = \sum_{i=1}^{I} \hat{\omega}_i \hat{p}(\hat{Q}_i), \quad \hat{p} \in \hat{P},$$

and that the set of quadrature points

$$\{\hat{Q}_i\}_{i=1}^{I} \text{ in (6.1) contains a } P_1(\hat{R})\text{-unisolvent subset},$$

where $P_1(\hat{R})$ is the set of all linear polynomials over $\hat{R}$.

The conditions (6.2) and (6.3) are satisfied by the quadrature schemes

- **Scheme 1:** $I = 6$, all $\hat{\omega}_i = \frac{4}{3}$, $\{\hat{Q}_i\}_{i=1}^{6} = \{(\pm\hat{q}, 0, 0), (0, \pm\hat{q}, 0), (0, 0, \pm\hat{q})\}$;
- **Scheme 2:** $I = 8$, all $\hat{\omega}_i = 1$, $\{\hat{Q}_i\}_{i=1}^{8} = \{(\xi, \eta, \zeta) : \xi, \eta, \zeta = \pm\hat{q}\}$,

where $0 < \hat{q} \leq 1$. The computations for the dynamics of martensitic microstructure reported in [15] used Scheme 1 with $\hat{q} = 1$, in which case the nodes of the quadrature scheme are identical to the nodes of the finite element with respect to the $V_h^F$-approximation. Scheme 2 with $\hat{q} = 1/\sqrt{3}$ is the Gaussian quadrature over $\hat{R}$ with eight nodes of quadrature.
Now, for an element \( R \equiv [a - r, a + r] \times [b - s, b + s] \times [c - t, c + t] \in \tau_h \), let \( K_R : \tilde{R} \rightarrow R \) be the invertible affine mapping given by (3.7). Then, the quadrature scheme (6.1) induces automatically the following quadrature scheme over the element \( R \in \tau_h \),

\[
\int_R g(x, y, z) \, dx \, dy \, dz = \sum_{i=1}^I \omega_{i,R} g(Q_{i,R}), \quad g \in C(R),
\]

where

\[
\omega_{i,R} = \det(\nabla K_R) \tilde{w}_i, \quad Q_{i,R} = K_R(\tilde{Q}_i), \quad i = 1, \ldots, I.
\]

To apply the numerical quadrature to the finite element formulation (3.2), in what follows we assume that in (1.1) \( c \in C(\Omega) \) and \( f \in C(\Omega) \). Let us now define \( a_h^*(\cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R} \) by

\[
a_h^*(v_h, w_h) = \sum_{R \in \tau_h} \sum_{i=1}^I \omega_{i,R} \left[ \left( a_1 \frac{\partial v_h}{\partial x} \frac{\partial w_h}{\partial x} \right)(Q_{i,R}) + \left( a_2 \frac{\partial v_h}{\partial y} \frac{\partial w_h}{\partial y} \right)(Q_{i,R}) + \left( a_3 \frac{\partial v_h}{\partial z} \frac{\partial w_h}{\partial z} \right)(Q_{i,R}) + (c v_h w_h)(Q_{i,R}) \right], \quad v_h, w_h \in V_h,
\]

and define \( f_h^* : V_h \rightarrow \mathbb{R} \) by

\[
f_h^*(v_h) = \sum_{R \in \tau_h} \sum_{i=1}^I \omega_{i,R} (f v_h)(Q_{i,R}), \quad v_h \in V_h.
\]

Obviously, \( a_h^*(\cdot, \cdot) \) and \( f_h^*(\cdot) \) are discrete approximations for \( a_h(\cdot, \cdot) \) and \( f_h(\cdot) \equiv (f, \cdot) \), respectively.

By the uniform \( V_{0h} \)-ellipticity of \( a_h(\cdot, \cdot) \) given by (3.1) and the conditions (6.1) and (6.2), we have the following uniform \( V_{0h} \)-ellipticity of \( a_h^*(\cdot, \cdot) \) (cf. Theorem 4.1.2 in [6]).

**Lemma 6.1.** There exists a constant \( \alpha^* > 0 \), independent of \( h \), such that

\[
a_h^*(v_h, v_h) \geq \alpha^* \|v_h\|_{1,h}^2, \quad \forall v_h \in V_{0h}.
\]

\( \square \)

It is now a direct consequence of the Lax-Milgram lemma that there exists a unique \( u_h^* \in V_{0h} \), the discrete solution to (1.2), such that

\[
a_h^*(u_h^*, v_h) = f_h^*(v_h), \quad \forall v_h \in V_{0h}.
\]

Our main result in this section is that, with a certain smoothness of the coefficients in (1.1), the discrete solution \( u_h^* \in V_{0h} \) converges to the exact solution \( u \in H^1_0(\Omega) \cap H^2(\Omega) \) with the same rates as the solution \( u_h \in V_{0h} \) does.

**Theorem 6.2.** Assume that in (1.1), in addition, \( c \in W^{1,\infty}(\Omega) \) and \( f \in W^{1,\infty}(\Omega) \). Let \( u \in H^1_0(\Omega) \cap H^2(\Omega) \) and \( u_h^* \in V_{0h} \) be the solutions to (1.2) and (6.9), respectively. Then,

\[
\|u - u_h^*\|_{1,h} \leq C h \|f\|_{1,\infty,\Omega}.
\]
Theorem 6.3. Assume that in (1.1), in addition, all $a_1, a_2, a_3, c$ and $f$ are in $W^{2, \infty}(\Omega)$. Then,

\begin{equation}
\|u - u_h^*\|_{0, \Omega} \leq C h^2 \|f\|_{2, \infty, \Omega}.
\end{equation}

To prove these two theorems, we need to estimate the errors induced by the quadrature schemes (6.1) and (6.4). Thus, we first define the quadrature error functionals

\begin{equation}
\hat{E}(\hat{g}) \equiv \int_{\hat{R}} \hat{g} \, d\hat{\xi} \, d\hat{n} \, d\hat{\zeta} - \sum_{i=1}^{I} \omega_i \hat{g}(\hat{Q}_i), \quad \hat{g} \in C(\hat{R}),
\end{equation}

\begin{equation}
E_R(g) \equiv \int_{R} g \, dx \, dy \, dz - \sum_{i=1}^{I} \omega_{i,R} g(Q_{i,R}), \quad g \in C(R), \quad R \in \tau_h.
\end{equation}

Obviously,

\begin{equation}
E_R(g) = \text{det}(\nabla K_R) \hat{E}(\hat{g}), \quad \hat{g} = g \circ K_R \in C(\hat{R}).
\end{equation}

Recall that $P_R$ is the finite element polynomial space over the element $R \in \tau_h$ (cf. (2.1)).

Lemma 6.4. Let $a_1, a_2, a_3$ and $c$ be given in (1.1). Suppose $c \in W^{1, \infty}(\Omega)$. Then, for any $R \in \tau_h$ and any $v, w \in P_R$, we have

\begin{equation}
\begin{align*}
|E_R \left( a_1 \frac{\partial v}{\partial x} \frac{\partial w}{\partial z} \right) | + |E_R \left( a_2 \frac{\partial v}{\partial y} \frac{\partial w}{\partial z} \right) | + |E_R \left( a_3 \frac{\partial v}{\partial z} \frac{\partial w}{\partial z} \right) | + |E_R (cvw)| \\
\leq C h \|v\|_{2,R} \|w\|_{1,R}.
\end{align*}
\end{equation}

Proof. Let $R \equiv [a-r, a+r] \times [b-s, b+s] \times [c-t, c+t]$. As before, let the mapping $K_R : \hat{R} \rightarrow R$ be defined by (3.7). Write $\hat{\phi} = \varphi \circ K_R$ for $\varphi \in W^{1, \infty}(\hat{R})$. By (6.12), we have

\begin{equation}
|\hat{E}(\hat{\varphi} \hat{p})| \leq C \|\hat{\varphi}\|_{1, \infty, \hat{R}} \|\hat{p}\|_{0, \hat{R}}, \quad \forall \hat{\varphi} \in W^{1, \infty}(\hat{R}), \hat{p} \in P_{\hat{R}}.
\end{equation}

Replacing $\hat{\varphi}$ in (6.16) by $\hat{\varphi} + \hat{c}$ with $\hat{c}$ an arbitrary constant, by (6.2) we obtain

\begin{equation}
|\hat{E}(\hat{\varphi} \hat{p})| \leq C \inf_{\hat{c}} \|\hat{\varphi} + \hat{c}\|_{1, \infty, \hat{R}} \|\hat{p}\|_{0, \hat{R}} \leq C \|\hat{\varphi}\|_{1, \infty, \hat{R}} \|\hat{p}\|_{0, \hat{R}}, \quad \forall \hat{\varphi} \in W^{1, \infty}(\hat{R}), \hat{p} \in P_{\hat{R}}.
\end{equation}

Now let $a \in \{a_1, a_2, a_3, c\}$, $q \in \{v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\}$, and $p \in \{w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\}$ where $v, w \in P_R$. Note that $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \in P_R$ if $w \in P_R$. Setting $\hat{\varphi} = \hat{a} \hat{q}$ in (6.17), by (6.14), (3.7) and Theorem 3.2, we get

\begin{equation}
|E_R(aqp)| \leq C h^3 \|\hat{E}(\hat{a} \hat{q} \hat{p})\| \leq C h^3 \|\hat{a} \hat{q}\|_{1, \infty, \hat{R}} \|\hat{p}\|_{0, \hat{R}} \leq C h^{3/2} \|a\|_{1, \infty, R} \|q\|_{1,R} \|p\|_{0,R} \leq C h^{3/2} \|q\|_{1, \infty, R} \|p\|_{0,R} \leq C h^{3/2} \|v\|_{2,R} \|w\|_{1,R}.
\end{equation}

This proves (6.15). \(\square\)
Lemma 6.5. Suppose $f \in W^{1,\infty}(\Omega)$. Then, for any $R \in \tau_h$ and any $v \in P_R$, we have

$$|E_R(fv)| \leq Ch^\frac{3}{2} \|f\|_{1,\infty,R} \|v\|_{1,R}. \tag{6.19}$$

Proof. Since $\dot{E}(\phi) = 0$ for any constant polynomial $\phi$, by the Bramble-Hilbert lemma we have

$$\left| \dot{E}(\phi) \right| \leq C \|\phi\|_{1,\infty,\hat{R}}, \quad \forall \phi \in W^{1,\infty}(\hat{R}).$$

Taking $\phi = \dot{f}v$, by (6.14) and Theorem 3.2 we have

$$|E_R(fv)| \leq Ch^3 \left| \dot{E} \left( \dot{f}v \right) \right| \leq Ch^3 \left| \dot{f}v \right|_{1,\infty,\hat{R}} \leq Ch^4 \|v\|_{1,\infty,R} \leq Ch^\frac{3}{2} \|f\|_{1,\infty,R} \|v\|_{1,R},$$

completing the proof. □

Lemma 6.6. Let $R \in \tau_h$ and $a \in W^{2,\infty}(R)$. Then for any $v, w \in P_R$ we have

$$|E_R(av)| \leq Ch^\frac{3}{2} \|a\|_{2,\infty,R} \|v\|_{2,R}, \tag{6.20}$$

$$|E_R(aw)| \leq Ch^2 \|a\|_{2,\infty,R} \|v\|_{2,R} \|w\|_{2,R}. \tag{6.21}$$

Proof. By (6.12) and the imbedding $H^2(\hat{R}) \hookrightarrow C(\hat{R})$, we have

$$\left| \dot{E}(\dot{a}v) \right| \leq C \|\dot{a}v\|_{2,\hat{R}}. \tag{6.22}$$

By (6.14), (6.2), the Bramble-Hilbert lemma, and (3.7), we thus get

$$|E_R(av)| \leq Ch^3 \|\dot{a}v\|_{2,\hat{R}} \leq Ch^\frac{3}{2} \|av\|_{2,R} \leq Ch^\frac{3}{2} \|a\|_{2,\infty,R} \|v\|_{2,R}, \tag{6.23}$$

obtaining (6.20). Now, replacing $a$ in (6.20) by $aw$, we have, by Theorem 3.2, that

$$|E_R(aw)| \leq Ch^\frac{3}{2} \|aw\|_{2,\infty,R} \|v\|_{2,R} \leq Ch^\frac{3}{2} \|a\|_{2,\infty,R} \|w\|_{2,\infty,R} \|v\|_{2,R} \leq Ch^2 \|a\|_{2,\infty,R} \|v\|_{2,R} \|w\|_{2,R},$$

leading to (6.21). □

Notice that in the two-dimensional case, the orders $h^{\frac{3}{2}}$ in (6.19) and $h^{\frac{3}{2}}$ in (6.20) should be replaced by $h^2$ and $h^3$, respectively.

Proof of Theorem 6.2. By (6.8), (3.2), (6.9), Lemma 6.4, and Lemma 6.5, we have

$$\alpha^* \|u_h - u_h^*_h\|_{2,h}^2 \leq a_h^* (u_h - u_h^*, u_h - u_h^*) \leq a_h (u_h - u_h^*, u_h - u_h^*) + [f_h (u_h - u_h^*) - f_h^* (u_h - u_h^*)] \leq Ch \sum_{R \in \tau_h} \|u_h\|_{2,R} \|u_h - u_h^*\|_{1,R} + Ch^{\frac{3}{2}} \sum_{R \in \tau_h} \|f\|_{1,\infty,R} \|u_h - u_h^*\|_{1,R} \leq Ch \left( \|u_h\|_{2,h} + \|f\|_{1,\infty,\Omega} \right) \|u_h - u_h^*\|_{1,h}.$$
where we also used the Cauchy-Schwarz inequality and (5.9). It then follows from
Theorem 3.1—Theorem 3.3 and Theorem 1.1 that
\[ \|u - u_h^*\|_{1,h} \leq \|u - u_h\|_{1,h} + \|u_h - u_h^*\|_{1,h} \]
\[ \leq Ch\|u\|_{2,\Omega} + Ch \left( \|u_h - I_h u\|_{2,h} + \|I_h u - u\|_{2,h} + \|f\|_{1,\infty,\Omega} \right) \]
\[ \leq Ch \left( \|u\|_{2,\Omega} + \|f\|_{1,\infty,\Omega} \right) \leq Ch\|f\|_{1,\infty,\Omega}. \]
which is the result of Theorem 6.2. □

**Proof of Theorem 6.3.** For any \( g \in L^2(\Omega) \), by (3.2) and (6.9), the following identity holds,
\[ (u_h - u_h^*, g) = a_h (u_h - u_h^*, \varphi_{gh} - \psi_h) - [a_h (u_h^*, \psi_h) - a_h^* (u_h^*, \psi_h)] \]
\[ + \left[ f_h (\psi_h) - f_h^* (\psi_h) \right], \quad \forall \psi_h \in V_{0h}, \]
where \( \varphi_{gh} \in V_{0h} \) satisfies
\[ a_h (\varphi_{gh}, v_h) = (g, v_h), \quad \forall v_h \in V_{0h}, \]
and \( \varphi_g \in H^1_0(\Omega) \cap H^2(\Omega) \) satisfies (3.20) and (3.21). Consequently,
\[ \|u - u_h^*\|_{0,\Omega} \leq \|u - u_h\|_{0,\Omega} + \sup_{g \notin L^2(\Omega)} \frac{1}{\|g\|_{0,\Omega}} \inf_{\psi_h \in V_{0h}} \left[ \|a_h (u_h - u_h^*, \varphi_{gh} - \psi_h)\| \right] \]
\[ + \|a_h (u_h^*, \psi_h) - a_h^* (u_h^*, \psi_h)\| + \|f_h (\psi_h) - f_h^* (\psi_h)\| \]
(6.25)

Now let us fix \( g \in L^2(\Omega) \) and choose \( \psi_h = I_h \varphi_g \). By Theorem 3.1, Theorem 6.2, Theorem 3.3 and (3.21), we get
\[ \|a_h (u_h - u_h^*, \varphi_{gh} - \psi_h)\| \leq C \|u - u_h^*\|_{1,h} \|\varphi_{gh} - \psi_h\|_{1,h} \]
\[ \leq Ch^2 \|f\|_{1,\infty,\Omega} \|g\|_{0,\Omega}. \]
(6.26)

It follows from the definitions of \( a_h(\cdot, \cdot) \), \( a_h^*(\cdot, \cdot) \), and \( E_R(\cdot) \) that
\[ \|a_h (u_h^*, \psi_h) - a_h^* (u_h^*, \psi_h)\| \leq \sum_{R \in \tau_h} \left[ E_R \left( a_1 \frac{\partial u_h^*}{\partial x} \frac{\partial \psi_h}{\partial x} \right) + E_R \left( a_2 \frac{\partial u_h^*}{\partial y} \frac{\partial \psi_h}{\partial y} \right) \right] \]
\[ + E_R \left( a_3 \frac{\partial u_h^*}{\partial z} \frac{\partial \psi_h}{\partial z} \right) + |E_R (\psi_h)|. \]
(6.27)

Notice that \( \partial u_h^*|_{R}, \partial \psi_h|_{R} \in P_R \), for \( R \in \tau_h \), where \( \partial = \partial_x, \partial_y \) or \( \partial_z \). Therefore, by Lemma 6.6, the Cauchy-Schwarz inequality, Theorem 6.2, (3.21) and (1.3), we have
\[ \|a_h (u_h^*, \psi_h) - a_h^* (u_h^*, \psi_h)\| \leq Ch^2 \sum_{R \in \tau_h} \|u_h^*\|_{2,R} \|\psi_h\|_{2,R} \leq Ch^2 \|u_h^*\|_{2,h} \|I_h \varphi_g\|_{2,h} \]
\[ \leq Ch^2 \left( \|u\|_{2,\Omega} + \|f\|_{1,\infty,\Omega} \right) \|g\|_{0,\Omega} \]
\[ \leq Ch^2 \|f\|_{1,\infty,\Omega} \|g\|_{0,\Omega}. \]
By Lemma 6.6, (5.9), Theorem 3.1 and (3.21), we have

$$
|f_h(\psi_h) - f_h^*(\psi_h)| \leq C \sum_{R \in T_h} |E_R(f \psi_h)| \leq C h^{\frac{1}{2}} \sum_{R \in T_h} \|f\|_{2,\infty,R} \|\psi_h\|_{2,R}
$$

(6.28)

$$
\leq C h^2 \|f\|_{2,\infty,R} \|\psi_h\|_{2,R} \leq C h^2 \|f\|_{2,\infty,R} \|g\|_{0,R}.
$$

Now (6.11) is a direct consequence of the combination of (6.24)–(6.28) and Theorem 3.3. The proof is complete. □

REFERENCES


206 Church St. SE, School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA

*Email: kloucek@math.umn.edu*

206 Church St. SE, School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA

*Email: bli@math.umn.edu*

206 Church St. SE, School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455, USA

*Email: luskin@math.umn.edu*
<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1121</td>
<td>Nahum Shimkin &amp; Adam Shwartz</td>
<td>Asymptotically efficient adaptive strategies in repeated games, part II: Asymptotic optimality</td>
</tr>
<tr>
<td>1122</td>
<td>M.E. Bradley</td>
<td>Well-posedness and regularity results for a dynamic Von Kármán plate</td>
</tr>
<tr>
<td>1123</td>
<td>Zhangxin Chen</td>
<td>Finite element analysis of the 1D full drift diffusion semiconductor model</td>
</tr>
<tr>
<td>1124</td>
<td>Gang Bao &amp; David C. Dobson</td>
<td>Diffractive optics in nonlinear media with periodic structure</td>
</tr>
<tr>
<td>1125</td>
<td>Steven Cox &amp; Enrique Zuazua</td>
<td>The rate at which energy decays in a damped string</td>
</tr>
<tr>
<td>1126</td>
<td>Anthony W. Leung</td>
<td>Optimal control for nonlinear systems of partial differential equations related to ecology</td>
</tr>
<tr>
<td>1127</td>
<td>H.J. Sussmann</td>
<td>A continuation method for nonholonomic path-finding problems</td>
</tr>
<tr>
<td>1128</td>
<td>Yung-Jen Guo &amp; Walter Littman</td>
<td>The null boundary controllability for semilinear heat equations</td>
</tr>
<tr>
<td>1129</td>
<td>Q. Zhang &amp; G. Yin</td>
<td>Turnpike sets in stochastic manufacturing systems with finite time horizon</td>
</tr>
<tr>
<td>1130</td>
<td>I. Győri, F. Hartung &amp; J. Turi</td>
<td>Approximation of functional differential equations with time- and state-dependent delays by equations with piecewise constant arguments</td>
</tr>
<tr>
<td>1131</td>
<td>I. Győri, F. Hartung &amp; J. Turi</td>
<td>Stability in delay equations with perturbed time lags</td>
</tr>
<tr>
<td>1132</td>
<td>F. Hartung &amp; J. Turi</td>
<td>On the asymptotic behavior of the solutions of a state-dependent delay equation</td>
</tr>
<tr>
<td>1133</td>
<td>Pierre-Alain Gremaud</td>
<td>Numerical optimization and quasiconvexity</td>
</tr>
<tr>
<td>1134</td>
<td>Jie Tai Yu</td>
<td>Resultants and inversion formula for $N$ polynomials in $N$ variables</td>
</tr>
<tr>
<td>1135</td>
<td>Avner Friedman &amp; J.L. Velázquez</td>
<td>The analysis of coating flows in a strip</td>
</tr>
<tr>
<td>1136</td>
<td>Eduardo D. Sontag</td>
<td>Control of systems without drift via generic loops</td>
</tr>
<tr>
<td>1137</td>
<td>Yuan Wang &amp; Eduardo D. Sontag</td>
<td>Orders of input/output differential equations and state space dimensions</td>
</tr>
<tr>
<td>1138</td>
<td>Scott W. Hansen</td>
<td>Boundary control of a one-dimensional, linear, thermoelastic rod</td>
</tr>
<tr>
<td>1139</td>
<td>Robert Lipton &amp; Bogdan Vernescu</td>
<td>Homogenization of two phase emulsions with surface tension effects</td>
</tr>
<tr>
<td>1140</td>
<td>Scott Hansen &amp; Enrique Zuazua</td>
<td>Exact controllability and stabilization of a vibrating string with an interior point mass</td>
</tr>
<tr>
<td>1141</td>
<td>Bei Hu &amp; Jiongmin Yong</td>
<td>Pontryagin Maximum principle for semilinear and quasilinear parabolic equations with pointwise state constraints</td>
</tr>
<tr>
<td>1142</td>
<td>Mark H.A. Davis</td>
<td>A deterministic approach to optimal stopping with application to a prophet inequality</td>
</tr>
<tr>
<td>1143</td>
<td>M.H.A. Davis &amp; M. Zervos</td>
<td>A problem of singular stochastic control with discretionary stopping</td>
</tr>
<tr>
<td>1144</td>
<td>Bernardo Cockburn &amp; Pierre-Alain Gremaud</td>
<td>An error estimate for finite element methods for scalar conservation laws</td>
</tr>
<tr>
<td>1145</td>
<td>David C. Dobson &amp; Fadil Santosa</td>
<td>An image enhancement technique for electrical impedance tomography</td>
</tr>
<tr>
<td>1146</td>
<td>Jin Ma, Philip P롯ter, &amp; Jiongmin Yong</td>
<td>Solving forward-backward stochastic differential equations explicitly — a four step scheme</td>
</tr>
<tr>
<td>1147</td>
<td>Yong Liu</td>
<td>The equilibrium plasma subject to skin effect</td>
</tr>
<tr>
<td>1148</td>
<td>Ulrich Hornung</td>
<td>Models for flow and transport through porous media derived by homogenization</td>
</tr>
<tr>
<td>1149</td>
<td>Avner Friedman, Chaocheng Huang, &amp; Jiongmin Yong</td>
<td>Effective permeability of the boundary of a domain</td>
</tr>
<tr>
<td>1150</td>
<td>Gang Bao</td>
<td>A uniqueness theorem for an inverse problem in periodic diffractive optics</td>
</tr>
<tr>
<td>1151</td>
<td>Angelo Favini, Mary Ann Horn, &amp; Irena Lasiecka</td>
<td>Global existence and uniqueness of regular solutions to the dynamic von Kármán system with nonlinear boundary dissipation</td>
</tr>
<tr>
<td>1152</td>
<td>E.G. Kalnins &amp; Willard Miller, Jr.</td>
<td>Models of $q$-algebra representations: $q$-integral transforms and “addition theorems”</td>
</tr>
<tr>
<td>1153</td>
<td>E.G. Kalnins, V.B. Kuznetsov &amp; Willard Miller, Jr.</td>
<td>Quadrics on complex Riemannian spaces of constant curvature, separation of variables and the Gaudin magnet</td>
</tr>
<tr>
<td>1154</td>
<td>A. Kersch, W. Morokoff &amp; Chr. Werner</td>
<td>Selfconsistent simulation of sputtering with the DSMC method</td>
</tr>
<tr>
<td>1155</td>
<td>Bing-Yu Zhang</td>
<td>A remark on the Cauchy problem for the Korteweg-de Vries equation on a periodic domain</td>
</tr>
<tr>
<td>1156</td>
<td>Gang Bao</td>
<td>Finite element approximation of time harmonic waves in periodic structures</td>
</tr>
<tr>
<td>1157</td>
<td>Tao Lin &amp; Hong Wang</td>
<td>Recovering the gradients of the solutions of second-order hyperbolic equations by interpolating the finite element solutions</td>
</tr>
<tr>
<td>1158</td>
<td>Zhangxin Chen</td>
<td>$L^p$-posteriori error analysis of mixed methods for linear and quasilinear elliptic problems</td>
</tr>
<tr>
<td>1159</td>
<td>Todd Arbogast &amp; Zhangxin Chen</td>
<td>Homogenization of compositional flow in fractured porous media</td>
</tr>
<tr>
<td>1160</td>
<td>L. Qiu, B. Bernhardsson, A. Rantzer, E.J. Davison, P.M. Young &amp; J.C. Doyle</td>
<td>A formula for computation of the real stability radius</td>
</tr>
<tr>
<td>1161</td>
<td>Maria Inés Troparevsky</td>
<td>Adaptive control of linear discrete time systems with external disturbances under inaccurate modelling: A case study</td>
</tr>
<tr>
<td>1162</td>
<td>Petr Klouček &amp; Franz S. Rys</td>
<td>Stability of the fractional step $\Theta$-scheme for the nonstationary Navier-Stokes equations</td>
</tr>
<tr>
<td>1163</td>
<td>Eduardo Casas, Luis A. Fernández &amp; Jiongmin Yong</td>
<td>Optimal control of quasilinear parabolic equations</td>
</tr>
<tr>
<td>1164</td>
<td>Darrell Duffie, Jin Ma &amp; Jiongmin Yong</td>
<td>Black’s consol rate conjecture</td>
</tr>
<tr>
<td>1165</td>
<td>D.G. Aronson &amp; J.L. Vazquez</td>
<td>Anomalous exponents in nonlinear diffusion</td>
</tr>
</tbody>
</table>
Ruben D. Spies, Local existence and regularity of solutions for a mathematical model of thermomechanical phase transitions in shape memory materials with Landau-Ginzburg free energy

Pu Sun, On circular pipe Poiseuille flow instabilities

Angelo Favini, Mary Ann Horn, Irena Lasiecka & Daniel Tataru, Global existence, uniqueness and regularity of solutions to a Von Kármán system with nonlinear boundary dissipation

A. Dontchev, Tz. Donchev & I. Slavov, On the upper semicontinuity of the set of solutions of differential inclusions with a small parameter in the derivative

Jin Ma & Jiongmin Yong, Regular-singular stochastic controls for higher dimensional diffusions — dynamic programming approach

Alex Solomonoff, Bayes finite difference schemes

Todd Arbogast & Zhangxin Chen, On the implementation of mixed methods as nonconforming methods for second order elliptic problems

Zhangxin Chen & Bernardo Cockburn, Convergence of a finite element method for the drift-diffusion semiconductor device equations: The multidimensional case

Boris Mordukhovich, Optimization and finite difference approximations of nonconvex differential inclusions with free time

Avner Friedman, David S. Ross, and Jianhua Zhang, A Stefan problem for reaction-diffusion system

Alex Solomonoff, Fast algorithms for micromagnetic computations

Nikan B. Firoozye, Homogenization on lattices: Small parameter limits, H-measures, and discrete Wigner measures

G. Yin, Adaptive filtering with averaging

Wlodzimierz Byrc and Amir Dembo, Large deviations for quadratic functionals of Gaussian processes

Ilja Schmelzer, 3D anisotropic grid generation with intersection-based geometry interface

Alex Solomonoff, Application of multipole methods to two matrix eigenproblems

A.M. Latypov, Numerical solution of steady euler equations in streamline-aligned orthogonal coordinates

Bei Hu & Hong-Ming Yin, Semilinear parabolic equations with prescribed energy

Bei Hu & Jianhua Zhang, Global existence for a class of Non-Fickian polymer-penetrant systems

Rongze Zhao & Thomas A. Posrbergh, Robust stabilization of a uniformly rotating rigid body

Mary Ann Horn & Irena Lasiecka, Uniform decay of weak solutions to a von Kármán plate with nonlinear boundary dissipation

Mary Ann Horn, Irena Lasiecka & Daniel Tataru, Well-posedness and uniform decay rates for weak solutions to a von Kármán system with nonlinear dissipative boundary conditions

Mary Ann Horn, Nonlinear boundary stabilization of a von Kármán plate via bending moments only

Frank H. Shaw & Charles J. Geyer, Constrained covariance component models

Tomasz Luczaka, A greedy algorithm estimating the height of random trees

Timo Seppäläinen, Maximum entropy principles for disordered spins

Yuandan Lin, Eduardo D. Sontag & Yuan Wang, Recent results on Lyapunov-theoretic techniques for nonlinear stability

Svante Janson, Random regular graphs: Asymptotic distributions and contiguity

Rachid Ababou, Random porous media flow on large 3-D grids: Numerics, performance, & application to homogenization

Moshe Fridman, Hidden Markov model regression

Petr Klouček, Bo Li & Mitchell Luskin, Analysis of a class of nonconforming finite elements for Crystalline microstructures

Steven P. Lalley, Random series in inverse Pisot powers

Rudy Yaksick, Expected optimal exercise time of a perpetual American option: A closed-form solution

Rudy Yaksick, Valuation of an American put catastrophe insurance futures option: A Martingale approach

János Pach, Farhad Shahrokhi & Mario Szegedy, Application of the crossing number

Avner Friedman & Chaocheng Huang, Averaged motion of charged particles under their self-induced electric field

Joel Spencer, The Erdős-Hanani conjecture via Talagrand’s inequality

Zhangxin Chen, Superconvergence results for Galerkin methods for wave propagation in various porous media

Russell Lyons, Robin Pemantle & Yuval Peres, When does a branching process grow like its mean? Conceptual proofs of L log L criteria

Robin Pemantle, Maximum variation of total risk

Robin Pemantle & Yuval Peres, Galton-Watson trees with the same mean have the same polar sets

Robin Pemantle, A shuffle that mixes sets of any fixed size much faster than it mixes the whole deck

Itai Benjamini, Robin Pemantle & Yuval Peres, Martini capacity for Markov chains and random walks in varying dimensions

Wlodzimierz Bryc & Amir Dembo, On large deviations of empirical measures for stationary Gaussian processes

Martin Hildebrand, Some random processes related to affine random walks

Alexander E. Mazel & Yuri M. Suhov, Ground states of a Boson quantum lattice model