ON THE UNIQUENESS IN THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT

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IMA Preprint Series # 463

December 1988
ON THE UNIQUENESS IN THE INVERSE CONDUCTIVITY PROBLEM WITH ONE MEASUREMENT*

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Introduction. We shall consider an inverse problem for electrically conductive material occupying a domain $\Omega$ in $\mathbb{R}^n$, $n \geq 2$. Let $D$ be a subdomain of $\Omega$ and suppose that the conductivity coefficient of $D$ is 2 and of $\Omega \setminus D$ is 1. We wish to determine the location of $D$ by injecting current with density $g$ across $\partial \Omega$ and measuring the voltage $u$ on a portion $\Gamma_0$ of the boundary $\partial \Omega$. The voltage $u$ satisfies the refraction problem:

$$\text{div} \left((1 + \chi(D))\nabla u\right) = 0 \quad \text{in} \quad \Omega,$$

$$\frac{\partial u}{\partial N} = g \quad \text{on} \quad \partial \Omega \quad (N = \text{normal to } \partial \Omega)$$

where $\chi(D)$ is the indicator function of $D$, and $u$ is normalized by

$$\int_{\partial_0 \Omega} u = 0 \quad \text{where } \partial_0 \Omega \text{ is a compact subset of } \partial \Omega.$$

Thus the problem is to determine $D$ either by one measurement, i.e., by using one specific choice of $g$, or perhaps by a finite number of measurements $g_1, g_2, \ldots, g_N$.

If any number of measurements is allowed or, more precisely, if the Dirichlet-to-Neumann operator

$$u|_{\partial \Omega} \rightarrow \frac{\partial u}{\partial N}|_{\partial \Omega}$$

is known for all functions $u$ defined on $\partial \Omega$, then $D$ is uniquely determined; this was proved, for $\partial D$ piecewise analytic, by Kohn and Vogelius [6], and, for $\partial D$ Lipschitz, by Isakov [5]; see also [4].

The case of a single measurement $g$ was studied by Friedman and Gustafsson [2] and Bellout and Friedman [1]. They proved, roughly, that the local dependence of the data $u|_{\Gamma_0}$ on the domain $D$ is a function whose Fréchet derivative does not vanish. Friedman and Vogelius [3] established a Lipschitz estimate for the location of extreme inhomogeneities in a smooth conductive medium using one measurement only.

In this paper we prove that if $D$ is known to be a convex polyhedron (although its specific shape is not known) then the shape and location of $D$ are determined by one measurement only.

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*This work is partially supported by National Science Foundation Grant DMS-86–12880.
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§1. Statement of main results. Let $\Omega$ be either a bounded domain in $\mathbb{R}^n$ ($n = 2, 3$) with Lipschitz and piecewise $C^{1,1}$ boundary $\partial \Omega$, or a half space in $\mathbb{R}^n$. Let $D$ be a bounded subdomain of $\Omega$ with Lipschitz and piecewise $C^{1,1}$ boundary $\partial D$, such that $\overline{D} \subset \Omega$.

Let $g$ be any function in $L^\infty(\partial \Omega)$ such that $g \neq 0, g$ has compact support, $\int g = 0$, and let $S$ be a compact subset of $\partial \Omega$ containing the support of $g$. Consider the refraction problem

\begin{align*}
\text{div} \left((1 + \chi(D))\nabla u\right) &= 0 \quad \text{in} \quad \Omega, \\
\frac{\partial u}{\partial N} &= g \quad \text{on} \quad \partial \Omega \tag{1.2}
\end{align*}

where $N$ is the outward normal; $u$ is normalized, say, by

\begin{equation}
\int_S u = 0. \tag{1.3}
\end{equation}

It is well known (see, for instance, [7]) that if $\Omega$ is bounded then this problem has a unique solution in $H^1(\Omega) \cap C^\alpha(\overline{\Omega})$ for some $\alpha > 0$; further, setting

\begin{align*}
D^e &= \Omega \setminus \overline{D}, \quad D^i = D, \quad u^e = u|_{D^e}, \quad u^i = u|_{D^i};
\end{align*}

we have

\begin{align*}
\Delta u^e &= 0 \quad \text{in} \quad D^e, \quad \Delta u^i = 0 \quad \text{in} \quad D^i, \\
u^e &= u^i \quad \text{on the smooth part of} \quad \partial D, \tag{1.4}
\end{align*}

\begin{align*}
\frac{\partial u^e}{\partial N} &= 2 \frac{\partial u^i}{\partial N} \quad \text{on the smooth part of} \quad \partial D, \tag{1.5}
\end{align*}

where the "smooth part" refers to the portion of $\partial D$ which is $C^{1,1}$. The same is true of $D$ is a half space. Indeed, to prove existence we set $\Omega_\rho = \Omega \cap \{|x| < \rho\}$ and denote by $u_n$ ($n = n_0, n_0 + 1, \ldots$) the solution of (1.1) in $\Omega_n$ satisfying (1.2) on $\partial \Omega_n$ ($g = 0$ on $\partial \Omega_n \setminus S$) and normalized by (1.3); $n_0$ is such that $S \subset \{|x| < n_0\}$. Clearly

\begin{equation}
\frac{1}{2} \int_{\Omega_n} (1 + \chi(D)) |\nabla u_n|^2 \leq \int_S |g| |u_n|, \tag{1.6}
\end{equation}

and using Poincaré's inequality we get

\begin{equation}
\int_{\Omega_{n_0}} (|u_n|^2 + |\nabla u_n|^2) \leq C_{n_0} \quad \text{if} \quad n \geq n_0.
\end{equation}

Using this to estimate $\int_S |g| |u_n|$ in the previous inequality, we find that

\begin{equation}
\int_{\Omega_n} (1 + \chi(D)) |\nabla u_n|^2 \leq C, \quad C \text{ independent of } n.
\end{equation}

It follows that a subsequence of $u_n$ is convergent to a solution $u$ as asserted above. Uniqueness follows by multiplying the difference of the equations (for solutions $u_1$ and $u_2$) by $u_1 - u_2$ and integrating over $\Omega_\rho$, and then taking an appropriate sequence $\rho \to \infty$. 

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Theorem 1.1. Let $D_1, D_2$ be two convex polyhedrons such that $\overline{D_j} \subset \Omega$ and

\begin{equation}
\text{diam } D_j < \text{dist } (D_j, \partial \Omega) \quad (j = 1, 2),
\end{equation}

and denote by $u_j$ the solution of the refraction problem (1.1)-(1.3) for $D = D_j$. If

\begin{equation}
u_1 = u_2 \quad \text{on } \Gamma_0
\end{equation}

where $\Gamma_0$ is a nonempty open subset of $\partial \Omega$, then $D_1 = D_2$.

The restriction (1.7) can be relaxed provided some geometric assumptions are made on $D$. In particular:

Theorem 1.2. If $\Omega$ is a half space then the assertion of Theorem 1.1 is valid even if the condition (1.7) is dropped.

The uniqueness results of Theorems 1.1, 1.2 imply, by a compactness argument, the following stability property:

Theorem 1.3. Let $D_j$ $(j = 1, 2, \ldots)$ and $D_\infty$ be convex polyhedrons with closure in $\Omega$ and denote by $u_j, u_\infty$ the solutions of (1.1)-(1.3) corresponding to $D_j, D_\infty$ respectively. Assume further that the number of vertices of $D_j$ is bounded by a constant independent of $j$. If

\begin{equation}|u_j - u_\infty|_{L^\infty(\Gamma_0)} \to 0 \quad \text{as } j \to \infty
\end{equation}

where $\Gamma_0$ is a nonempty open subset of $\partial \Omega$, then

\begin{equation}D_j \to D_\infty \quad \text{as } j \to \infty.
\end{equation}

The convergence in (1.10) is taken in the sense that there is a pairing of the vertices $v_{jm}$ of $D_j$ to the vertices $v_{\infty m}$ of $D_\infty$ such that

\begin{equation}v_{jm} \to v_{\infty m} \quad \text{as } j \to \infty, \quad \forall m.
\end{equation}

It would be interesting to replace (1.10) by a quantitative estimate if the $D_j, D_\infty$ were uniformly $C^{1,1}$ smooth then a Lipschitz continuity estimate

\begin{equation}d(D_j, D_\infty) \leq C |u_j - u_\infty|_{L^\infty(\Gamma_0)} \quad (C \text{ constant}),
\end{equation}

where $d$ in the Hausdorff distance, has been derived by Bellout and Friedman [1]. However the results in [1] do not extend to the case where $\partial D_j, \partial D_\infty$ are not in $C^{1,1}$.

In §2 we establish some auxiliary results needed in the proofs of Theorems 1.1, 1.2. Theorems 1.1, 1.2 for $n = 2$ are proved in Section 3, and for $n = 3$ in Section 4; the proof for $n = 3$ can easily be extended to any $n \geq 3$. In §5 we extend Theorem 1.2 to the case where $D$ is a circle. Finally in §6 we establish (1.11) in some (very special) cases.
\section{Auxiliary results.}

**Definition 2.1.** Two bounded domains $D_1, D_2$ with closure in $\Omega$ are said to have i-
contact (interior contact) if

\begin{equation}
\begin{aligned}
&\text{the sets } \Omega \setminus \overline{D}_j, \Omega \setminus (\overline{D}_1 \cup \overline{D}_2), D_1 \cap D_2 \text{ are connected,} \\
&\text{the set } \partial D_1 \cap \partial D_2, \text{ int } (\overline{D}_1 \cup \overline{D}_2) \text{ are disjoint, and} \\
&\text{there is a nonempty } C^2 \text{ hypersurface } \Gamma \text{ which} \\
&\text{belongs to the boundaries of both } \Omega \setminus (\overline{D}_1 \cup \overline{D}_2) \text{ and } D_1 \cap D_2. \\
\end{aligned}
\end{equation}

**Remark 2.1.** If $D_1, D_2$ are convex polyhedrons with int $(D_1 \cap D_2) \neq \emptyset$ such that their boundaries have nonempty interior and such that $\overline{D}_1, \overline{D}_2$ are contained in $\Omega$, then they have i-
contact.

**Lemma 2.1.** Suppose $D_1$ and $D_2$ have i-
contact and $D_1 \neq D_2$. Denote by $u_j$ the
solution of (1.1)-(1.3) with $D = D_j$. Then (1.8) cannot hold.

**Proof.** If (1.8) holds then $u_1, u_2$ have the same Cauchy data on $\Gamma_0$ and, consequently,
by uniqueness for the Cauchy problem for harmonic functions, $u_1 \equiv u_2$ in $\Omega \setminus (\overline{D}_1 \cup \overline{D}_2)$. Using (1.5), (1.6) we conclude that $u_1 = u_2, \partial u_1 / \partial N = \partial u_2 / \partial N$ on $\Gamma$ and therefore, again
by uniqueness for the Cauchy problem,

\begin{equation}
\begin{aligned}
&u_1 = u_2 \quad \text{in } D_1 \cap D_2.
\end{aligned}
\end{equation}

Since $D_1 \neq D_2$, we may assume that $D_2 \setminus \overline{D}_1$ is nonempty. The functions $u_1, u_2$ are
harmonic in $D_2 \setminus \overline{D}_1$. From (2.1) we have that $\partial(D_2 \setminus \overline{D}_1)$ consists only of points which
belong to either $\partial(D_1 \cup D_2)$ or $\partial(D_1 \cap D_2)$. On $\partial(D_1 \cup D_2)$ we have $u_1 = u_2 = u_2$ (using
(1.5)), and on $\partial(D_1 \setminus \overline{D}_2) \cap \partial(D_1 \cap D_2)$ we have $u_1 = u_1 = u_1$ (by (1.5) and (2.2)). Thus

\begin{equation}
\begin{aligned}
&u_1 = u_2 \quad \text{on } \partial(D_2 \setminus \overline{D}_1)
\end{aligned}
\end{equation}

and, by the maximum principle,

\begin{equation}
\begin{aligned}
&u_1 = u_2 \quad \text{in } D_2 \setminus \overline{D}_1.
\end{aligned}
\end{equation}

This implies that

\begin{equation}
\begin{aligned}
&\frac{\partial u_1}{\partial N} = \frac{\partial u_2}{\partial N} \quad \text{on } \partial(D_2 \setminus \overline{D}_1).
\end{aligned}
\end{equation}

Since $u_1 = u_2$ in $\Omega \setminus (D_1 \cup D_2)$,

\begin{equation}
\begin{aligned}
&\frac{\partial u_1}{\partial N} = \frac{\partial u_2}{\partial N} \quad \text{on } \partial D_2 \setminus \overline{D}_1.
\end{aligned}
\end{equation}
Combining this with (2.4) we get

$$\frac{\partial u^*_2}{\partial N} = \frac{\partial u^*_1}{\partial N} \quad \text{on} \quad \partial D_2 \setminus \overline{D}_1$$

and, by comparison with (1.6),

(2.5) $$\frac{\partial u^*_3}{\partial N} = 0 \quad \text{on} \quad \partial D_2 \setminus D_1.$$  

Next, from (2.2),

$$\frac{\partial u^*_1}{\partial N} = \frac{\partial u^*_2}{\partial N} \quad \text{on} \quad \partial(D_1 \cap D_2).$$

Recalling (2.4), we deduce that

(2.6) $$\frac{\partial u^*_1}{\partial N} = \frac{\partial u^*_1}{\partial N} \quad \text{on} \quad \partial(D_2 \setminus \overline{D}_1) \cap \partial D_1.$$  

Since, by (1.6),

$$\frac{\partial u^*_1}{\partial N} = 2\frac{\partial u^*_1}{\partial N} \quad \text{on} \quad \partial(D_2 \setminus \overline{D}_1) \cap \partial D_1,$$

we deduce that

$$\frac{\partial u^*_1}{\partial N} = 0 \quad \text{on} \quad \partial(D_2 \setminus \overline{D}_1) \cap \partial D_1.$$  

From this relation and from (2.5) it follows that $\partial u^*_3/\partial N = 0$ on the boundary of $D_2 \setminus \overline{D}_1$. Hence $u^*_3 \equiv \text{const.}$ in $D_2 \setminus \overline{D}_1$. This easily implies that also $u^*_1 \equiv \text{const.}$ in $\Omega \setminus \overline{D}_1$ and, in particular,

$$g = \frac{\partial u^*_1}{\partial N} = 0 \quad \text{on} \quad \partial \Omega,$$

which is a contradiction.

**Lemma 2.2.** For any domain $D$, the solution $u$ of (1.1)-(1.3) cannot be harmonic in all of $\Omega$; consequently $u^*$ does not have harmonic continuation into all of $\Omega$.

**Proof.** Otherwise, the gradient of $u$ is continuous across $\partial D$ and, in particular,

$$\frac{\partial u^*_i}{\partial N} = \frac{\partial u^*_i}{\partial N}$$

on the smooth part of $\partial D$. Recalling (1.6) we deduce that $\partial u^*_1/\partial N = 0$ on the smooth part of $\partial D$. This implies that $u^*_1 = \text{const.}$ in $D$ and then also $u = \text{const.}$ in $\Omega \setminus D$. Consequently $g = \partial u/\partial N = 0$ on $\partial \Omega$, a contradiction.
Lemma 2.3. If solutions $u_1, u_2$ of (1.1)-(1.3) corresponding to two subdomains $D_1, D_2$ satisfy the relation (1.8), then $\overline{D_1} \cap \overline{D_2}$ is nonempty.

Proof. If $\overline{D_1} \cap \overline{D_2}$ is empty then $u^e_2$ is harmonic in a neighborhood of $\overline{D_1}$. Since $u^e_1 = u^e_2$ in $\Omega \setminus (\overline{D_1} \cup \overline{D_2})$, it follows that $u^e_1$ has harmonic continuation into all of $\Omega$, a contradiction to Lemma 2.2.

Lemma 2.3 shows that in order to prove Theorems 1.1, 1.2 we may assume, without loss of generality, that $\overline{D_1} \cap \overline{D_2} \neq \emptyset$.

Notation. We denote by $B_\varepsilon(a)$ a ball with center $a$ and radius $\varepsilon$.

Lemma 2.4. Consider a solution $u$ to the refraction problem (1.1)-(1.3) and suppose $a_0$ is a point of $\partial D$ such that

$$D \cap B_\varepsilon(a_0) = P_1 \cap \cdots \cap P_k \cap B_\varepsilon(a_0) \quad (\varepsilon > 0)$$

where $P_j$ are half-spaces $\{x_1(x - a_0) \cdot N_j > 0\}$, for some unit vectors $N_j$. If $u^e$ has harmonic continuation from $\Omega \setminus \overline{D}$ into $B_\varepsilon(a_0)$, then $u^i$ has harmonic continuation from $D$ into $B_\varepsilon(a_0)$, for some $0 < \varepsilon_1 < \varepsilon$.

Proof. We may assume that $\partial P_1$ has a common part with $\partial D \cap B_\varepsilon(a_0)$. Consider the Cauchy problem

$$\begin{align*}
\Delta w &= 0 \quad \text{in} \quad B_{\varepsilon_1}(a_0), \\
 w &= u^e, \quad \frac{\partial w}{\partial N} = \frac{1}{2} \frac{\partial u^e}{\partial N} \quad \text{on} \quad \partial P_1 \cap B_{\varepsilon_1}(a_0).
\end{align*}$$

(2.7)

Since $u^e$ is analytic in $B_{\varepsilon_1}(a_0)$, by the Cauchy–Kowalewski theorem there exists a unique analytic solution of (2.7) if $\varepsilon_1$ is small enough. In view of (1.4)-(1.6), $u^i$ satisfies the same differential equation in $D \cap B_{\varepsilon_1}(a_0)$ and the same Cauchy condition on $\partial P_1 \cap \overline{D}$. Hence, by uniqueness for the Cauchy problem, $w = u^i$ in $D \cap B_{\varepsilon_1}(a_0)$, and the lemma follows.

§3. Proof of Theorems 1.1, 1.2 for $n = 2$. Denote by $x(\beta)$ the image under rotation $(r, \phi) \rightarrow (r, \phi + \beta)$ about the origin of a point $x = (r, \phi)$. Similarly we define the image $x(\beta)$ of $x$ under rotation with respect to any center $x_0$.

Lemma 3.1. Suppose $a_0 \in \partial D$ and $D \cap B_\varepsilon(a_0)$ is a nonempty convex set given by $P_1 \cap P_2 \cap B_\varepsilon(a_0)$ where $P_1, P_2$ are two distinct half-planes. If $u^e$ has harmonic continuation from $D^e$ into $D^e \cup B_\varepsilon(a_0)$ then there is a rotation $x \rightarrow x(2\pi/p) \ (p = 2, 3, \ldots)$ about $a_0$ such that

$$u^e(x) = u^e \left( x \left( \frac{2\pi}{p} \right) \right) \quad \text{for} \quad x \in B_\varepsilon(a_0).$$

(3.1)
Proof. We may assume that \( a_0 = 0 \) and that

\[
D \cap B_\varepsilon(a_0) = \{0 < \phi < \alpha, \quad r < \varepsilon\}
\]

where \( 0 < \alpha < \pi \). According to Lemma 2.4 the function \( u^j \) has harmonic continuation into a disk \( B_\varepsilon(a_0) \). We thus have, in \( B_\varepsilon(a_0) \),

\[
\begin{align*}
  u^e(x) &= \sum_{k=0}^{\infty} (a_k^e \cos k\phi + b_k^e \sin k\phi)r^k, \\
  u^i(x) &= \sum_{k=0}^{\infty} (a_k^i \cos k\phi + b_k^i \sin k\phi)r^k
\end{align*}
\]

(3.2)

where \( b_0^e = b_0^i = 0 \).

From the conditions (1.5), (1.6) at \( \phi = 0 \) we get

(3.3)

\[
  a_k^e = a_k^i, \quad b_k^e = 2b_k^i
\]

and from the same conditions at \( \phi = \alpha \) we get, after using (3.3),

\[
  (b_k^e - b_k^i) \sin k\alpha = 0, \quad (a_k^e - 2a_k^i) \sin k\alpha = 0
\]

or

(3.4)

\[
b_k^e \sin k\alpha = a_k^e \sin k\alpha = 0.
\]

If \( \alpha/\pi \) is irrational then \( \sin k\alpha \neq 0 \) and consequently \( b_k^e = a_k^e = 0 \) for \( k = 1, 2, \ldots; \)

hence \( u^e \equiv \text{const.} \) which is a contradiction. We conclude that \( \alpha/\pi \) is a rational number, say

\[
\frac{\alpha}{\pi} = \frac{q}{p} \quad \text{when} \quad p = 2, 3, \ldots, q \geq 1, \quad (p, q) = 1.
\]

For all nonzero coefficients \( a_k^e, b_k^e \) we must have \( \sin(k\pi q/p) = 0 \), i.e., \( k\pi q/p \) is a positive integer; since \( (p, q) = 1 \), it follows that \( k = mp \) where \( m = 0, 1, 2, \ldots \). Recalling the representation (3.2) and noting that

\[
\cos(mp(\phi + \frac{2\pi}{p})) = \cos(mp\phi), \quad \sin(mp(\phi + \frac{2\pi}{p})) = \sin(mp\phi),
\]

the assertion (3.1) follows.
LEMMA 3.2. Let \( D \) and \( u^* \) satisfy the conditions of Lemma 3.1 and assume that \( u^* \) has harmonic continuation into an open set \( S \) containing \( B_{\varepsilon}(a_0) \), and that the images \( S_j \) of \( S \) under the rotation \( \phi \rightarrow \phi + 2\pi j/p \) (about \( a_0 \)) satisfy:

\[
S_j \cap S_k \text{ are connected for all } j, k.
\]

Then \( u^* \) has harmonic continuation into the open set \( \bigcup_{j=1}^p S_j \).

Proof. Define \( u_j^*(r, \phi) = u^*(r, \phi - (j - 1)2\pi/p) \). The function \( u_j^*(r, \phi) \) is harmonic in \( S_j \). In view of Lemma 3.1 the harmonic function \( u_j^* \) and \( u_k^* \) agree on \( S_j \cap S_k \cap B_{\varepsilon}(a_0) \). Since the intersection \( S_j \cap S_k \) is connected and contains \( B_{\varepsilon}(a_0) \), \( u_j^* \) and \( u_k^* \) agree on \( S_j \cap S_k \). It follows that the function defined by \( u = u_j^* \) on \( S_j \), for \( 1 \leq j \leq p \), is a harmonic continuation of \( u^* \) into \( S_1 \cup \cdots \cup S_p \).

Proof of Theorem 1.1. If \( D_1 \neq D_2 \) then we may assume that the origin \( 0 \) is a vertex of the convex hull of \( D_1 \cup D_2 \) and that it belongs to \( \overline{D}_1 \) but not to \( \overline{D}_2 \). As in Lemma 2.1, \( u_1^* = u_2^* \) in \( \Omega \setminus (\overline{D}_1 \cup \overline{D}_2) \). Since \( u_1^* \) is harmonic in a neighborhood of \( 0 \), \( u_1^* \) has harmonic continuation from \( \Omega \setminus D_1 \) into \( B_{\varepsilon}(0) \), for some \( \varepsilon > 0 \). From the condition (1.7) it follows that there are two different half-discs of radius \( d > \text{diam } D_1 \), centered at \( 0 \), which belong to \( \Omega \setminus D_1 \). Let \( \bar{S} \) denote the union of these two half-discs and \( B_{\varepsilon}(0) \). By Lemma 3.2 \( u_1^* \) has harmonic continuation into \( S_1 \cup \cdots \cup S_p = B_d(0) \). Since \( \overline{D}_1 \subset B_d(0) \), \( u_1^* \) has harmonic continuation into all of \( \Omega \), which is a contradiction to Lemma 2.2.

Proof of Theorem 1.2. We assume that \( D_1 \neq D_2 \) and take \( \Omega = \{ x_2 < 0 \} \). Let \( a(1) = (a_1(1), a_2(1)) \) be a vertex in the convex hull \( D_\ast \) of \( D_1 \cup D_2 \) with the largest distance to \( \partial \Omega \). If \( u_1^* \) (or \( u_2^* \)) has harmonic continuation into a neighborhood of \( a(1) \) then using the fact that \( u_1^* \) is harmonic in the half-plane \( \{ x_2 < a_2(1) \} \) we can construct two half-discs in \( \Omega \setminus D_\ast \) with radius larger than the diameter of \( D_\ast \) and center at \( a(1) \) and show, as in the previous proof, that \( u_1^* \) (or \( u_2^* \)) has harmonic continuation into \( \Omega \), which is a contradiction. Thus in the sequel we may assume that

\[
\text{(3.5)} \quad u_1^* \text{ and } u_2^* \text{ do not have harmonic continuation from } \Omega \setminus D_\ast \text{ into a neighborhood of } a(1).
\]

It follows that \( a(1) \) is a vertex of both \( D_1 \) and \( D_2 \). Consider the vertices \( a(0) = (a_1(0), a_2(0)) \), \( a(2) = (a_1(2), a_2(2)) \) of \( D_\ast \) adjacent to \( a(1) \). If \( a(0) \) is a vertex of both \( D_1 \) and \( D_2 \), then the convex polygons \( D_1 \) and \( D_2 \) have i-contact (with \( \Gamma = a(0)a(1) \)), which is a contradiction to Lemma 2.1. Therefore

\[
\text{(3.6)} \quad a(0) \text{ is not a common vertex of } D_1, D_2, \text{ and the same holds for } a(2).
\]
We may assume that $a_1(0) \leq a_1(2)$. Consider the case

\[ (3.7) \quad a_1(1) < a_1(2). \]

Since $a(2)$ is not a common vertex, the functions $u_1^*, u_2^*$ have harmonic continuation into an \( \varepsilon \)-neighborhood of $a(2)$. We can therefore apply Lemma 3.1 to $u_1^*$, considering rotations with respect to $a(2)$. Denote by $\bar{S}$ the set $B_\varepsilon(a(2)) \cup (\Omega \setminus \overline{D_3})$. If the related rotation number is $p = 2$ then the sets $S_1 = \bar{S}$ and $S_2$ satisfy:

\[ S_1 \cup S_2 \supset \mathbb{R}^2 \setminus \overline{D_3} \]

where $D_3$ is the union of $D_1$ and its image under rotation $x \rightarrow x(\pi)$ about $a(2)$. We conclude that $u_1^*$ has harmonic continuation into $\mathbb{R}^2 \setminus \overline{D_3}$.

We now use $a(0)$ as a center of rotation for $u_1^*$, taking

\[ (3.8) \quad \bar{S} = B_\varepsilon(a(0)) \cup (\mathbb{R}^2 \setminus \overline{D_3}). \]

It is easy to check that $S_1 \cup \cdots \cup S_p$ contains $a(1)$, no matter what $p$ is. Hence $u_1^*$ has harmonic continuation to a neighborhood of $a(1)$, which is a contradiction to (3.5).

Consider next the case where the rotation number at $a(2)$ is $p = 3$. In this case $u_1^*$ has harmonic continuation into $S_1 \cup S_2 \cup S_3$. The boundary of this union contains the images $a(1; 2)$, $a(1; 3)$ of the point $a(1)$ under the rotation. Since $u_1^*$ has no harmonic continuation into a neighborhood of $a(1)$, also

\[ (3.9) \quad u_1^* \text{ has no harmonic continuation} \]

\[ \text{into a neighborhood of } a(1; 2), \text{ or } a(1; 3). \]

We now resort again to rotation about $a(0)$. If $p = 2$ then taking $\bar{S}$ as in (3.8) we easily check that $S_1 \cup S_2$ contains either $a(1; 2)$ or $a(1; 3)$ and therefore $u_1^*$ has harmonic continuation into a neighborhood of one of these points, a contradiction to (3.9). If $p \geq 3$ then the set $S_1 \cup S_2 \cup S_3$ contains a sector of the plane with vertex at $a(0)$ and opening larger than $2\pi/3$. Thus $u_1^*$ has harmonic continuation to all of $\mathbb{R}^2$, a contradiction.

It remains to consider the case where the rotation number at $a(2)$ is $p \geq 4$. In this case $u_1^*$ has harmonic continuation to all of $\mathbb{R}^2$ which is obtained by rotation about $a(2)$ of a sector $\Sigma$ of the plane with vertex at $a(2)$ and opening $\sigma_1 + \varepsilon_1$ ($\varepsilon_1 > 0$) which is entirely contained in $\Omega$.

We have thus shown that the case (3.7) gives a contradiction. If (3.7) is not true, then $a_1(0) \leq a_1(2) \leq a_1(0)$ and both equalities cannot hold simultaneously. Hence $a_1(0) < a_1(1)$ and we can proceed as above, interchanging the roles of $a(0)$ and $a(2)$.
§4. Proofs of Theorem 1.1, 1.2 in case \( n = 3 \). For a convex polyhedron \( D \) and an edge \( \gamma \) in a direction \( \sigma \), we shall use the representation

\[ s\sigma + x' \quad \text{where} \quad x' \perp \sigma, \quad s \in \mathbb{R}^1 \]

for any \( x \in \mathbb{R}^3 \).

**Lemma 4.1.** Assume that the origin 0 belongs to \( \gamma \) and denote by \( u = (u^e, u^i) \) the solution of (1.1)-(1.3). If \( u^e \) has harmonic continuation into \( B_\varepsilon(0) \), for some \( \varepsilon > 0 \), then there is a rotation \( x'(2\pi/p), \quad p = 2, 3, \ldots \), such that

\[ u^e(x) = u^e(s\sigma + x'(2\pi/p)) \quad \text{for} \quad x \in B_\varepsilon(0). \]

**Proof.** From Lemma 2.4 it follows that the function \( u^i \) has harmonic continuation into a ball \( B_{\varepsilon_1}(0), 0 < \varepsilon_1 < \varepsilon \). Introduce polar coordinates \((r, \phi)\) in the plane \( E \equiv \{(x^i, 0)\} \). We may assume that \( E \cap D \) is bounded by \( \phi = 0 \) and \( \phi = \phi_1 \) for some \( 0 < \phi_1 < \pi \). Consider the functions

\[ u_1 = u^e - u^i \quad \text{and} \quad u_2 = u^e - 2u^i. \]

From the relations (1.5), (1.6) we have

\[ u_1 = \frac{\partial u_2}{\partial \phi} = 0 \quad \text{for} \quad \phi = 0, \phi_1. \]

Since \( u_1, u_2 \) are harmonic in \( B_{\varepsilon_1}(0) \), the refraction formulas

\[ u_1(x^i, s) = -u_1(x^i, s), \quad u_2(x^i, s) = u_2(x^i, s) \]

hold for all points \( x^i, x^i \) symmetric with respect to the lines \( \phi = 0, \phi = \phi_1 \). This implies, in particular, the relations (4.2) for all \( \phi = k\phi_1, \quad k = 1, 2, \ldots \).

Let \( \phi_0 \) be the smallest nonnegative \( \tilde{\phi}_1 \) such that (4.2) holds for \( \phi = \tilde{\phi}_1 \). If \( \phi_0 = 0 \) then both \( u_1 \) and \( u_2 \) do not depend on \( \phi \). If \( \phi_0 > 0 \) then \( \phi_0 = \pi/p \) for some \( p = 2, 3, \ldots \) and, from the relation (4.3), we conclude that

\[ u_j(x^i, s) = u_j(x^i \left(\frac{2\pi}{p}\right), s). \]

Since \( u^e = 2u_1 - u_2 \), the proof is complete.

**Lemma 4.2.** Let \( D, u \) be as in Lemma 4.1 and let \( B \) be a ball with a center at some point of \( \gamma \) such that \( B \supset \overline{D} \). Let \( B_0 \) be a hemi-ball of \( B \) (i.e., \( B_0 \subset B, \quad \text{radius} \ B_0 = \text{radius} \ B \)) and let \( B_0 \subset S_\varepsilon \subset B \) be such that

\[ S_\varepsilon \text{ contains all sufficiently small rotations of } B_0 \text{ about } \gamma. \]
Assume that \( u^e \) has harmonic continuation into \( \overline{S}_* \setminus \gamma_0 \) where \( \gamma_0 \) is the straight line containing \( \gamma \). Then \( u^e \) has harmonic continuation into a neighborhood of \( \overline{D} \).

**Proof.** Let \( p \) be the rotation number from Lemma 4.1. We may assume that \( S_* \) contains \( B \cap \{ 0 < \phi < \frac{2\pi}{p} + \epsilon' \} \) for some \( \epsilon' > 0 \). The function \( u^e \) thus has harmonic continuation into the set

\[
S^* = (\{ 0 < \phi < \frac{2\pi}{p} + \epsilon' \} \cap S_*) \cup B_4(0).
\]

Denote by \( S^*_j \) the image of \( S^* \) under the rotation \( \phi \to \phi + 2(j - 1)\pi/p \). Then

\[
B \setminus \gamma_0 \subseteq S^*_1 \cup \cdots \cup S^*_p.
\]

Define functions \( u_j(r, \phi, s) \) as \( u^e(r, \phi - 2(j - 1)\pi/p, s) \). Then \( u_j \) is harmonic in \( S^*_j \). In view of Lemma 4.1, \( u_j = u_k \) on \( S^*_j \cap S^*_k \cap B_4(0) \) and therefore \( u_j \) and \( u_k \) agree on \( S^*_j \cap S^*_k \). Hence the function \( w \) defined by \( w = u_j \) on \( S^*_j \) is a harmonic continuation of \( u^e \) into \( B \setminus \gamma_0 \). Since \( w \in H^1(B) \), the singularity of \( w \) on \( B \cap \gamma_0 \) is removable (as can be seen by writing

\[
\int \nabla w \cdot \nabla (\zeta_j \psi) = 0, \quad \text{for any} \quad \psi \in C_0^\infty(B),
\]

where \( \zeta_j \in C_0^1(B), \zeta_j = 0 \) in a neighborhood of \( \gamma_0 \), \( \zeta_j \to 1 \) as \( j \to \infty \), and taking \( j \to \infty \)).

**Proof of Theorem 1.1.** We shall assume that \( D_1 \neq D_2 \) and derive a contradiction. As before, \( u_1 = u_2 \) in \( \Omega \setminus (\overline{D}_1 \cup \overline{D}_2) \). We may assume that the origin belongs to an edge \( \gamma \) of the convex hull of \( D_1 \cup D_2 \) and that the interior of \( \gamma \) does not belong to \( \overline{D}_1 \).

As above the function \( u^e \) has harmonic continuation from \( \Omega \setminus \overline{D}_1 \) into a ball \( B_4(0), 0 \in \text{int} \gamma \). From the condition (1.7) it follows that there are two different hemiballs of radius \( d, d > \text{diam} \ D_1 \), centered at 0, that do not intersect \( D_1 \) and belong to \( \Omega \). Let \( \mathcal{S} \) be the union of these balls and \( B_4(0) \). Repeating the proof of Theorem 1.1 for \( n = 2 \) but using Lemma 4.2 instead of Lemma 3.2, we get a contradiction.

**Proof of Theorem 1.2.** We again assume that \( D_1 \neq D_2 \) and proceed to derive a contradiction. We take \( \Omega = \{ x_3 < 0 \} \). Since the function \( g \) has compact support, the functions \( u^e_1, u^e_2 \) have harmonic extensions to \( \mathbb{R}^3 \setminus (\overline{D}_1 \cup \overline{D}_2 \cup S_0) \) where \( D_* \) is the convex hull of \( D_1 \cup D_2, D_* \) in its symmetric image under the map \( (x_1, x_2, x_3) \to (x_1, x_2, -x_3) \), and \( S_0 \) is a bounded set in \( \{ x_3 = 0 \} \) containing the support of \( g \).

Let \( a \) be a vertex of \( D_* \) with the largest distance to \( \partial \Omega \). Then there is a face \( \Gamma \) of \( \partial D_* \) whose exterior normal has negative \( x_3 \)-coordinate. Since, by Lemma 2.1, \( D_1 \) and \( D_2 \) do not have \( i \)-contact there must be an edge \( \gamma \) of \( \partial D_* \) with \( a \in \overline{\gamma}, \gamma \subset \overline{\Gamma} \) such that \( \gamma \) is not contained in either \( \overline{D}_1 \) or \( \overline{D}_2 \); for definiteness we take \( \gamma \not\subset \overline{D}_1 \), and choose a point \( b \in \gamma, b \not\in \overline{D}_1 \).
The plane containing $\Gamma$ divides $\mathbb{R}^3$ into two half-spaces and, according to the choice of $\Gamma$, one of them, call it $P$, does not intersect $\overline{D}_+ \cap \overline{D}_-$. Since $D_+, D_-$ are convex polyhedrons, any small rotation of $P$ about $\gamma$ also does not intersect $\overline{D}_+ \cup \overline{D}_-$. We can now apply the proof of Lemma 4.2 to $u^*_1$ and conclude that $u^*_1$ has harmonic continuation into $\mathbb{R}^3 \setminus \hat{S}$ where $\hat{S}$ consists of $S_0$ and its image $S_1$ under the appropriate rotation. But since $\nabla u^*_1$ is in $L^2_{\text{loc}}$, the singularities of $u^*_1$ on $\hat{S}$ are removable (as in the proof of Lemma 4.2). Thus $u^*_1$ is harmonic in $\mathbb{R}^3$, which is a contradiction.

§5. The case where $D$ is a circle. In this section we assume that $\Omega$ is a half-space in $\mathbb{R}^2$, say $\{x_2 < 0\}$. We shall prove:

THEOREM 5.1. Let $B_1, B_2$ be two open circles with $\overline{B}_j \subset \Omega$ and denote by $u_j$ the solution of (1.1)–(1.3) with $D = B_j$. If (1.8) holds where $\Gamma_0$ is a nonempty open subset of $\partial \Omega$, then $B_1 = B_2$.

For any ball $B = B(a, r)$ with center $a$ and radius $r$, let $x^*(B)$ denote the inversion of a point $x$ with respect to $B$, and set $K^*(B) = \{x^*(B) ; x \in K\}$. Also let $\sigma(A)$ denote the image of the set $A$ under the mapping $(x_1, x_2) \mapsto (x_1, -x_2)$.

LEMMA 5.2. Let $S$ be any compact set on $x_2 = 0$ containing the support of $g$. Denote by $u$ a solution of (1.1)–(1.3) when $D$ is a circle $B = B(a, r)$ with $\overline{B} \subset \Omega$. Then $u^*$ has harmonic continuation from $\Omega \setminus \overline{B}$ into

$$\Omega \setminus (S(1) \cup \cdots \cup S(k) \cup \ldots \cup \{b\}, \{a\})$$

where $S(1)$ is $S^*(B), S(k)$ is $(\sigma(S(k - 1)))^*(B)$, and $b$ is some point of $B$; further, for any $k$, the first derivatives of the continuation are bounded in some neighborhood of $S(1) \cup \cdots \cup S(k)$.

Proof. Since $\frac{\partial u}{\partial x_2} = 0$ on $\partial \Omega \setminus S, u^*$ has harmonic continuation onto $\mathbb{R}^2 \setminus (S \cup \overline{B} \cup \sigma(B))$. The inversion $u^*_1(x) = u^*(x^*(B))$ yields a function $u^*_1$ harmonic in $\mathbb{R}^2 \setminus B$ and from the refraction conditions (1.5), (1.6),

$$u^*_1 = u^*, -2 \frac{\partial u^*_1}{\partial r} = \frac{\partial u^*}{\partial r} \text{ on } \partial B.$$ 

Let

$$u_1 = u^*_1 - u^*, \quad u_2 = 2u^*_1 + u^*.$$ 

Then $u_1, u_2$ are harmonic in $\mathbb{R}^2 \setminus (S \cup \overline{B} \cup \sigma(B))$ and

$$u_1 = 0, \quad \frac{\partial u_2}{\partial r} = 0 \text{ on } \partial B.$$
By inversion (with respect to $B$) the functions $u_1, u_2$ and therefore also $u^*_1, u^*_2$ have harmonic continuation onto

$$
\Omega \setminus (S(1) \cup E_1 \cup \{a\}), E_1 = (\sigma B)^*(B);
$$

Moreover, the first derivatives of the continuation of $u^*$ are bounded in a neighborhood of $S(1) = S^*(B)$.

We now repeat the previous argument with $u^*$ being the harmonic continuation of the original $u^*$ into the region (5.1). This leads to a harmonic continuation of $u^*$ into the set

$$
\Omega \setminus (S(1) \cup S(2) \cup E_2 \cup \{a\})
$$

where $E_2 = (\sigma(E_1))^*(B)$.

Proceeding step by step we continue $u^*$ harmonically into

$$
\Omega \setminus (S(1) \cup \cdots \cup S(k) \cup E_k \cup \{a\})
$$

where $E_k = (\sigma(E_{k-1}))^*(B)$. Clearly $E_k$ converge to a single point $b$ in $B$ as $k \to +\infty$, and the proof is complete.

**Proof of Theorem 5.1.** Suppose $B_1 \neq B_2$. Let $S_1(k), S_2(k), b_1, b_2, a_1, a_2$ be the sets from Lemma 5.2 related to $B_1, B_2$ respectively. We may assume $S$ is symmetric with respect to the $x_1$-coordinate of the center of $B_2$ and that the distance from $S_2(1)$ to $S$ is not greater than the distance from $S_1(1)$ to $S$.

Since $B_1 \neq B_2$ the intersection $S_0$ of $S_2(1)$ with $S_1(1) \cup \cdots \cup S_1(k) \cup \{b_1\} \cup \{a_1\}$ has no more than one accumulation point $a_\infty$. As above $u^*_1 = u^*_2$ on $\Omega \setminus (\overline{B_1} \cup \overline{B_2})$ and therefore from the Lemma 5.2 we conclude that $u^*_2$ has a bounded harmonic continuation into the set $V \setminus S_0$ where $V$ is some neighborhood of $S_2(1)$. Since $u^*_2$ is bounded near any isolated point of $S_0$ this point is a removable singularity and $u^*_2$ has a harmonic continuation into $V \setminus \{a_\infty\}$; similarly $a_\infty$ is also removable singularity. It follows that $u^*_2$ has a harmonic continuation into a neighborhood of $S_2(1)$.

The set $S$ is the image of $S_2(1)$ under the inversion with respect to $\partial B_2$. Since $u^*_2$ has harmonic continuation into a neighborhood of $S_2(1)$, it has also harmonic continuation into a neighborhood of $S$. This is a contradiction since $(\partial u^*_2 / \partial x_2)(x_1, 0) = g$ and $g$ is not analytic in a neighborhood of $S$.

§6. A Lipschitz stability estimate. The results of Bellout and Friedman [1] can be used to derive a Lipschitz stability estimate of the form (1.11) for a family of balls $B(h)$ as $h \to 0$, with $D_j = B(h_j)$; it is assumed here that $B(h)$ is monotone in $h$ (For non-monotone family $B(h)$, results are given in [1] only in case $n = 2$). We shall subsequently deal only with the case where the domains $D(h)$ are convex polyhedrons. We shall further
assume that \( n = 2 \), that \( \Omega = \{(x, y); y < 0\} \) and that \( D(h) \) obtained from \( D(\overline{D} \subset \Omega) \) by the mapping
\[
(x, y) \mapsto (x, y + h).
\]

Denote by \( u(x, y; h) \) the solution of the refraction problem (1.1)–(1.3) corresponding to \( D(h) \) (we assume that \( \overline{D(h)} \subset \Omega \)) and set
\[
D_1 = D_2, \quad D_2 = D(h).
\]
We shall also denote by \( u_j \) the solutions of (1.1)–(1.3) corresponding to \( D_j \).

**Theorem 6.1.** Assume that one edge of \( D \) is parallel to the \( y \)-axis. Then the estimate
\[
(6.1) \quad \text{dist} (D_1, D_2) \leq C \|u_1 - u_2\|_{L^\infty(\Gamma_0)}
\]
holds, where \( C \) is a positive constant independent of \( h; \Gamma_0 \) is any nonempty open subset of \( \partial \Omega \).

**Remark 1.1.** A similar result can be established in case \( D(h) \) is obtained from \( D \) by a translation \( (x, y) \mapsto (x + h, y) \), and one edge of \( D \) is parallel to the \( x \)-axis.

**Remark 6.2.** The complimentary estimate to (6.1), namely,
\[
\|u_1 - u_2\|_{L^\infty(\partial \Omega)} \leq C' \text{dist}(D_1, D_2),
\]
follows by standard elliptic estimates applied to the function \( w_h \) defined below.

**Proof.** If the assertion of the theorem is not true then we have
\[
\frac{u(x, 0; h) - u(x, 0)}{h} \rightarrow 0 \quad \text{uniformly on } \Gamma_0.
\]

The function
\[
w_h(x, y) = \frac{u(x, y - h; h) - u(x, y)}{h}
\]
satisfies (1.1) and, as easily seen,
\[
\frac{\partial w_h}{\partial N} \quad \text{is bounded on } \partial D.
\]

It follows that \( w_h \) has a subsequence which is convergent (weakly in \( H^1(\Omega) \)) to a solution \( w \) of (1.1). On \( \Gamma_0 \),
\[
w_h(x, 0) = \{u(x, 0; h) - u_x(x, 0; h)h - u(x, 0+) + o(h)\}/h
\]
\[
\rightarrow -g(x), \quad \text{by (6.2)},
\]
\[
\frac{\partial}{\partial y} w(x, 0) = -u_{y y}(x, 0) + o(1) \rightarrow -u_{y y}(x, 0).
\]
Since \( w_h(z, y) \) is a solution of (1.1) with Cauchy data converging to \((-u_y, -u_{yy})\) on \( \Gamma_0 \), \( u \) is harmonic in \( \Omega \setminus D \) having the Cauchy data \((-u_y, -u_{yy})\) on \( \Gamma_0 \). But \(-u_y^\varepsilon\) is also harmonic in \( \Omega \setminus D \) and it has the same Cauchy data on \( \Gamma_0 \); consequently

\[
(6.3) \quad w = -u_y^\varepsilon \quad \text{in} \quad \Omega \setminus D.
\]

The refraction conditions for \( w \) and \( u \) are:

\[
(6.4) \quad w^i = w^\varepsilon, \quad w_N^i = \frac{1}{2} w_N^\varepsilon, \\
\quad u^i = u^\varepsilon, \quad u_N^i = \frac{1}{2} u_N^\varepsilon \quad \text{on} \quad \partial D
\]

where \( w^\varepsilon = u\big|_{\Omega \setminus D} \), \( w^i = w\big|_D \). Let us write

\[
\frac{\partial}{\partial N} = N_x \frac{\partial}{\partial x} + N_y \frac{\partial}{\partial y}
\]

and introduce the tangential derivative

\[
\frac{\partial}{\partial N_\perp} = -N_y \frac{\partial}{\partial x} + N_x \frac{\partial}{\partial y} \quad \text{along} \quad \partial D.
\]

We shall use (6.4) in order to obtain expressions for \( w^i \) and \( w_N^i \) in terms of derivatives of \( u^i \) on \( \partial D \). First we derive from (6.4), (6.3) the relations

\[
(6.5) \quad -N_y u^i_x + N_x w^i_y = N_y u^\varepsilon_{x}\!y - N_x u_{y}\!y^\varepsilon, \\
N_x u^i_x + N_y w^i_y = -\frac{1}{2} N_x u^\varepsilon_{x}\!y - \frac{1}{2} N_y u_{y}\!y^\varepsilon, \\
-N_y u^i_y + N_x w^i_x = -N_y u^\varepsilon_x + N_x u^\varepsilon_y, \\
N_x u^i_y + N_y w^i_x = \frac{1}{2} N_x u^\varepsilon_{x}\!y + \frac{1}{2} N_y u^\varepsilon_{y}\!x.
\]

Using (6.3) and expressing \( u_y^\varepsilon \) from the third and fourth equations in (6.5), we get

\[
(6.6) \quad w^i = (N_y N_x - 2 N_x N_y) u^i_x - (N_x^2 + 2 N_y^2) u^i_y \\
= -N_x N_y u^i_x - (1 + N_x^2) u^i_y \quad \text{on} \quad \partial D.
\]

Next, the second equation in (6.4) can be written in the form

\[
(6.7) \quad w_N^i = -\frac{1}{2} N_x u_{x}\!y^\varepsilon - \frac{1}{2} N_y u_{y}\!y^\varepsilon \quad \text{on} \quad \partial D.
\]
Applying $\partial/\partial N_i$ to the third and fourth equations in (6.5), we obtain

\[
N_y^2 u_{xx}^i - N_x N_y u_{xy}^i - N_y N_x u_{xy}^i + N_x^2 u_{yy}^i = -N_y^2 u_{xx}^e - 2N_x N_y u_{xz}^e + N_x^2 u_{yy}^e
\]

and

\[
-N_x N_y u_{xx}^i + N_x^2 u_{xy}^i - N_y^2 u_{xy}^i + N_x N_y u_{yy}^i = \frac{1}{2} N_x N_y u_{xx}^e + \frac{1}{2} N_x^2 u_{xy}^e - \frac{1}{2} N_y^2 u_{xy}^e + N_x N_y u_{yy}^e
\]

or

\[
-2N_x N_y u_{xy}^i + (N_x^2 - N_y^2) u_{yy}^i = -2N_x N_y u_{xy}^e + (N_x^2 - N_y^2) u_{yy}^e
\]

(6.8)

\[
2N_x N_y u_{yy}^i + (N_x^2 - N_y^2) u_{xy}^i = N_x N_y u_{yy}^e + \frac{1}{2} (N_x^2 - N_y^2) u_{xy}^e.
\]

(6.9)

From these equations we can solve for $u_{xy}^e$ and $u_{yy}^e$:

\[
u_{xy}^e = 2(N_x^2 + N_y^2) u_{xy}^i + 2N_x N_y (N_x^2 - N_y^2) u_{yy}^i,
\]

(6.10)

\[
u_{yy}^e = 2N_x N_y (N_x^2 - N_y^2) u_{xy}^i + (1 + 4N_x^2 N_y^2) u_{yy}^i.
\]

(6.11)

Substituting these expressions into (6.7) we easily get

\[
N_x^3 u_{xy}^i = \frac{1}{2} N_y^3 u_{xy}^i - \frac{1}{2} N_y^3 N_x u_{yy}^i
\]

(6.12)

on $\partial D$.

Let $\sigma$ be an edge of $D$ parallel to the $y$-axis. Then $N_x = 1$, $N_y = 0$ on $\sigma$ and (6.6), (6.12) yields:

\[
w_i = -u_y, \quad w_N = -u_{xy} = -\frac{\partial}{\partial N} u_y^i \quad \text{on } \sigma.
\]

It follows that

\[
w_i = -u_y \quad \text{in } D.
\]

(6.13)

Now let $\tau$ be any edge of $D$ which is not parallel to the $y$-axis; then $N_y \neq 0$ along $\tau$. Using (6.13) we obtain from (6.6), (6.12),

\[
-N_x u_x^i + N_y u_y^i = 0,
\]

(6.14)
\[(6.15) \quad N_x(1 - N_x^2)u_{xx}^i + \left( \frac{1}{2} - N_x^2 \right) N_y u_{yy}^i = 0 \]

along \( \tau \). Applying \( \partial / \partial N \) to (6.14) results in

\[(6.16) \quad (N_x^2 - N_y^2)u_{xy}^i + 2N_x N_y u_{yy}^i = 0. \]

Consider (6.15), (6.16) as a linear system for the unknown variables \( u_{xy}^i, u_{yy}^i \) along \( \tau \). Since the determinant of the coefficients is equal to \(-1/2\), it follows that

\[ u_{xy}^i = u_{yy}^i = 0 \quad \text{along} \ \tau . \]

This implies that \( Vw^i = 0 \) along \( \tau \) and therefore \( w^i = \text{const. in} \ D \). It follows that also \( w^* = \text{const. and, in particular,} \quad g = u_x^i = -w^* = \text{const. on} \ \partial \Omega, \ a \ contradiction. \]

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