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By

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IMA Preprint Series # 531

June 1989

ON THE SEMILINEAR ELLIPTIC EQUATION
$\Delta w - \frac{1}{2}y\cdot\nabla w + \lambda w - w^{-\beta} = 0$ IN $\mathbf{R}^n$

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§0. Introduction. The purpose of this paper is to study positive radial solutions of the equation

$$\Delta w - \frac{1}{2}y\cdot\nabla w + f(w) = 0 \quad \text{in} \quad \mathbf{R}^n, \tag{0.1}$$

where $n \geq 1$, $f(w) = \lambda w - w^{-\beta}$, $\lambda > 0$ and $\beta > 1$. Equation (0.1) arises in the study of the quenching behavior of the solution of problem

$$\begin{aligned} &u_t - \Delta u = \frac{1}{(1-u)^\beta} \quad \text{if} \quad x\in\Omega, t>0,\\ &u(x,t) = 0 \quad \text{for} \quad x\in\partial\Omega, t\geq 0,\\ &u(x,0) = 0 \quad \text{for} \quad x\in\Omega, \end{aligned} \tag{0.2}$$

where $\Omega = B_R(0)$ is a ball of radius R with center zero in $\mathbf{R}^n$. We say that u quenches if u reaches 1 in finite time. Given a fixed constant $\beta > 1$, by a result of Acker and Walter [1], there is a positive constant $R_* = R_*(\beta)$ such that the solution u of (0.2) quenches at finite time T if $R > R_*$. By introducing the variables

$$y = \frac{x}{\sqrt{T-t}}, \quad s = -\ln(T-t), \quad w(y,s) = [1-u(x,t)]\,(T-t)^{-\gamma},$$

$w(y,s)$ satisfies the equation

$$w_s - (\Delta w - \frac{1}{2}y\cdot\nabla w + f(w)) = 0, \tag{0.3}$$

in the set $W = \{(y,s); |y| < R e^{s/2}, s > s_0 \equiv -\ln T\}$ with $\lambda = \gamma = \frac{1}{\beta+1}$. Set $\xi = |x|$ and $r = |y|$. Recall that the solution u of (0.2) is radially symmetric and strictly decreasing in ξ, hence $w(y,s)$ is also radially symmetric and strictly increasing in r(cf. [7]). We say that $w(r)$ is a positive radial solution or simply radial solution of (0.1) if there exists an $\eta > 0$ such that $w(0) = \eta, w'(0) = 0$, and that w satifies the equation

$$w'' + (\frac{n-1}{r} - \frac{r}{2})\,w' + f(w) = 0, r > 0. \tag{0.4}$$

Hereafter the prime will represent the derivative with respect to r. For convenience, we let

$$\begin{cases} k = \lambda^{-\gamma}, \quad \gamma = \frac{1}{\beta+1},\\ F(w) = \int_k^w f(t)\,dt, w > 0,\\ \rho(r) = \exp(-\frac{r^2}{4}), \quad \sigma(r) = r^{n-1}\,\rho(r). \end{cases}$$

Notice that k is the only positive zero of $f(w)$. Also,

$$\begin{cases} F \geq 0, F(w) \sim w^2 \quad \text{if} \quad w \to \infty, \\ F(w) \sim w^{1-\beta} \quad \text{if} \quad w \to 0, \\ \rho'(r) = -\frac{r}{2}\,\rho(r), \\ \sigma'(r) = (\frac{n-1}{r} - \frac{r}{2})\,\sigma(r). \end{cases}$$

This paper is organized as follows. In §1, we study radial solutions of (0.1) and prove that every non-constant radial solution $w(r)$ of (0.1) must be strictly increasing for all r sufficiently large. Moreover, $w(r) \to \infty$ as $r \to \infty$. In §2, we apply the method of [4] to study the asymptotic behavior of radial solutions of (0.1) as $r \to \infty$. We prove that any non-constant radial solution $w(r)$ of (0.1) grows either exponentially or polynomially and we obtain a precise asymptotic expansion of $w(r)$ at $r = \infty$. In §3, we use the results obtained in §2 to prove some uniqueness results. We show that every non-constant radial solution of (0.1) must grow exponentially, if $n = 1, \beta \geq 3$, and $\lambda = \gamma$. For $n \geq 2$ and $\lambda = \gamma$, we apply a method in [2] to prove that every non-constant radial solution of (0.1) with $\eta < k$ must intersect a certain singular solution at least twice. Finally, in §4, we apply the results of §§1-3 to study the quenching rate of the solution u of (0.2) near the quenching point $x = 0$.

§1. Preliminaries.

Let $n \geq 1, \beta > 1, \lambda > 0$ be three real parameters. The following lemma is motivated by [9].

LEMMA 1.1. *For any $\eta > 0$, there is a unique positive global solution $w(r) = w(r;\eta)$ of (0.4) with $w(0) = \eta$ and $w'(0) = 0$.*

Proof. Since $r = 0$ is a regular singular point and $f(w)$ is real analytic at $w = \eta$ the local existence and uniqueness follow. Let $[0, a)$ be the maximal interval on which w is defined. Multiplying the equation (0.4) by $\rho^2\, w'$, we obtain

$$(\frac{\rho^2\, w'^2}{2})' + \frac{n-1}{r}\,\rho^2\, w'^2 + \rho^2\, F(w)' = 0. \tag{1.1}$$

Integrating (1.1) over the interval $[0, r]$, where $r < a$, we obtain

$$\frac{\rho^2(r)\, w'^2(r)}{2} + \int_0^r \frac{n-1}{s}\,\rho^2(s)\, w'^2(s)\, ds + \int_0^r \rho^2(s)\, F(w(s))'\, ds = 0. \tag{1.2}$$

Integrating the last integral in (1.2) by parts gives

$$\frac{\rho^2(r)\, w'^2(r)}{2} + \int_0^r \frac{n-1}{s}\,\rho^2(s)\, w'^2(s)\, ds + \int_0^r s\,\rho^2(s)\, F(w(s))\, ds + \rho^2(r)\, F(w(r)) - F(\eta) = 0.$$

Hence we get

$$\frac{1}{2}\,\rho^2(r)\, w'(r)^2 + \rho^2(r)\, F(w(r)) \leq F(\eta). \tag{1.3}$$

From (1.3) and the standard continuation theorem it follows that $a = \infty$. Moreover, by (1.3), $w > 0$ for all $r \geq 0$. □

LEMMA 1.2. *Any critical point* r_0 *of a radial solution* w *of (0.1) is a local maximal point if* $w(r_0) > k$*, and a local minimal point if* $w(r_0) < k$*. Moreover, there cannot exist a point* r *with* $w(r) = k$ *and* $w'(r) = 0$ *except when* $w \equiv k$*.*

Proof. From the uniqueness of the initial value problem for ordinary differential equations it follows that there cannot exist a point r with $w(r) = k$ and $w'(r) = 0$ except when $w \equiv k$. Since at any critical point we have $w'' = -f(w)$, the lemma follows. □

LEMMA 1.3. *Any non-constant radial solution* w *of (0.1) which takes the value* k *only finitely many times must be strictly increasing for all* r *sufficiently large. Moreover,* $w(r) \to \infty$ *as* $r \to \infty$*.*

Proof. If w is a non-constant radial solution of (0.1) which takes the value k only finitely many times, then there exists a number $\bar{r} > 0$ such that either $w(r) > k$, or $w(r) < k$ for all $r \geq \bar{r}$. Suppose that $w(r) > k, \forall r \geq \bar{r}$. We claim that w is monotone for all r sufficiently large. If w is not monotone for all r sufficiently large, then there exist a local maximum and a local minimum which are both greater than k. This contradicts to Lemma 1.2. Hence w must be monotone for all r sufficiently large. The other case is similar. Applying Lemma 1.2, it is easily seen that there is $r_0 > \bar{r}$ such that either $w'(r) > 0$ or $w'(r) < 0$ for all $r \geq r_0$. We claim that the second case cannot happen.

Suppose that $w'(r) < 0, \forall r \geq r_0$. Then the limit $\lim_{r\to\infty} w(r) = l$ exists. We shall show that $l < k$. Indeed, if $l \geq k$, then $w(r) \geq k, \forall r \geq r_0$. Then

$$(\sigma\, w')' = -\sigma\, f(w) \leq 0, \forall r \geq r_0. \tag{1.4}$$

It follows that the limit $\lim_{r\to\infty}(\sigma\, w')(r) = l_1$ exists and $l_1 \leq 0$. If $l_1 < 0$ then w' is unbounded and hence w is also unbounded which is a contradiction. It follows that $l_1 = 0$. Now, we integrate (1.4) from r_0 to ∞ to obtain

$$\int_{r_0}^{\infty} \sigma\, f(w) = \sigma(r_0)\, w'(r_0) < 0,$$

which is a contradiction, since $\sigma\, f(w) \geq 0, \forall r \geq r_0$. Therefore, $l < k$.

Following the idea in [9], we rewrite equation (0.4) as

$$\frac{w''}{r} + (\frac{n-1}{r^2} - \frac{1}{2})\, w' = -\frac{1}{r}\, f(w). \tag{1.5}$$

Since $\int_{r_0}^{\infty} w'(r)\, dr = l - w(r_0)$, which is finite, there exists a sequence $r_j \to \infty$ such that $w'(r_j) \to 0$ as $j \to \infty$. Integrating (1.5) from r_0 to r_j, we get

$$\int_{r_0}^{r_j} \frac{w''}{r}\, dr + \int_{r_0}^{r_j} (\frac{n-1}{r^2} - \frac{1}{2})\, w'\, dr = -\int_{r_0}^{r_j} \frac{1}{r}\, f(w)\, dr. \tag{1.6}$$

The first integral in (1.6) can be computed as

$$\int_{r_0}^{r_j} \frac{w''}{r}\,dr = \int_{r_0}^{r_j} (\frac{w'(r)}{r^2})\,dr + \frac{w'(r_j)}{r_j} - \frac{w'(r_0)}{r_0},$$

which is finite as $j \to \infty$. The second integral in (1.6) is also finite. On the other hand, since $l < k$, there is a positive number R such that $w(r) \le w(R) < k, \forall r \ge R$. Thus we get

$$\int_{R}^{r_j} \frac{1}{r} f(w)\,dr \le f(w(R)) \int_{R}^{r_j} \frac{dr}{r} \to -\infty \quad \text{as} \quad j \to \infty,$$

a contradiction. Hence w must be strictly increasing for all $r \ge r_0$ and the limit,

$$\lim_{r \to \infty} w(r) = l,$$

exists. Moreover, $l = \infty$ by a similar reasoning as above. □

Consider the phase plane of the system

$$\begin{aligned} w' &= v \\ v' &= (\frac{r}{2} - \frac{n-1}{r})v - f(w). \end{aligned} \tag{1.7}$$

We have the following positively invariant lemma (cf. [4]).

LEMMA 1.4. *For any $\alpha > 0$, there exists a number $R = R(\alpha, \lambda)$ such that if $r_1 \ge R$ and $(w(r_1), v(r_1)) \in A_\alpha \equiv \{(w, v); w \ge k, v \ge \alpha\, w\}$, then $(w(r), v(r)) \in A_\alpha, \forall r \ge r_1$.*

Proof. Choose $R = R(\alpha, \lambda)$ so that $\frac{R}{2} - \frac{n-1}{R} = \alpha + \frac{\lambda}{\alpha}$. In the phase plane of the system (1.7), we have

$$\begin{aligned} &w' > 0, \\ &v' = (\frac{r}{2} - \frac{n-1}{r})v \ge (\alpha + \frac{\lambda}{\alpha})v > 0, \\ &\frac{v'}{w'} = (\frac{r}{2} - \frac{n-1}{r}) \ge (\alpha + \frac{\lambda}{\alpha}) > \alpha, \end{aligned} \tag{1.8}$$

on the line $\{w = k, v \ge \alpha\, w\}$ and for $r \ge R$. Also, on the line $\{w > k, v = \alpha\, w\}$, we have

$$\begin{aligned} &w' > 0, \\ &v' = (\frac{r}{2} - \frac{n-1}{r})\alpha\, w - f(w) \ge \alpha^2 w > 0, \\ &\frac{v'}{w'} = (\frac{r}{2} - \frac{n-1}{r}) - \frac{\lambda}{\alpha} + \frac{w^{-\beta}}{v} > \alpha, \end{aligned} \tag{1.9}$$

if $r \ge R$. Hence the lemma follows from (1.8) and (1.9). □

We say that a non-constant radial solution w of (0.1) is oscillatory about k if w takes the value k infinitely many times.

LEMMA 1.5. *There does not exist a non-constant radial solution which is oscillatory about k.*

Proof. Let w be a non-constant radial solution of (0.1). We rewrite equation (0.4) as

$$\left(\frac{w'^2}{2}+F(w)\right)'=\left(\frac{r}{2}-\frac{n-1}{r}\right)w'^2. \tag{1.10}$$

If $r \geq R_0 \equiv \sqrt{2(n-1)}$, then the right-hand side of (1.10) is non-negative, hence the limit,

$$\lim_{r\to\infty}\left(\frac{w'^2}{2}+F(w)\right)=l,$$

exists and $l \geq 0$, since $F \geq 0$. We shall claim that $l=\infty$.

If $l=0$, then $w'(r)\to 0$ and $w(r)\to k$ as $r\to\infty$. Integrating (1.10) from r to ∞, we obtain

$$-\frac{w'(r)^2}{2}-F(w(r))=\int_r^\infty\left(\frac{s}{2}-\frac{n-1}{s}\right)w'(s)^2\,ds, \tag{1.11}$$

for $r \geq R_0$. Since the right-hand side of (1.11) is non-negative, the left-hand side of (1.11) is non-positive, and r is arbitrary, we conclude that $w' \equiv 0$ and $w \equiv k$. This contradiction leads to the conclution $l>0$.

Suppose that $l \in (0,\infty)$. Then w must be oscillatory about k by Lemma 1.3. For any sequence of extremal points $r_m \to \infty$, we have

$$F(w(r_m)) \to l \quad \text{as} \quad m\to\infty. \tag{1.12}$$

In particular, for any sequence of maximal points $r_m\to\infty$, we have

$$w(r_m)\to F^{-1}(l)\cap\{w>k\} \quad \text{as} \quad m\to\infty,$$

hence w is bounded from above. Similarly, we can prove that w is bounded from below and away from zero. That is,

$$0<c_1\leq w\leq c_2<\infty, \forall r>0, \tag{1.13}$$

for some constants c_1 and c_2.

Let $v=w'$. Then v satisfies the equation

$$v''+\left(\frac{n-1}{r}-\frac{r}{2}\right)v'+\left(\lambda-\frac{1}{2}-\frac{n-1}{r^2}+\beta\, w^{-\beta-1}\right)v=0. \tag{1.14}$$

Define $u(\tau)=v(r)$, where $\tau=\tau(r)=\int_1^r\frac{ds}{\sigma(s)}$, for $r\geq 1$. Notice that τ strictly increases to ∞ as r increases to ∞. Then u satisfies the equation

$$\frac{d^2\,u}{d\,\tau^2}+\sigma^2(r)\left(\lambda-\frac{1}{2}-\frac{n-1}{r^2}+\beta\, w^{-\beta-1}\right)u=0, \tau>0. \tag{1.15}$$

Applying L'Hôpital's rule, we get

$$\lim_{r\to\infty}[r^2\,\sigma^2(r)] = \lim_{r\to\infty}\left[\frac{\left(\int_1^r \frac{ds}{\sigma(s)}\right)^2}{\sigma^{-2}(r)}\right]$$

$$= 0.$$

Since $(\lambda - \frac{1}{2} - \frac{n-1}{r^2} + \beta\, w^{-\beta-1})$ is bounded by (1.13), we conclude that

$$\lim_{r\to\infty} r^2\,\sigma^2(r)\,(\lambda - \frac{1}{2} - \frac{n-1}{r^2} + \beta\, w^{-\beta-1}) = 0.$$

Applying a non-oscillatory criterion of Hartman (cf. [11, p.362]), we see that v can only have finitely many zeros. This contradicts the fact that w is oscillatory about k. Consequently, $l = \infty$, i.e.,

$$\lim_{r\to\infty}(\frac{w'^2}{2} + F(w)) = \infty. \tag{1.16}$$

Suppose now w is oscillatory about k. We can choose $R_m \to \infty$ so that $w(R_m) = k$ and $w'(R_m) > 0$. Thus, by (1.16), $w'(R_m) \to \infty$ as $m \to \infty$. Given $\alpha > 0$, choose m_0 so large that $R_{m_0} \geq R(\alpha, \lambda)$ and $w'(R_{m_0}) > \alpha\, k$. Then we obtain $(w(R_{m_0}), v(R_{m_0})) \in A_\alpha$. So, by Lemma 1.4, $(w(r), v(r)) \in A_\alpha, \forall r \geq R_{m_0}$. In particular, we have $w'(r) > 0$, for all $r \geq R_{m_0}$, a contradiction. Hence the lemma is proved. □

As a corollary of the previous lemmas, we have the following theorem.

THEOREM 1.6. *Every non-constant radial solution w of (0.1) must be strictly increasing for all r sufficiently large, and $w(r) \to \infty$ as $r \to \infty$.*

COROLLARY 1.7. *The only radial solution of (0.1) which is bounded from above is the constant solution $w \equiv k$.*

§2. Asymptotic behavior of $w(r)$ as $r \to \infty$.

In this section we apply the method used in [4] which is based on using L'Hôpital's rule to study the asymptotic behavior of w as $r \to \infty$, where w is a radial solution of (0.1). The results of this section are quite similar to the results in [4]. First, we shall prove a lemma analogous to Lemma 7 in [4].

LEMMA 2.1. *The limit, $\alpha = \lim_{r\to\infty} \frac{w'(r)}{w(r)}$, exists and $\alpha \in \{0, \infty\}$.*

Proof. Let

$$\alpha_1 = \liminf_{r\to\infty} \frac{w'(r)}{w(r)}, \alpha_2 = \limsup_{r\to\infty} \frac{w'(r)}{w(r)}.$$

Then we have $0 \leq \alpha_1 \leq \alpha_2 \leq \infty$. Suppose that $\alpha_1 < \alpha_2$. Then, for $\alpha \in (\alpha_1, \alpha_2)$, there is a sequence $r_m \to \infty$ as $m \to \infty$, such that

$$\frac{w'(r_m)}{w(r_m)} > \alpha \quad \text{for all} \quad m.$$

Choose m_0 so large that $w(r_{m_0}) \geq k$ and $r_{m_0} \geq R(\alpha, \lambda)$. Then, by Lemma 1.4, $\frac{w'(r)}{w(r)} \geq \alpha, \forall r \geq r_{m_0}$. Thus we obtain that $\liminf_{r\to\infty} \frac{w'(r)}{w(r)} \geq \alpha > \alpha_1$, a contradiction. Therefore, the limit, $\alpha = \lim_{r\to\infty} \frac{w'(r)}{w(r)}$, exists and is non-negative by Theorem 1.6.

Suppose that $\alpha \in (0, \infty)$. Then $w'(r) \to \infty$ as $r \to \infty$, since $w(r) \to \infty$ as $r \to \infty$. Applying L'Hôpital's rule, we get

$$\begin{aligned} \alpha &= \lim_{r\to\infty} \frac{w''(r)}{w'(r)} \\ &= \lim_{r\to\infty} \left[\frac{(\frac{r}{2} - \frac{n-1}{r})\, v - \lambda\, w + w^{-\beta}}{v} \right] \\ &= \frac{1}{2} \lim_{r\to\infty} r - \frac{\lambda}{\alpha}, \end{aligned}$$

a contradiction. Hence α must be either 0 or ∞. □

In the sequel, we use the following definition which is similar to a definition in [4].

DEFINITION 2.2. *A non-constant radial solution $w(r)$ of (0.1) is called a fast orbit if $\alpha = \infty$, and is called a slow orbit if $\alpha = 0$.*

Starting with Lemma 2.1, we can prove the following theorems on asymptotic behavior of w at $r = \infty$. The proofs are almost the same as the proofs in [4] and hence will not be repeated here.

THEOREM 2.3. *If $\lambda > \gamma$ and w is a slow orbit, then*

$$w(r) = A_1 \, r^{2\lambda} [1 - 2\lambda (2\lambda + n - 2) r^{-2} + o(r^{-2})]$$

as $r \to \infty$ for some positive constant A_1.

THEOREM 2.4. *If $\lambda \leq \gamma$ and w is a slow orbit, then*

$$w(r) = A_3 \, r^{2\lambda} [1 - \frac{A_4}{\delta} r^{-\delta} + o(r^{-\delta})]$$

as $r \to \infty$, where $\delta = 2\lambda(\beta + 1)$ and A_3 and A_4 are some positive constants.

THEOREM 2.5. *Let w be a fast orbit. Then there exists a positive constant A such that*

$$w(r) = A \exp\left(\frac{r^2}{4}\right) r^{-(n+2\lambda)} [1 + 2(n + 2\lambda)(\lambda + 1) r^{-2} + o(r^{-2})]$$

as $r \to \infty$.

§**3. Uniqueness results.** In this section, we restrict our attention to the case when $\lambda = \gamma$. Hence we have $k = \gamma^{-\gamma}$ and $f(w) = \gamma w - w^{-\beta}$.

3.1. The case when $n = 1$**.** We shall prove that every non-constant radial solution of (0.1) must be a fast orbit, if $\beta \geq 3$. In fact, we have the following slightly more general theorem.

THEOREM 3.1. *If* $\frac{\beta+1}{\beta-1} \leq \frac{2}{n}$, *then every non-constant orbit must be a fast orbit.*

Theorem 3.1 will follow from the following identity of Pohozaev's type.

LEMMA 3.2. *If* w *is a slow orbit, then*

$$\left(1 - \frac{n(\beta+1)}{2(\beta-1)}\right) \int_0^\infty \sigma (w')^2 + \left(\frac{1}{2(\beta-1)} + \frac{1}{4}\right) \int_0^\infty r^2 \sigma (w')^2 = 0. \tag{3.1}$$

Proof. Multiplying the equation (0.4) by σw, $r^2 \sigma w$ and $r \sigma w'$ respectively and integrating from 0 to ∞, we obtain, using Theorem 2.4, the following identities

$$\int_0^\infty \sigma (w')^2 - \gamma \int_0^\infty \sigma w^2 + \int_0^\infty \sigma w^{1-\beta} = 0, \tag{3.2}$$

$$\int_0^\infty r^2 \sigma (w')^2 - n \int_0^\infty \sigma w^2 + (\frac{1}{2} - \gamma) \int_0^\infty r^2 \sigma w^2 + \int_0^\infty r^2 \sigma w^{1-\beta} = 0, \tag{3.3}$$

$$\begin{aligned}(1 - \tfrac{n}{2}) \int_0^\infty \sigma (w')^2 + \frac{1}{4} \int_0^\infty r^2 \sigma (w')^2 + \frac{\gamma n}{2} \int_0^\infty \sigma w^2 - \frac{\gamma}{4} \int_0^\infty r^2 \sigma w^2 \\ + \frac{n}{\beta - 1} \int_0^\infty \sigma w^{1-\beta} - \frac{1}{2(\beta-1)} \int_0^\infty r^2 \sigma w^{1-\beta} = 0.\end{aligned} \tag{3.4}$$

Then the lemma follows by taking

$$(3.2) \times \left(\frac{-n}{\beta - 1}\right) + (3.3) \times \left(\frac{1}{2(\beta-1)}\right) + (3.4) = 0.$$

□

Remark 3.3. For $n = 1$, the condition on β in Theorem 3.1 is equivalent to $\beta \geq 3$. Since $\beta > 1$, we see that Theorem 3.1 is only true for $n < 2$ and for some β.

3.2. The case when $n \geq 2$**.** We shall prove the following theorem which is similar to a result in [2].

THEOREM 3.4. *Let*

$$\phi(r) = K\, r^{2\gamma}, \quad K = [2\gamma(2\gamma + n - 2)]^{-\gamma}. \tag{3.5}$$

Then the only radial solution $w = w(r;\eta)$ of (0.1) with $\eta \leq k$, which intersects ϕ exactly once, is $w \equiv k$.

The proof of this theorem is based on the idea of [2] which is called the Wronskion argument in [5]. Note that ϕ is a singular solution of the equation

$$w'' + (\frac{n-1}{r} - \frac{r}{2})\, w' + \gamma\, w - w^{-\beta} = 0, r > 0. \tag{3.6}$$

In fact, ϕ satisfies

$$\gamma\,\phi = \frac{1}{2}\, r\, \phi', \quad \phi'' + \frac{n-1}{r}\,\phi' - \phi^{-\beta} = 0,$$

for $r > 0$. For any radial solution w of (0.1), we define

$$g(r) = \gamma\, w - \frac{1}{2}\, r\, w', \quad h(r) = w'' + \frac{n-1}{r}\, w'.$$

For convenience, we set

$$p(r) = \frac{n-1}{r} - \frac{r}{2}.$$

By a simple computation,

$$g'' + p\, g' = \beta\, w^{-1}\, f\, g. \tag{3.7}$$

Using the identity $h = w^{-\beta} - g$, we obtain

$$h'' + p\, h' = \beta\,(\beta + 1)\, w^{-(\beta+2)}\, w'^2 + \beta\, w^{-(\beta+1)}\, f - \beta\, w^{-1}\, f\, g. \tag{3.8}$$

LEMMA 3.5. *g must have a zero for any fast orbit or any slow orbit with $\eta < k$.*

Proof. Note that for a given fast orbit w, by Theorem 2.5, we have $g(r) \to -\infty$ as $r \to \infty$. Then g must have a zero, since $g(0) = \gamma\,\eta > 0$. This proves the lemma for the case of fast orbit.

The proof for the other case is more difficult. Given any slow orbit w with $\eta < k$, we define

$$W(r) = g(r)\, h'(r) - g'(r)\, h(r).$$

It follows from (3.7) and (3.8) that

$$W' + p\, W = \beta\,(\beta + 1)\, w^{-(\beta+2)}\, (w')^2\, g. \tag{3.9}$$

Note that $W(0) = 0$, $g(0) > 0$, and $h(0) = -f(\eta)$, which is positive since $\eta < k$. From (3.9), we obtain

$$W(r) = \sigma^{-1}(r) \int_0^r \sigma(t)\, \beta\, (\beta + 1)\, w^{-(\beta+2)}(t)\, [w'(t)]^2\, g(t)\, dt.$$

Hence $W(r) > 0$ if $g > 0$ in the interval $(0, r)$. Notice that

$$(\frac{h}{g})' = \frac{W}{g^2}. \tag{3.10}$$

Therefore, we have

$$h(r) = \frac{h(0)}{g(0)}\, g(r) + g(r) \int_0^r \frac{W(t)}{g^2(t)}\, dt. \tag{3.11}$$

Suppose that g does not have any zeros. Then $g(r) > 0$ for all $r > 0$, since $g(0) > 0$.

Case 1. $\liminf_{r\to\infty} g(r) > 0$, i.e., there exists $\delta > 0$ such that $g(r) \geq \delta$ for all $r > 0$. By (3.11),

$$h(r) \geq \frac{h(0)}{g(0)}\, \delta \equiv c > 0, \forall r > 0.$$

Hence

$$\left(r^{n-1}\, w'(r)\right)' \geq c\, r^{n-1}, \forall r > 0.$$

Therefore, we get

$$w'(r) \geq \frac{c}{n}\, r, \forall r > 0,$$

which is a contradiction, since $w' \to 0$ as $r \to \infty$ by Theorem 2.4.

Case 2. $\liminf_{r\to\infty} g(r) = 0$. Note that g cannot be nondecreasing for all r sufficiently large. Otherwise, g is bounded below and away from zero, which is not the case. First, we claim that $g'(r) < 0$ for all r sufficiently large. If not, then there exists a sequence $\{r_m\}$ such that $r_m \to \infty$ as $m \to \infty$, and g takes a local maximum at the point r_m for each m. Recall that g satisfies the equation

$$g'' + p\, g' = \beta\, (\gamma - w^{-(\beta+1)})\, g, \tag{3.12}$$

and $w \to \infty$ as $r \to \infty$. Then $g''(r_m) > 0$, if m is sufficiently large. This is impossible, since $g''(r_m) \leq 0$ for all m. Therefore, there exists a $r_0 > 0$ such that $g'(r) \leq 0$ for all $r \geq r_0$. Hence, by choosing $r_1 > r_0$ such that $w(r) > k$ for all $r \geq r_1$, we get $g' < 0$ for all $r \geq r_1$. Otherwise, if there exists a $r > r_1$ such that $g'(r) = 0$, then $g''(r) > 0$ by (3.12), which is a contradiction. Consequently, $g(r) \to 0$ as $r \to \infty$, and $\limsup_{r\to\infty} g'(r) = 0$.

Next, we claim that $g' \to 0$ as $r \to \infty$. Otherwise, we have $\liminf_{r\to\infty} g'(r) < 0$. Then there exist $r_m \to \infty$ and $\delta > 0$ such that

$$g'(r_m) \leq -\delta, \quad g''(r_m) = 0, \quad \forall m.$$

Hence, by (3.12),

$$p(r_m)\,g'(r_m) = \beta\,[\gamma - w^{-(\beta+1)}(r_m)]\,g(r_m). \tag{3.13}$$

This is impossible, since the left-hand side of (3.13) is unbounded and the right-hand side of (3.13) tends to zero as $m \to \infty$. Hence we conclude that $g'(r) \to 0$ as $r \to \infty$.

Now, using (3.11) and applying L'Hôpital's rule, we obtain

$$\begin{aligned}\lim_{r\to\infty} h(r) &= \lim_{r\to\infty}\left[g(r)\int_0^r \frac{W(t)}{g^2(t)}\,dt\right]\\ &= -\lim_{r\to\infty}\frac{W(r)}{g'(r)}\\ &= \infty.\end{aligned}$$

This is impossible, since w is a slow orbit. Hence g must have a zero. □

Theorem 3.4 will follow from the following lemma.

LEMMA 3.6. *For any fast orbit w or any slow orbit w with $\eta < k$, w intersects ϕ at least twice.*

Proof. Let

$$W(r) = w(r)\,\phi'(r) - w'(r)\,\phi(r).$$

Then W satisfies the equation

$$W(r) = 2\,K\,r^{2\,\gamma-1}\,g(r) \quad \text{for} \quad r > 0. \tag{3.14}$$

Also, $W \to \infty$ as $r \to 0$ and W satisfies

$$W' + p\,W = w\,\phi\,[\phi^{-(\beta+1)} - w^{-(\beta+1)}] \quad \text{for} \quad r > 0. \tag{3.15}$$

First, we claim that w intersects ϕ at least once. Otherwise, we have $w > \phi$ for all $r > 0$. It follows from (3.15) that $(\sigma\,W)'(r) > 0$ for all $r > 0$. Hence $W > 0$ for all $r > 0$. Therefore, by (3.14), $g > 0$ for all $r > 0$, which is a contradiction to Lemma 3.5.

Let w be a fast orbit. If w only intersects ϕ at one point r_0, then $w > \phi$ for all $r \neq r_0$, since $w > \phi$ for all r sufficiently large by Theorem 2.5. This is impossible by the same reasoning as above. Hence w intersects ϕ at least twice.

Next, let w be a slow orbit with $\eta < k$. If w only intersects ϕ at the point r_0, then we have

$$w < \phi, \forall r > r_0 \quad \text{and} \quad w > \phi, \forall r < r_0. \tag{3.16}$$

Let r_1 be the first zero of g. Then $g(r) > 0$ for all $r < r_1$. Thus, by (3.14), we have

$$W(r) > 0, \forall r < r_1, \quad \text{and} \quad W(r_1) = 0. \tag{3.17}$$

It follows from (3.17) that

$$W'(r_1) \le 0. \tag{3.18}$$

We claim that $r_1 > r_0$ and $W'(r_1) < 0$. Suppose that $r_1 \le r_0$. Then, by (3.15)-(3.18), $W'(r_1) = 0$ and $w(r_1) = \phi(r_1)$. Hence, by the definition of W, we have $w'(r_1) = \phi'(r_1)$. It follows from the uniqueness of the initial value problem for ordinary differential equations that $w \equiv \phi$, which is a contradiction. Therefore, we must have $r_1 > r_0$. Moreover, by (3.15)-(3.17), we have $W'(r_1) < 0$.

Now, let $r_2 > r_1$ such that $W(r_2) < 0$. By (3.15) and (3.16), we obtain that $(\sigma W)'(r) < 0$ for all $r > r_0$. Hence we have

$$(\sigma W)(r) < (\sigma W)(r_2) < 0, \quad \forall r > r_2.$$

Since

$$\left(\frac{\phi}{w}\right)'(r) = \frac{W(r)}{w^2(r)},$$

we get

$$\frac{\phi(r)}{w(r)} \le \frac{\phi(r_2)}{w(r_2)} + (\sigma W)(r_2) \int_{r_2}^{r} \frac{1}{\sigma(t)\, w^2(t)}\, dt, \tag{3.19}$$

for all $r > r_2$. Since w is a slow orbit, by Theorem 2.4, we see that the second term in the right-hand side of (3.19) tends to $-\infty$ as $r \to \infty$. Consequently, $\phi(r) < w(r)$ for some $r > r_2$, which is a contradiction to (3.16). Therefore, w must intersect ϕ at least twice. $\square$

§4. Application.

In this section, we shall always assume that $R > R_*$. Hence the solution u of (0.2) quenches at time $T < \infty$. By the strong maximum principle, we have

$$u_t > 0$$

in $Q_T \equiv \Omega \times (0, T)$. We say that x is a quenching point for u, if there are sequences $x_m \to x$ and $t_m \uparrow T$ as $m \to \infty$ such that $u(x_m, t_m) \to 1$ as $m \to \infty$. We shall use the method of blow-up to prove that the solution u of (0.2) quenches only at the point $x = 0$ and to study the quenching rate of u at this point.

4.1. One point quenching.

Let $v = \frac{1}{1-u}$. Then v satisfies

$$\begin{aligned} &v_t - \Delta v + 2\frac{|\nabla v|^2}{v} - v^{2+\beta} = 0 \quad \text{in} \quad Q_T, \\ &v(x,t) = 1 \quad \text{for} \quad x \in \partial\Omega, t \ge 0, \\ &v(x,0) = 1 \quad \text{for} \quad x \in \Omega. \end{aligned} \tag{4.1}$$

Since v is radial, we see that $v = v(\xi, t)$ satisfies

$$\begin{aligned} & v_t - v_{\xi\xi} - \frac{n-1}{\xi} v_\xi + 2\frac{{v_\xi}^2}{v} - v^{2+\beta} = 0 \quad \text{in} \quad Q_T, \\ & v(\xi,t) = 1 \quad \text{for} \quad \xi = R, t \geq 0, \\ & v(\xi,0) = 1 \quad \text{for} \quad 0 \leq \xi \leq R, \end{aligned} \tag{4.2}$$

where $\xi = |x|$, $v_\xi = \frac{\partial v}{\partial \xi}$, and $v_{\xi\xi} = \frac{\partial^2 v}{\partial \xi^2}$.

We say that v blows up at time T if $\max_{|x| \leq R} v(x,t) \to \infty$ as $t \uparrow T$. Also, a is a blow-up point for v if there exists a sequence $\{(x_n, t_n)\}$ with $x_n \to a$ and $t_n \uparrow T$ such that $v(x_n, t_n) \to \infty$ as $n \to \infty$ (see [6]). Note that u quenches at time T if and only if v blows up at time T; a is a quenching point for u if and only if a is a blow-up point for v.

Lemma 4.1. *For any positive constant δ such that $\delta < \min(R, T)$, we have*

$$v_\xi \leq -\eta(\xi - \delta)^2 v^2$$

in $Q_T \cap \{\delta \leq |x| \leq R, t \geq \delta\}$ for some positive constant $\eta = \eta(\delta, R)$.

Proof. Comparing the solution $u(x,t)$ with the solution $\psi(x,t)$ of

$$\begin{aligned} & \psi_t - \Delta \psi = 0 \quad \text{in} \quad Q_T \cap \{t \geq \delta\}, \\ & \psi(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega, t \geq \delta, \\ & \psi(x,\delta) = u(x,\delta) \quad \text{for} \quad x \in \Omega, \end{aligned}$$

we obtain that $u \geq \psi > 0$ in $\Omega \times (\delta, T)$. Therefore, there is a constant $\alpha > 0$ such that

$$u_\xi \leq \psi_\xi \leq -\alpha$$

on $\partial\Omega \times (\delta, T)$. Since $v_\xi = u_\xi$ on $\partial\Omega \times (\delta, T)$, we obtain that

$$v_\xi \leq -\alpha$$

on $\partial\Omega \times (\delta, T)$.

Following [6], we consider the function

$$J(\xi,t) = v_\xi(\xi,t) + \eta(\xi - \delta)^2 v^2(\xi,t) \quad \text{in} \quad (\delta, R) \times (\delta, T),$$

where the positive constant $\eta = \eta(\delta, R)$ is chosen so small that $J(\xi, t) \leq 0$ on the parabolic boundary of $(\delta, R) \times (\delta, T)$. Since

$$\begin{aligned} & J_t - J_{\xi\xi} - \frac{n-1}{\xi} J_\xi + b_1 J_\xi + b_2 J \\ & = -\beta\eta(\xi - \delta)^2 v^{\beta+3} - 2\eta v^2 - \eta(n-1)\left[2\left(\frac{\xi - \delta}{\xi}\right) - \left(\frac{\xi-\delta}{\xi}\right)^2\right] v^2 \\ & \leq 0 \end{aligned}$$

for some functions b_1 and b_2, we conclude from the maximum principle that $J \leq 0$ in the set $(\delta, R) \times (\delta, T)$. Hence the lemma follows. □

THEOREM 4.2. *The solution u of (0.2) quenches only at $x = 0$.*

Proof. Since $u(0,t) \to 1$ as $t \uparrow T$, u quenches at $x = 0$. Suppose that u quenches at some point $x_0 \neq 0$. Let $\xi_0 = |x_0|$. Then v blows up in $|x| \leq \xi_0$, since v is radially decreasing. Let $\delta = \min(\xi_0/2, T/2)$. By Lemma 4.1, there exists a constant $\eta = \eta(\delta, R)$ such that

$$\frac{v_\xi(\xi,t)}{v^2(\xi,t)} \leq -\eta(\xi-\delta)^2 \tag{4.3}$$

for $\delta \leq \xi \leq R$ and $t \geq \delta$. Integrating (4.3) from δ to $\xi \in (3\delta/2, R)$, we obtain

$$\frac{1}{v(\xi,t)} \geq \frac{1}{3}\eta\,(\xi-\delta)^3 \geq \frac{1}{24}\eta\delta^3$$

for any $t \geq \delta$. This shows that v cannot blow up at ξ_0, a contradiction. □

4.2. Quenching rate. We shall follow the method used in [10] to obtain the quenching rate of u at $x = 0$. The proof of the following estimate is the same as the proof of Lemma 3.1 in [10] and we omit it.

LEMMA 4.3. *We have the estimate*

$$v(x,t) \leq B\,(T-t)^{-\gamma} \tag{4.4}$$

in Q_T for some positive constant B.

Notice that the partial differential equation in (4.1) is invariant under the transformation

$$V(x,t) = \lambda^{2\gamma} v(\bar{x} + \lambda\, x, T - \lambda^2\,(T-t)),$$

where $\lambda = \sqrt{2\,(T-\bar{t})/T}$ and $(\bar{x}, \bar{t})$ is given. Applying an argument as Proposition 1 in [8] (see also [10, Lemma 3.2]), we can obtain the following estimates.

LEMMA 4.4. *There exists a positive constant M such that*

$$|D_x\, v(x,t)| \leq M\,(T-t)^{-(\gamma+1/2)}$$
$$|D_x^2\, v(x,t)| \leq M\,(T-t)^{-(\gamma+1)}$$
$$|v_t(x,t)| \leq M\,(T-t)^{-(\gamma+1)}$$

hold for $|x| \leq R$ and $0 \leq t < T$, where $D_x = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$ and $D_x^2 = (\partial^2/\partial x_i \partial x_j)$.

Set $z(y,s) = v(x,t)(T-t)^\gamma$. Note that $z(y,s) = 1/w(y,s)$. Then z satisfies the equation

$$z_s - (\Delta z - \frac{1}{2}\,y \cdot \nabla z - 2\frac{|\nabla z|^2}{z} - \gamma\, z + z^{2+\beta}) = 0 \quad \text{in} \quad W. \tag{4.5}$$

Recall that z is radially symmetric and strictly decreasing in r. Hence z satisfies the equation

$$z_s - (z_{rr} + \frac{n-1}{r}\, z_r - \frac{1}{2}\, r\, z_r - 2\,\frac{{z_r}^2}{z} - \gamma\, z + z^{2+\beta}) = 0 \tag{4.6}$$

in the set $\{(r,s); r < R\, e^{s/2}, s > s_0\} \equiv W$.

As a result of Lemmas 4.3 and 4.4, we have the following estimates

$$e^{-\gamma s} \leq z(y,s) \leq B \quad \text{in} \quad W, \tag{4.7}$$

$$|D_y\, z| \leq M, |D_y^2\, z| \leq M, |z_s| \leq M(1 + \frac{1}{2}\, |y|) + \gamma\, B \quad \text{in} \quad W, \tag{4.8}$$

where B is the constant in Lemma 4.3 and M is the constant in Lemma 4.4.

Since $v(0,t)$ is a global maximum for each t, we have

$$v_t(0,t) \leq v^{2+\beta}(0,t) \quad \text{or} \quad v^{-(2+\beta)}(0,t) v_t(0,t) \leq 1.$$

Integrating the last inequality from t to T, we obtain

$$v(0,t) \geq k^{-1}\, (T-t)^{-\gamma},$$

and hence

$$z(0,s) \geq 1/k \quad \text{or} \quad w(0,s) \leq k. \tag{4.9}$$

The following lemma can be proved in the same way as Lemma 3.4 in [10].

LEMMA 4.5. *There exists a constant* $t_0 \in (0,T)$ *such that for any* $t_* \in (t_0, T)$, *we have*

$$\frac{1}{2}\, |\nabla\, u|^2 \leq H(u) \quad \textit{in} \quad Q_{t_*} \equiv \{(x,t); |x| < R, t < t_*\},$$

where

$$H(u) = \frac{1}{\beta - 1}\, [\frac{1}{(1-u(0,t_*))^{\beta-1}} - \frac{1}{(1-u)^{\beta-1}}].$$

COROLLARY 4.6. *There is a constant* C *depending only on the constants* B *and* β *such that*

$$|\nabla\, w| \leq C \quad \textit{in} \quad W. \tag{4.10}$$

By (4.9), (4.10) and the mean value theorem, we obtain

$$w(r,s) \leq C\, r + k \quad \text{in} \quad W, \tag{4.11}$$

or equivalently,

$$z(r,s) \geq (C\, r + k)^{-1} \quad \text{in} \quad W. \tag{4.12}$$

We rewrite equation (4.6) as

$$\sigma\frac{z_s}{z^2}-[(\sigma\frac{z_r}{z^2})_r-\gamma\,\sigma\,z^{-1}+\sigma\,z^{\beta}]=0,$$

and define the energy of z over the interval $(0,L)$, where $L\le R\,e^{s/2}$, at time s by

$$E_L[z](s)=\int_0^L\sigma\frac{|z_r|^2}{z^4}\,dr-\gamma\int_0^L\sigma\frac{1}{z^2}\,dr-\frac{2}{\beta-1}\int_0^L\sigma\,z^{\beta-1}\,dr. \tag{4.13}$$

4.2.1. The case when $n=1$. The proof of the following theorem is similar to the proof of Theorem 3.10 in [10] and will not be repeated here; it uses the estimates (4.7), (4.8), (4.12), and Theorem 3.1(note that the energy function is given by (4.13)).

THEOREM 4.7. *If $\beta\ge 3$, then*

$$\lim_{t\uparrow T}[1-u(x,t)]\,(T-t)^{-\gamma}=k$$

uniformly for $|x|\le C\sqrt{T-t}$ for any positive constant C.

4.2.2. The case when $n\ge 2$. Recall that

$$\phi(r)=K\,r^{2\,\gamma},\quad K=[2\,\gamma\,(2\,\gamma+n-2)]^{-\gamma},$$

is a singular solution of (3.6).

THEOREM 4.8. *If $R^2>2\,\gamma\,(2\,\gamma+n-2)$, then*

$$\lim_{t\uparrow T}[1-u(x,t)]\,(T-t)^{-\gamma}=k$$

uniformly for $|x|\le C\sqrt{T-t}$ for any positive constant C.

In order to prove Theorem 4.8, we need the following lemma which is similar to a result in [2].

LEMMA 4.9. *Under the assumption of Theorem 4.8, $w(r,s)$ intersects $\phi(r)$ exactly once for each $s\ge s_0$.*

Proof. Set

$$D(r,s)=w(r,s)-\phi(r).$$

Since $w>0$, we have $D(0,s)>0$ for all $s\ge s_0$. By assumption, $D(R\,e^{s/2},s)<0$. Hence, for each $s\ge s_0$, there exists a first point $r_1(s)\in(0,R\,e^{s/2})$ such that $D(r,s)>0$, for all

$r < r_1(s)$ and $D(r_1(s), s) = 0$. Notice that $r_1(s)$ is a C^1 curve (cf. [2]). By a comparison principle, we obtain that

$$D(r,s) < 0 \quad \text{in} \quad Q \equiv \{(r,s); r_1(s) < r < R e^{s/2}, s \geq s_0\}.$$

Hence the lemma follows. □

Theorem 4.8 can now be proved by using (4.7), (4.8), (4.12), Lemma 4.9, and Theorem 3.4, where the energy function is given by (4.13). The details can be found in [2,10] and we omit it here.

Acknowledgment. I wish to thank Professor Avner Friedman for his advice and encouragement.

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