ONE-DIMENSIONAL THERMOMECHANICAL PHASE TRANSITIONS WITH NON-CONVEX POTENTIALS OF GINZBURG-LANDAU TYPE

By

Jürgen Sprekels

IMA Preprint Series # 505
April 1989
ONE-DIMENSIONAL THERMOMECHANICAL PHASE TRANSITIONS WITH NON-CONVEX POTENTIALS OF GINZBURG-LANDAU TYPE

Jürgen Sprekels*

Abstract. In this paper we study the system of partial differential equations governing the nonlinear thermomechanical processes in non-viscous, heat-conducting, one-dimensional solids. To allow for both stress- and temperature-induced solid-solid phase transitions in the material, possibly accompanied by hysteresis effects, a non-convex free energy of Ginzburg-Landau form is assumed. Results concerning the well-posedness of the problem, as well as the numerical approximation and the optimal control of the solutions, are presented in the paper, in particular in connection with the austenitic-martensitic phase transitions in the so-called “shape memory alloys”.

Key words. phase transitions, non-convex potentials, Ginzburg-Landau theory, shape memory alloys, hysteresis, conservation laws.

AMS(MOS) subject classifications. 35L65, 35K60, 73U05, 73B30

1. Introduction. In this paper we consider thermomechanical processes in non-viscous, one-dimensional heat-conducting solids of constant density $\rho$ (assumed normalized to unity) that are subjected to heating and loading. We think of metallic solids that do not only respond to a change of the strain $\varepsilon = u_x$ ($u$ stands for the displacement) by an elastic stress $\sigma = \sigma(\varepsilon)$, but also react to changes of the curvature of their metallic lattices by a couple stress $\mu = \mu(\varepsilon_x)$. Thus, the corresponding free energy $F$ is assumed in Ginzburg-Landau form, i.e.,

\begin{equation}
F = F(\varepsilon, \varepsilon_x, \theta),
\end{equation}

where $\theta$ is the absolute temperature. In the framework of the Landau theory of phase transitions, the strain $\varepsilon$ plays the role of an “order parameter”, whose actual value determines what phase is prevailing in the material (see [3]).

Since we are interested in solid-solid phase transitions, driven by loading and/or heating, which are accompanied by hysteresis effects, we do not assume that $F(\varepsilon, \varepsilon_x, \theta)$ is a convex function of the order parameter $\varepsilon$ for all values of $(\varepsilon_x, \theta)$. A particularly interesting

*Fachbereich Bauwesen, Universität-GHS-Essen, D-4300 Essen 1, West Germany (visiting at IMA). This work was supported by Deutsche Forschungsgemeinschaft (DFG), SPP “Anwendungsbezogene Optimierung und Steuerung”.
class of materials are the metallic alloys exhibiting the so-called "shape memory effect". Among those there are alloys like CuZn, CuSn, AuCuZn₂, AgCd and, most important, TiNi (so-called Nitinol). In these materials, the metallic lattice is deformed by shear, and the assumption of a constant density is justified. The relation between shear stress and shear strain ($\sigma - \varepsilon$-curves) of shape memory alloys exhibit a rather spectacular temperature-dependent hysteretic behavior (see [2] for an account of the properties of shape memory alloys):

![Diagram of $\sigma - \varepsilon$-curves in shape memory alloys](image)

Fig 1. Typical $\sigma - \varepsilon$-curves in shape memory alloys, with temperature $\theta$ increasing from a) to c).

In addition, for sufficiently small shear stresses $\sigma$ another hysteresis occurs in the $\varepsilon - \theta$-diagrams:

![Diagram of $\varepsilon - \theta$-curves for different values of $\sigma$.](image)

Fig 2. $\varepsilon - \theta$ curves in shape memory alloys for different values of $\sigma$. 

2
On the microscopic scale, this hysteric behavior is ascribed to first-order stress-induced (fig. 1a,b) or temperature-induced (fig. 2) phase transitions between different configurations of the metallic lattice, namely the symmetric high-temperature phase "austenite" (taken as reference configuration) and its two oppositely oriented sheared versions termed "martensitic twins", which prevail at low temperatures (cf., [6], [7]).

The simplest form for the free energy $F$ which matches the experimental evidence given by figs. 1,2 quite well and takes interfacial energies into account is given by (cf., [4], [5])

$$ F(\epsilon, \epsilon_x, \theta) = -C_V \theta \log(\theta / \theta_2) + C_V \theta + \bar{C} + \kappa_1(\theta - \theta_1)\epsilon^2 - \kappa_2 \epsilon^4 + \kappa_3 \epsilon^6 + \frac{\gamma}{2} \epsilon_x^2, $$

where $C_V$ denotes the specific heat, $\bar{C}$ is some constant, $\theta_1$ and $\theta_2$ are (positive) temperatures and $\kappa_1, \kappa_2, \kappa_3, \gamma$ are positive constants. A complete set of data for the alloy AuCuZn$_2$ is given in [5]. Note that within the range of interesting temperatures, for $\theta \rightarrow \theta_1$, $F$ is not convex as function of $\epsilon$.

In the sequel, we assume $F$ in the somewhat more general form (with positive $\kappa_1, \kappa_2, \gamma$)

$$ F(\epsilon, \epsilon_x, \theta) = -C_V \theta \log(\theta / \theta_2) + C_V \theta + \bar{C} + \kappa_1 \theta F_1(\epsilon) + \kappa_2 F_2(\epsilon) + \frac{\gamma}{2} \epsilon_x^2, $$

where $F_1$ and $F_2$ satisfy the hypothesis:

(H1) $F_1, F_2 \in C^4(\mathbb{R})$; $F_2(\epsilon) \geq \tilde{c}_1 |\epsilon| - \tilde{c}_2$, $\forall \epsilon \in \mathbb{R}$, with positive constants $\tilde{c}_1, \tilde{c}_2$.

The dynamics of thermomechanical processes in a solid are governed by the conservation laws of linear momentum, energy and mass. The latter may be ignored since $\rho$ is constant for the materials under consideration (we assume $\rho \equiv 1$). The two others read

$$
\begin{align}
(1.4a) \quad u_{tt} - \sigma_x + \mu_{xx} &= f, \\
(1.4b) \quad \epsilon_t + q_x - \sigma \epsilon_x - \mu \epsilon_{xt} &= g.
\end{align}
$$

Here the involved quantities have their usual meanings, namely: $\sigma$-elastic stress, $\mu$-couple stress, $u$-displacement, $f$ - density of loads, $\epsilon$ - density of internal energy, $q$ - heat flux, $g$ -density of heat sources or sinks.

We have the constitutive relations

$$
(1.5) \quad \sigma = \frac{\partial F}{\partial \epsilon}, \quad \mu = \frac{\partial F}{\partial \epsilon_x}, \quad \epsilon = F - \theta \frac{\partial F}{\partial \theta},
$$

and we assume the heat flux in the Fourier-form

$$
(1.6) \quad q = -\kappa \theta_x, \quad \text{where} \ \kappa > 0 \ \text{is the heat conductivity}.
$$
Notice that (1.6) implies that the second principle of thermodynamics in form of the Clausius-Duhem inequality is automatically satisfied.

Inserting (1.3), (1.5), (1.6) in the balance laws and assuming a one-dimensional sample of unit length, we obtain the system

\begin{align}
(1.7a) & \quad u_{tt} - (\kappa_1 \theta F'_1(\varepsilon) + \kappa_2 F'_2(\varepsilon))_x + \gamma u_{xxxx} = f, \\
(1.7b) & \quad C_V \theta_t - \kappa_1 \theta F'_1(\varepsilon) \varepsilon_t - \kappa \theta_{xx} = g, \\
(1.7c) & \quad \varepsilon = u_x,
\end{align}

to be satisfied in the space-time cylinder $\Omega_T$, where $T > 0$, $\Omega = (0,1)$, and, for $t > 0$, $\Omega_t := \Omega \times (0,t)$.

In addition, we prescribe the initial and boundary conditions

\begin{align}
(1.7d) & \quad u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), x \in \overline{\Omega}, \\
(1.7e) & \quad u(0,t) = u_{xx}(0,t) = u(1,t) = u_{xx}(1,t) = 0, \ t \in [0,T], \\
(1.7f) & \quad \theta_x(0,t) = 0, \quad -\kappa \theta_x(1,t) = \beta(\theta(1,t) - \theta_T(t)), \ t \in [0,T],
\end{align}

where $\beta > 0$ is a heat exchange coefficient, and $\theta_T$ stands for the outside temperature at $x = 1$.

In the following sections we state some results concerning the well-posedness of the system (1.7a–f), including a convergent numerical algorithm for its approximate solution. To abbreviate the exposition, all constants in (1.7a–f) are assumed to equal unity; this will have no bearing on the mathematical analysis.

2. Well-posedness. We consider (1.7a–f). In addition to (H1), we generally assume:

(H2) $u_0 \in \tilde{H}^4(\Omega) = \{ u \in H^4(\Omega) | u(0) = u(1) = 0 = u''(0) = u''(1) \}$;

$u_1 \in \tilde{H}^1(\Omega) \cap H^2(\Omega)$; $\theta_0 \in H^2(\Omega)$, $\theta_0(x) > 0$, $\forall x \in \overline{\Omega}$.

(H3) $\theta'_0(0) = 0$, $\theta_T(0) = \theta_0(1) + \frac{\kappa}{\beta} \theta'_0(1) > 0$ (compatibility).

(H4) $f, g \in H^1(0,T; H^1(\Omega))$, $\theta_T \in H^1(0,T)$, where $g(x,t) \geq 0$ on $\Omega_T$ and $\theta_T(t) > 0$ on $[0,T]$.

We have the result:

**Theorem 2.1.** Suppose (H1)–(H4) hold. Then (1.7a–f) has a unique solution $(u, \theta)$ which satisfies

\begin{align}
(2.1a) & \quad u \in W^{2,\infty}(0,T; L^2(\Omega)) \cap W^{1,\infty}(0,T; H^1(\Omega) \cap H^2(\Omega)) \cap L^\infty(0,T; \tilde{H}^4(\Omega)), \\
(2.1b) & \quad \theta \in H^1(0,T; H^1(\Omega)) \cap L^2(0,T; H^3(\Omega)), \\
(2.1c) & \quad \theta(x,t) > 0, \text{ on } \overline{\Omega}_T.
\end{align}
Moreover, the operator \((f, g, \theta_\Gamma) \mapsto (u, \theta)\) maps bounded subsets of \(H^1(0, T; H^1(\Omega)) \times H^1(0, T; H^1(\Omega)) \times \mathcal{M}\) into bounded subsets of \(X \times Y\), where \(\mathcal{M} := \{ z \in [H^1(0, T)] | z(t) > 0 \text{ on } [0, T], z(0) = \theta_0(1) + \theta'_0(1) \} \), \(X := W^{2, \infty}(0, T; L^2(\Omega)) \cap W^{1, \infty}(0, T; \bar{H}^1(\Omega) \cap H^2(\Omega)) \cap L^\infty(0, T; \bar{H}^4(\Omega)) \) and \(Y := H^1(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega))\), respectively.

**Proof.** The existence result is easily obtained by combining the Galerkin approximation employed in the proof of Theorem 2.1 in [9] with the a priori estimates derived in the proof of Theorem 2.1 in [12]; the uniqueness is a direct consequence of the subsequent Theorem 2.3. Finally, the boundedness of the mapping \((f, g, \theta_\Gamma) \mapsto (u, \theta)\) follows from the above-mentioned a priori estimates. 

A sharper existence result, with regards to the smoothness properties of the solution \((u, \theta)\), has been established in [12]:

**Theorem 2.2.** Suppose that, in addition to (H1)-(H4), the following assumptions on the data of (1.7a-f) are satisfied:

\[
\begin{align*}
\text{(2.2)} \\
u_0 \in H^5(\Omega), \quad u_1 \in H^3(\Omega), \quad \theta_0 \in H^4(\Omega), \\
f_{1t} \in L^2(\Omega_T), \quad g \in L^2(0, T; H^2(\Omega)), \quad \theta_\Gamma \in H^2(0, T).
\end{align*}
\]

Furthermore, suppose that \(\theta_0\) satisfies compatibility conditions of sufficiently high order. Then (1.7a-f) has a unique classical solution \((u, \theta)\), and all the partial derivatives appearing in (1.7a-c) belong to the Hölder class \(C^{\alpha, \alpha/2}(\overline{\Omega_T})\), for some \(\alpha \in (0, 1)\).

**Proof.** See Theorem 2.1 in [12]. 

We now derive a stability result with respect to the data \((f, g, \theta_\Gamma)\) which guarantees the uniqueness of the solution \((u, \theta)\).

**Theorem 2.3.** Suppose the general hypotheses (H1), (H2) and \(\theta_0(0) = 0\) are satisfied. We consider the variational problem
\[
(2.3a) \quad \int_0^t \int_{\Omega} u_t(x,t) \varphi(x,t) \, dx - \int_0^t \int_{\Omega} u_1(x) \varphi(x,0) \, dx - \int_0^t \int_{\Omega} u_t \varphi \, dx \, d\tau \\
+ \int_0^t \int_{\Omega} \left[ \frac{\partial F}{\partial \varepsilon}(u_x, \theta) \varphi_x + u_{xx} \varphi_{xx} - f \varphi \right] \, dx \, d\tau = 0, \\
\forall \varphi \in H^1(0,T; \hat{H}^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), \quad 0 \leq t \leq T,
\]

\[
(2.3b) \quad \int_0^t \int_{\Omega} \left[ \theta_t \eta - g \eta - \theta u_x u_x \eta + \theta_x \eta_x \right] \, dx \, d\tau \\
+ \int_0^t \left( \theta(1, \tau) - \theta_\Gamma(\tau) \right) \eta(1, \tau) \, d\tau = 0, \quad \forall \eta \in L^2(0,T; H^1(\Omega)), \quad 0 \leq t \leq T,
\]

\[
(2.3c) \quad \theta(x,0) = \theta_0(x), \quad u(x,0) = u_0(x), \quad x \in \overline{\Omega}.
\]

Suppose the data \((f^{(i)}, g^{(i)}, \theta^{(i)}_\Gamma), i = 1, 2,\) satisfy (H3) and (H4), and suppose that \((u^{(i)}, \theta^{(i)})\) are solutions to \((2.3a-c)\) corresponding to the data \((f^{(i)}, g^{(i)}, \theta^{(i)}_\Gamma), i = 1, 2,\) such that \(u^{(i)} \in W^{1,\infty}(0,T; \hat{H}^1(\Omega)) \cap L^\infty(0,T; H^3(\Omega)), \theta^{(i)} \in H^1(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)), i = 1, 2.\) Then there is some \(C > 0\) such that

\[
(2.4) \quad \sup_{t \in (0,T)} \left( \|u_t(t)\|^2 + \|u_{xx}(t)\|^2 + \|\theta(t)\|^2 \right) + \int_0^T \int_{\Omega} \theta^2 \, dx \, dt \\
+ \int_0^T \theta^2(1, t) \, dt \leq C \left( \|\theta_\Gamma\|^2_{L^2(0,T)} + \|g\|^2_{L^2(\Omega_T)} + \|f\|^2_{L^2(\Omega_T)} \right),
\]

where \(\theta = \theta^{(1)} - \theta^{(2)}, u = u^{(1)} - u^{(2)}, \theta_\Gamma = \theta^{(1)}_\Gamma - \theta^{(2)}_\Gamma, f = f^{(1)} - f^{(2)}, g = g^{(1)} - g^{(2)}.\)

**Remarks.**

1. Here (and throughout) we have omitted the arguments of the involved functions if no confusion may arise.
2. \(\| . \|\) denotes always the \(L^2(\Omega)\)-norm.
3. Obviously, any solution \((u, \theta)\) of (1.7a–f) with (2.1a,b) solves (2.3a–c); consequently, the solution of (1.7a–f) is unique.
4. From the upcoming proof it will become evident that a corresponding stability result holds with respect to the initial data \(u_0, u_1, \theta_0;\) we restrict ourselves to the
data \((f, g, \theta_T)\) as they are the natural candidates to serve as control variables if the system is to be controlled from the outside.

Proof. Let \(\epsilon^{(i)} = u_x^{(i)}\), \(i = 1, 2\), and \(\epsilon = \epsilon^{(1)} - \epsilon^{(2)}\). In terms of the variables \((u, \theta)\) introduced in the assertion, (2.3–c) can be rewritten as

\[
\begin{align*}
\int \lim_{t \to \infty} u_t(x,t) \varphi(x,t) dx &= 
\int_0^t \int \lim_{t \to \infty} u_t \varphi_t dx dt + \int_0^t \int \lim_{t \to \infty} (u_{xx} \varphi_{xx} - f \varphi) dx dt \\
&+ \int_0^t \int \left( \frac{\partial F}{\partial \epsilon} (\epsilon^{(1)}, \theta^{(1)}) - \frac{\partial F}{\partial \epsilon} (\epsilon^{(2)}, \theta^{(2)}) \right) \varphi_x dx dt = 0, \\
\forall \varphi \in H^1(0,T; \dot{H}^1(\Omega)) \cap L^2(0,T; H^2(\Omega)), \quad 0 \leq t \leq T,
\end{align*}
\]

(2.5b)

\[
\begin{align*}
\int \lim_{t \to \infty} [\theta_x \eta - g \eta + \theta_{xx} \eta_x] dx dt &= 
\int_0^t \int \lim_{t \to \infty} (\theta(1, \tau) - \theta_T(\tau)) \eta(1, \tau) d\tau \\
&+ \int_0^t \int \lim_{t \to \infty} [\theta^{(2)} \epsilon^{(2)} \epsilon^{(2)}_{\tau} - \theta^{(1)} \epsilon^{(1)} \epsilon^{(1)}_{\tau}] \eta dx d\tau = 0, \quad \forall \eta \in L^2(0,T; H^1(\Omega)), \quad 0 \leq t \leq T,
\end{align*}
\]

(2.5c)

\[
\theta(x,0) = u(x,0) = 0, \quad x \in \Omega.
\]

Next observe that, owing to our assumptions, \(u_x^{(i)}, u_{xx}^{(i)}\) and \(\theta^{(i)}\) belong to \(C(\Omega_T), i = 1, 2\). Thus, due to (H1), expressions of the form \(\frac{\partial^k}{\partial \epsilon^k} F_j(\epsilon^{(i)})\), \(1 \leq k \leq 4, j = 1, 2, i = 1, 2\), are bounded. Moreover \(u_t^{(i)} \in L^\infty(\Omega_T), i = 1, 2\). In the sequel, \(C_i, i \in \mathbb{N}\), always denote positive generic constants. We proceed in two steps:

STEP 1: Let \(\delta > 0\) be given (to be specified later). We insert \(\varphi = u_t\) in (2.5a), integrate by parts and use Young's inequality to arrive at the estimate

\[
\begin{align*}
\frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|u_{xx}(t)\|^2 &\leq \frac{1}{2} \|f\|_{L^2(\Omega_T)}^2 + C_1 \int_0^t \|u_t(\tau)\|^2 d\tau \\
&+ \delta \int_0^t \int \left( \frac{\partial F}{\partial \epsilon} (\epsilon^{(1)}, \theta^{(1)}) - \frac{\partial F}{\partial \epsilon} (\epsilon^{(2)}, \theta^{(2)}) \right) \varphi_x dx d\tau.
\end{align*}
\]

Now

\[
\begin{align*}
\frac{\partial}{\partial x} \left( \frac{\partial F}{\partial \epsilon} (\epsilon^{(1)}, \theta^{(1)}) - \frac{\partial F}{\partial \epsilon} (\epsilon^{(2)}, \theta^{(2)}) \right) &= \theta_{xx} F_1'(\epsilon^{(1)}) \\
&\quad + \theta_{xx} (F_1''(\epsilon^{(1)}) - F_1''(\epsilon^{(2)})) + F_2''(\epsilon^{(1)}) u_{xx} \\
&\quad + u_{xx} \theta_{xx} (F_2''(\epsilon^{(1)}) - F_2''(\epsilon^{(2)})) + \theta F_1''(\epsilon^{(1)}) u_{xx} \\
&\quad + u_{xx} \theta_2''(F_1''(\epsilon^{(1)}) - F_1''(\epsilon^{(2)})) + \theta_2'' F_1''(\epsilon^{(2)}) u_{xx}.
\end{align*}
\]
Consequently, invoking the mean value theorem,

\[ \int_0^t \int_{\Omega} \left( \frac{\partial F}{\partial \epsilon} (\epsilon^{(1)}, \theta^{(1)}) - \frac{\partial F}{\partial \epsilon} (\epsilon^{(2)}, \theta^{(2)}) \right)_z^2 \, dx \, d\tau \]

\[ \leq C_2 \int_0^t \int_{\Omega} (\theta^{(2)}_z + |\theta^{(2)}_z|^2 \epsilon^2 + u_{xx}^2 + \theta^2 + \epsilon^2) \, dx \, d\tau. \]

But

\[ \int_0^t \int_{\Omega} |\theta^{(2)}_z|^2 \epsilon^2 \, dx \, dt \leq \int_0^t \|e(\tau)\|^2_{L^\infty(\Omega)} \|\theta^{(2)}_z(\tau)\|^2 \, d\tau \]

\[ \leq C_3 \int_0^t \|e(\tau)\|^2_{L^\infty(\Omega)} \, d\tau, \]

since \( \theta^{(2)} \in L^\infty(0, T; H^1(\Omega)) \). Now observe that \( u(0, t) = 0 = u(1, t) \). Hence, to any \( t \in [0, T] \) there is some \( x(t) \in (0, 1) \) such that \( u_x(x(t), t) = 0 \). Thus,

\[ |e(x, \tau)| \leq \|u_{xx}(\tau)\|, \quad 0 \leq \tau \leq t, \quad x \in \overline{\Omega}. \]

Summarizing, we have shown the estimate

\[ \|u_t(t)\|^2 + \|u_{xx}(t)\|^2 \leq C_4 \|f\|^2_{L^2(\Omega T)} + C_5 \int_0^t \|u_t(\tau)\|^2 \, d\tau \]

\[ + C_6 \cdot \delta \cdot \int_0^t (\|\theta_x(\tau)\|^2 + \|\theta(\tau)\|^2 + \|u_{xx}(\tau)\|^2) \, d\tau. \]

**STEP 2:** Next we substitute \( \eta = \theta \) into (2.5b) to obtain via Young’s inequality:

\[ \frac{1}{2} \|\theta(t)\|^2 + \int_0^t \|\theta_x(\tau)\|^2 \, d\tau + \frac{1}{2} \int_0^t \theta^2(1, \tau) \, d\tau \]

\[ \leq \frac{1}{2} \int_0^t \|\theta(\tau)\|^2 \, d\tau + \frac{1}{2} \|g\|^2_{L^2(\Omega T)} + \frac{1}{2} \|\theta_T\|^2_{L^2(0, T)} + A, \]
\begin{align}
\text{(2.13)} \quad A &= \int_{\Omega} \int_{0}^{t} \theta(\theta^{(1)} \varepsilon^{(1)}_{t} - \theta^{(2)} \varepsilon^{(2)}_{t}) \, dx \, d\tau \\
&= \int_{\Omega} \int_{0}^{t} \frac{d}{dx} [\theta(\theta^{(1)} \varepsilon^{(1)} - \theta^{(2)} \varepsilon^{(2)}) u^{(1)}_{t} - \theta^{(2)} \varepsilon^{(2)}_{t} u^{(2)}_{t}] \, dx \, d\tau \\
&- \int_{\Omega} \int_{0}^{t} \theta x(\theta^{(1)} \varepsilon^{(1)}_{t} - \theta^{(2)} \varepsilon^{(2)}_{t} u^{(2)}_{t}) \, dx \, d\tau \\
&- \int_{\Omega} \int_{0}^{t} \theta(\theta^{(1)} \varepsilon^{(1)}_{t} + \theta^{(1)}_{xx} u^{(1)}_{t} - \theta^{(2)} \varepsilon^{(2)}_{t} u^{(2)}_{t}) \, dx \, d\tau.
\end{align}

Since \( u_{t}|_{x=0,1} = 0 \), the first integral vanishes; the other terms have to be treated individually.

\begin{enumerate}
\item a) We have, since \( \theta^{(i)}, \varepsilon^{(i)}, u^{(i)}_{t} \in L^{\infty}(\Omega_{T}), \ i = 1, 2, \)
\begin{align}
\text{(2.14)} \quad I_{1} &= \left| \int_{\Omega} \int_{0}^{t} \theta x(\theta^{(1)} \varepsilon^{(1)}_{t} - \theta^{(2)} \varepsilon^{(2)}_{t} u^{(2)}_{t}) \, dx \, d\tau \right| \\
&= \left| \int_{\Omega} \int_{0}^{t} \theta x[\theta e^{(1)} u^{(1)}_{t} + \theta^{(2)} e^{(2)} u^{(2)}_{t}] \, dx \, d\tau \right| \\
&\leq \delta \int_{\Omega} \int_{0}^{t} \theta^{2} \, dx \, d\tau + C_{7} \int_{\Omega} \int_{0}^{t} (\theta^{2} + e^{2} + u^{2}_{t}) \, dx \, d\tau.
\end{align}

\item b) Next we estimate
\begin{align}
\text{(2.15)} \quad |I_{2}| &= \left| \int_{\Omega} \int_{0}^{t} \theta x(\theta^{(1)} \varepsilon^{(1)}_{t} - \theta^{(2)} \varepsilon^{(2)}_{t} u^{(2)}_{t}) \, dx \, d\tau \right| \\
&\quad \left| \int_{\Omega} \int_{0}^{t} \theta x[\theta e^{(1)} u^{(1)}_{t} + \theta^{(2)} e^{(2)} u^{(2)}_{t}] \, dx \, d\tau \right| \\
&\leq \delta \int_{\Omega} \int_{0}^{t} \theta^{2} \, dx \, d\tau + C_{8} \int_{\Omega} \int_{0}^{t} (\theta^{2} + e^{2} + u^{2}_{t}) \, dx \, d\tau \\
&\quad + C_{9} \int_{\Omega} \int_{0}^{t} |\theta^{(2)}|^{2} \, dx \, d\tau.
\end{align}
\end{enumerate}
Recalling Nirenberg’s inequality in one space dimension (cf., [1]), we have with suitable \(\alpha_1 > 0, \alpha_2 > 0\):

\[
(2.16) \quad \|\theta(\tau)\|_{L^\infty(\Omega)}^2 \leq \alpha_1 \|\theta(\tau)\| \|\theta(\tau)\| + \alpha_2 \|\theta(\tau)\|^2 \leq \delta \|\theta_x(\tau)\|^2 + C_{10} \|\theta(\tau)\|^2.
\]

Since \(\theta^{(2)}_x \in L^\infty(0, T; L^2(\Omega))\), this implies that

\[
(2.17) \quad \int_0^t \int_\Omega |\theta^{(2)}_x|^2 \theta^2 \, dx \, d\tau \leq \int_0^t \|\theta(\tau)\|_{L^\infty(\Omega)}^2 \|\theta^{(2)}_x(\tau)\|^2 \, d\tau
\]

\[
\leq C_{10} \delta \int_0^t \int_\Omega \theta^2 \, dx \, d\tau + C_{11} \int_0^t \int_\Omega \theta^2 \, dx \, dt,
\]

whence

\[
(2.18) \quad |I_2| \leq C_{11} \delta \int_0^t \|\theta_x(\tau)\|^2 \, d\tau
\]

\[
+C_{12} \int_0^t (\|\theta(\tau)\|^2 + \|u_{xx}(\tau)\|^2 + \|u_t(\tau)\|^2) \, d\tau.
\]

c) Finally we have

\[
(2.19) \quad |I_3| = \left| \int_0^t \int_\Omega \theta^{(1)} u^{(1)}_x u^{(1)}_t - \theta^{(2)} u^{(2)}_{xx} u^{(2)}_t \, dx \, d\tau \right|
\]

\[
= \left| \int_0^t \int_\Omega \theta u^{(1)}_{xx} u^{(1)}_t + \theta^{(2)} u^{(2)}_{xx} u^{(1)}_t + \theta^{(2)} u^{(2)}_x u^{(2)}_t \, dx \, d\tau \right|
\]

\[
\leq C_{13} \int_0^t (\|\theta(\tau)\|^2 + \|u_{xx}(\tau)\|^2 + \|u_t(\tau)\|^2) \, dx \, d\tau.
\]

Summarizing the inequalities (2.12), (2.14), (2.18) and (2.19), we have shown that

\[
(2.20) \quad \frac{1}{2} \|\theta(t)\|^2 + \int_0^t \|\theta_x(\tau)\|^2 \, d\tau + \frac{1}{2} \int_0^t \theta^2(1, \tau) \, d\tau
\]

\[
\leq C_{14} \cdot \delta \int_0^t \|\theta_x(\tau)\|^2 \, d\tau + \frac{1}{2} \|\theta\|^2 \|L^2(\Omega, T)\| + \frac{1}{2} \|\theta_T\|^2 \|L^2(0, T)\|
\]

\[
+ C_{15} \int_0^t (\|\theta(\tau)\|^2 + \|u_{xx}(\tau)\|^2 + \|u_t(\tau)\|^2) \, d\tau.
\]
Adding (2.11) and (2.20), adjusting $\delta > 0$ sufficiently small, and invoking Gronwall's lemma, we have finally proved the assertion. \(\square\)

3. **Optimal Control.** We now turn our interest to optimal control problems associated with the system (1.7a–f). It is of considerable interest in the technological application of shape memory alloys to control the evolution of the austenitic-martensitic phase transitions in the material; in this connection, a typical object is to influence the system via the natural control variables $f, g, \theta_T$ in such a way, that a desired distribution of the phases in the material is produced. Since the phase transitions are characterized by the order parameter $\epsilon$, it is natural to use $\epsilon$ as the main variable in the cost functional. We consider the following control problem:

(CP)

Minimize $J(u, \theta; f, g, \theta_T) = \int_0^T \int_\Omega L_1(x, t, u_x(x, t), \theta(x, t), f(x, t), g(x, t)) \, dx \, dt$

$+ \int_0^T L_2(t, \theta_T(t)) \, dt + \int_\Omega L_3(x, u_x(x, T), \theta(x, T)) \, dx$

subject to (1.7a–f) and the side condition $(f, g, \theta_T) \in \mathcal{K}$, where $\mathcal{K}$ denotes some nonempty, bounded, closed and convex subset of $\mathcal{H}^1(0, T; H^1(\Omega)) \times \{ g \in H^1(0, T; H^1(\Omega)) | g(x, t) \geq 0 \}$ on $\Omega_T \times M$.

For $L_1 : \mathbb{R}^6 \to \mathbb{R}, L_2 : \mathbb{R}^2 \to \mathbb{R}, L_3 : \mathbb{R}^3 \to \mathbb{R}$, we assume:

(H5) (i) $L_1, L_2, L_3$ are measurable with respect to the variables $(x, t)$, resp. $t$, resp. $x$, and continuous with respect to the other variables.

(ii) $L_1$ is convex with respect to $f$ and $g$.

(iii) $L_2$ is convex with respect to $\theta_T$.

These assumptions are natural in the framework of optimal control; a typical form for $J$ would be

\[
J(u, \theta; f, g, \theta_T) = \beta_1 \|u_x - \overline{u}_x\|^2_{L^2(\Omega_T)} + \beta_2 \|\theta - \overline{\theta}\|^2_{L^2(\Omega_T)} \\
+ \beta_3 \|u_x(\cdot, T) - \overline{u}\|^2 + \beta_4 \|\theta(\cdot, T) - \overline{\theta}\|^2 \\
+ \beta_5 \|f\|^2_{L^2(\Omega_T)} + \beta_6 \|g\|^2_{L^2(\Omega_T)} + \beta_7 \|\theta_T\|^2_{L^2(0, T)}
\]

with $\beta_i \geq 0$, but not all zero, and functions $\overline{u}_x, \overline{\theta} \in L^2(\Omega_T), \overline{u}, \overline{\theta} \in L^2(\Omega)$, representing the desired strain and temperature distributions during the evolution and at $t = T$.

There holds
THEOREM 3.1. Suppose (H1)-(H5) are true, then (CP) has a solution \((\bar{u}, \bar{\vartheta}; \bar{f}, \bar{g}, \bar{\vartheta}_\Gamma)\).

Proof. Let \(\{(f_n, g_n, \theta_{\Gamma,n})\} \subseteq \mathcal{K}\) denote a minimizing sequence, and let \((u_n, \theta_n)\) denote the solution of (1.7a-f) associated with \((f_n, g_n, \theta_{\Gamma,n}), n \in \mathbb{N}\). Since \(\mathcal{K}\) is bounded, we may assume that

\[
\begin{align*}
    f_n &\rightarrow \bar{f}, \quad \text{weakly in } H^1(0,T; H^1(\Omega)), \\
g_n &\rightarrow \bar{g}, \quad \text{weakly in } H^1(0,T; H^1(\Omega)), \\
\theta_{\Gamma,n} &\rightarrow \bar{\vartheta}_\Gamma, \quad \text{weakly in } H^1(0,T).
\end{align*}
\]

Due to the weak closedness of the convex and closed set \(\mathcal{K}, (\bar{f}, \bar{g}, \bar{\vartheta}_\Gamma) \in \mathcal{K}\). Let \((\bar{u}, \bar{\vartheta})\) denote the associated solution of (1.7a-f). Now, owing to the boundedness of \(\mathcal{K}\) and Theorem 2.1, \(\{(u_n, \theta_n)\}_{n \in \mathbb{N}}\) is a bounded subset of \(X \times Y\). Therefore, we may assume that for some \((u, \vartheta) \in X \times Y\) there holds

\[
\begin{align*}
    u_{n,x} &\rightarrow u_x, \quad \text{uniformly on } \Omega_T, \\
\theta_n &\rightarrow \vartheta, \\
    u_{n,tt} &\rightarrow u_{tt}, \\
    u_{n,xx} &\rightarrow u_{xx}, \quad \text{weakly in } L^2(\Omega_T), \\
    u_{n,xt} &\rightarrow u_{xt}, \\
    u_{n,xxxx} &\rightarrow u_{xxxx},
\end{align*}
\]

as well as

\[
\begin{align*}
    \theta_{n,x} &\rightarrow \vartheta_x, \\
\theta_{n,t} &\rightarrow \vartheta_t, \quad \text{weakly in } L^2(\Omega_T), \\
\theta_{n,xx} &\rightarrow \vartheta_{xx},
\end{align*}
\]

Passing to the limit as \(n \rightarrow \infty\) in the equations (1.7a-f) shows that \((u, \vartheta)\) solves (1.7a-f) for the data \((\bar{f}, \bar{g}, \bar{\vartheta}_\Gamma)\), i.e., \(u = \bar{u}, \vartheta = \bar{\vartheta}\). Hence, \((\bar{u}, \bar{\vartheta}; \bar{f}, \bar{g}, \bar{\vartheta}_\Gamma)\) is admissible, and, in view of (H5),

\[
J(\bar{u}, \bar{\vartheta}; \bar{f}, \bar{g}, \bar{\vartheta}_\Gamma) \leq \liminf_{n \rightarrow \infty} J(u_n, \theta_n; f_n, g_n, \theta_{\Gamma,n}).
\]

Thus \((\bar{u}, \bar{\vartheta}; \bar{f}, \bar{g}, \bar{\vartheta}_\Gamma)\) is a solution of (CP). \(\square\)

REMARKS.

5. The above way of arguing follows the lines of [10] where a related result was derived for a much more restricted free energy \(F\).

6. It is natural to look for necessary conditions of optimality for the optimal controls of (CP). A corresponding result has not yet been derived.

7. The problem of the automatic self-regulation of the system via a fixed feedback control regulating the boundary temperature \(\vartheta_\Gamma\) has been considered in [11].
4. Numerical Approximation. In this section we follow the lines of [8]. We assume the free energy in the special form (see (1.2))

\begin{equation}
F(\epsilon, \epsilon_x, \theta) = -\theta \log \theta + \theta + \frac{1}{2} \theta \epsilon^2 - \frac{1}{4} \epsilon^4 + \frac{1}{6} \epsilon^6 + \frac{1}{2} \epsilon_x^2.
\end{equation}

Let

\begin{equation}
F_0(\epsilon, \theta) = \frac{1}{2} \theta \epsilon^2 - \frac{1}{4} \epsilon^4 + \frac{1}{6} \epsilon^6.
\end{equation}

Then, for \( \epsilon_1 \neq \epsilon_2 \),

\begin{equation}
\frac{F_0(\epsilon_1, \theta) - F_0(\epsilon_2, \theta)}{\epsilon_1 - \epsilon_2} = \frac{1}{2} \theta (\epsilon_1 + \epsilon_2) + \Psi(\epsilon_1, \epsilon_2),
\end{equation}

where \( \Psi(\epsilon_1, \epsilon_2) \) is a polynomial of degree 5 in \( \epsilon_1, \epsilon_2 \).

We are going to construct a numerical scheme for the approximate solution of (1.7a-f). To this end, we assume that (H1)-(H4) and (2.2) hold, so that Theorem 2.2 applies.

Now let \( K, N, M \in \mathbb{N} \) be chosen. We put \( h = \frac{T}{M}, t_m^{(M)} = mh, \) \( 0 \leq m \leq M \), and \( x_i^{(N)} = \frac{i}{N}, \) \( 0 \leq i \leq N \).

Define

\begin{equation}
Y_N = \{ \text{linear splines on } [0,1] \text{ corresponding to the partition } \{x_i^{(N)}\}_{i=0}^N \text{ of } [0,1]\},
\end{equation}

and let

\begin{equation}
Z_K = \text{span}\{z_1, \ldots, z_K\},
\end{equation}

where \( z_j \) denotes the \( j \)-th eigenfunction of the eigenvalue problem

\begin{equation}
z''' = \lambda z, \text{ in } (0,1), z(0) = z''(0) = 0 = z''(1) = z(1).
\end{equation}

We introduce the projection operators

\begin{equation}
P_K = H^4(0,1) - \text{orthogonal projection onto } Z_K,
\end{equation}

\begin{equation}Q_K = H^3(0,1) - \text{orthogonal projection onto } Z_K,
\end{equation}

\begin{equation}R_N = H^1(0,1) - \text{orthogonal projection onto } Y_N,
\end{equation}

and the averages

\begin{equation}
f_M^n(x) = \frac{1}{h} \int_{(m-1)h}^{mh} f(x, t) dt, \quad g_M^n(x) = \frac{1}{h} \int_{(m-1)h}^{mh} g(x, t) dt,
\end{equation}

\begin{equation}\Theta^n_{\Gamma, M} = \frac{1}{h} \int_{(m-1)h}^{mh} \theta_{\Gamma}(t) dt.
\end{equation}
We then consider the discrete problem

\[(D_{M,N,K})\text{ Find } u^m = \sum_{k=1}^{K} \alpha_k^m z_k, \theta^m = \sum_{k=0}^{N} \beta_k^m y_k^{(N)},\]

\[1 \leq m \leq M, \text{ such that}\]

\[(4.9a)\]

\[
\int_{\Omega} \left[ \frac{u^m - 2u^{m-1} + u^{m-2}}{h^2} \xi + \frac{1}{2} \theta^{m-1}(u_x^m + u_x^{m-1})\xi_x \right. \\
+ \Psi(u_x^m, u_x^{m-1})\xi_x + u_{xx}^m \xi_{xx} - f_M^m \xi] dx = 0, \quad \forall \xi \in Z_K, \\
\]

\[\int_{\Omega} \left[ \frac{\theta^m - \theta^{m-1}}{h} \eta - \frac{1}{2} \theta^{m-1} \frac{(u_x^m)^2 - (u_x^{m-1})^2}{h} \eta \right. \\
+ \theta_x^m \eta_x - g_M^m \eta \left. \right] dx + (\theta^m(1) - \theta_{1,M}^m)\eta(1) = 0, \quad \forall \eta \in Y_N, \\
\]

\[(4.9c)\]

\[u^0 = P_K(u^0), \frac{u^0 - u^{-1}}{h} = Q_K(u_1), \quad \theta^0 = R_N(\theta_0).\]

The following result has been shown in [8]:

**Theorem 4.1.** Suppose (H1)–(H4) and (2.2) are true, and let \(N\) be sufficiently large. Then there exist constants \(\tilde{C}_1 > 0, \tilde{C}_2 > 0\), which do not depend on \(M, N, K\), such that for \(\frac{1}{3N^2} < \frac{1}{M} \leq \tilde{C}_1\) the discrete problem \((D_{M,N,K})\) has a solution which satisfies

\[(4.10a)\]

\[\theta^m(x) \geq 0, \quad \forall x \in \overline{\Omega}, \quad 0 \leq m \leq M,\]

\[(4.10b)\]

\[\max_{0 \leq m \leq M} \left\{ \left\| \frac{u^m - u^{m-1}}{h} \right\| + \left\| \frac{u_x^m - u_x^{m-1}}{h} \right\| + \left\| u_{xxx}^m \right\| \right\} \leq \tilde{C}_2,\]

\[(4.10c)\]

\[\max_{0 \leq m \leq M} \left\{ \left\| \theta^m_x \right\|^2 + |\theta^m(1)|^2 \right\} + \sum_{m=1}^{M} h \left\| \frac{\theta^m - \theta^{m-1}}{h} \right\|^2 \leq \tilde{C}_2.\]

**Proof.** See Theorem 2.1 in [8]. \(\square\)

It now easy to derive convergent approximate solutions. To this end, let \(\varphi : \mathbb{N} \to \mathbb{N}\) denote some strictly increasing function. We put \(N = \varphi(K)\) and \(M = 2N^2\) (which implies \(\frac{1}{3N^2} < \frac{1}{M}\)) and take \(K \in \mathbb{N}\) large enough. Let \(\{(u^m_K, \theta^m_K)\}_{m=1}^M\) denote corresponding solutions of \((D_{M,N,K})\) with the above choice of \(N\) and \(M\).

We define the linear interpolations

\[(4.11)\]

\[u_K(x,t) = (Mt - m + 1)u^m_K(x) + (m - Mt)u^{m-1}_K(x),\]

\[\theta_K(x,t) = (Mt - m + 1)\theta^m_K(x) + (m - Mt)\theta^{m-1}_K(x),\]

\[0 \leq x \leq 1, \quad \frac{m-1}{M} \leq t \leq \frac{m}{M}, \quad m = 1, \ldots, M.\]
Then (4.10b,c) imply that, for any sufficiently large \( K \in \mathbb{N} \),

\[
\begin{align*}
(4.12a) & \quad \|u_K\|_{W^{1,\infty}(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))} \leq \tilde{C}_2, \\
(4.12b) & \quad \|\theta_K\|_{H^1(0,T;L^3(\Omega)) \cap L^\infty(0,T;H^1(\Omega))} \leq \tilde{C}_2.
\end{align*}
\]

From standard compactness arguments we conclude the existence of some \((\tilde{u}, \tilde{\theta})\) such that, for some subsequence,

\[
\begin{align*}
(4.13a) & \quad u_{K_n} \to \tilde{u}, \quad \text{weakly} \to \ast \text{ in } W^{1,\infty}(0,T;\tilde{H}^1(\Omega)) \text{ and} \\
& \quad \text{weakly} \to \ast \text{ in } L^\infty(0,T;H^3(\Omega)), \\
(4.13b) & \quad \theta_{K_n} \to \tilde{\theta}, \quad \text{weakly in } H^1(0,T;L^2(\Omega)) \text{ and} \\
& \quad \text{weakly} \to \ast \text{ in } L^\infty(0,T;H^1(\Omega)), \\
(4.13c) & \quad \frac{\partial}{\partial x} u_{K_n} \to \frac{\partial}{\partial x} \tilde{u}, \quad \text{uniformly on } \overline{\Omega}_T, \\
& \quad \theta_{K_n} \to \tilde{\theta}, \quad \text{uniformly on } \overline{\Omega}_T.
\end{align*}
\]

It is easy to see that the limit point \((\tilde{u}, \tilde{\theta})\) is a solution of the variational problem (2.3a–c).

By virtue of Theorem 2.3, \( \tilde{u} = u, \tilde{\theta} = \theta \), where \((u, \theta)\) is the (unique) solution of (1.7a–f).

It follows that the whole sequence \((u_K, \theta_K)\) converges to \((u, \theta)\) in the sense of (4.13a–c).

**Theorem 4.2.** Suppose (H1)–(H4) and (2.2) are true, and assume that to \( K \in \mathbb{N} \) we define \( N = \varphi(K) \) and \( M = 2N^2 \), where \( \varphi : \mathbb{N} \to \mathbb{N} \) is any strictly increasing function. For sufficiently large \( K \in \mathbb{N} \), let \((u_K, \theta_K)\) be defined by (4.11). Then \((u_K, \theta_K)\) converges in the sense of (4.13a–c) to the solution \((u, \theta)\) of (1.7a–f).

**Remark.**

8. Results concerning the order of convergence have not yet been established.

**Acknowledgement.** The author gratefully acknowledges the financial support and the stimulating atmosphere at the Institute for Mathematics and its Applications of the University of Minnesota.
REFERENCES


