INTEGRAL TYPE ASYMPTOTIC CONDITIONS FOR
THE SOLVABILITY OF A PERIODIC FOURTH ORDER
BOUNDARY VALUE PROBLEM

By

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Abstract:
The fourth order periodic boundary value problem

$$\frac{d^4 u}{dx^4} + f(u(x))u'(x) + g(x,u(x)) = e(x), \quad x \in [0,2\pi],$$

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,$$

is studied when the nonlinearity $g$ satisfies a more general sign condition of integral type instead of the usual sign condition, namely, there exists a $\rho > 0$ such that $g(x,u) u \geq 0$ for a.e. $x \in [0,2\pi]$ and all $u \in \mathbb{R}$ with $|u| \geq \rho$.

1. Introduction.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, $g : [0,2\pi] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying Caratheodory's conditions and $e(x) \in L^1[0,2\pi]$. This paper is concerned with a study of the boundary value problem

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\[
-\frac{d^4u}{dx^4} + f(u(x))u'(x) + g(x,u(x)) = e(x), \ x \in [0,2\pi],
\]
\[
u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0.
\]

The author has studied (1.1) in [1], [2] where, among other conditions, it is assumed that the function \( g \) satisfy the following sign condition:

\[\text{''there exists a } \rho > 0 \text{ such that } g(x,u)u \geq 0 \text{ for a.e. } x \in [0,2\pi] \text{ and all } u \in \mathbb{R} \text{ with } |u| \geq \rho.''
\]

The purpose of this paper is to replace (1.2) with the following more general sign condition of integral type, "there exists a \( \rho > 0 \) and a \( C^3 \)-function \( m: \mathbb{R} [\rho,\rho] \rightarrow \mathbb{R} \) with \( um(u) > 0, m'(u) \leq 0, m''''(u) \geq 0 \), such that

\[
\int_0^{2\pi} g(x,u(x))m(u(x))dx \geq 0
\]

for all \( C^3 \)-real valued function \( u(x) \) on \( [0,2\pi] \), with \( u'''' \) absolutely continuous on \( [0,2\pi] \), \( u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0 \) and \( \min_{x \in [0,2\pi]} |u(x)| \geq \rho. \)" We may note that (1.2) implies (1.3) with \( m(u) = sgn \ u = u/|u| \) for \( u \in \mathbb{R} [\rho,\rho] \).

The results of this paper are motivated by some of the results of Mawhin ([3]) for the second order Lienard's equation. The statement and proof of the basic result depend on three lemmas proved in Gupta [2] for which we need the following notations. Besides using the classical spaces \( C[0,2\pi], C^k[0,2\pi], L^k[0,2\pi] \) and \( L^k\infty[0,2\pi] \) of continuous, \( k \)-times continuously differentiable, measurable real-valued functions whose \( k \)-th power of the absolute value is Lebesgue integrable or measurable functions that are essentially-bounded on \( [0,2\pi] \), we shall use the Sobolev spaces \( H^k[0,2\pi] \), \( (k = 2,3,or 4) \) defined by

\[
H^k[0,2\pi] = \{u:[0,2\pi] \rightarrow \mathbb{R} \mid u^{(j)} \text{ abs.cont.on } [0,2\pi], \ j=0,1,\ldots,k-1, \ u^{(k)} \in L^2[0,2\pi]\}.
\]
with the inner product defined by

\[(u,v)_{H^k} = \sum_{j=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} u^{(j)}(x)v^{(j)}(x)\,dx + \left(\frac{1}{2\pi} \int_0^{2\pi} u(x)dx\right) \left(\frac{1}{2\pi} \int_0^{2\pi} v(x)dx\right),\]

and the corresponding norm denoted by \[|\cdot|_{H^k}.\] We also define, for the sake of convenience, the norm in \(L^k[0,2\pi]\) by

\[|u|_{L^k} = \left(\frac{1}{2\pi} \int_0^{2\pi} |u(x)|^k \,dx\right)^{\frac{1}{k}}.\]

We also use the Sobolev-space \(W^{4,1}[0,2\pi]\) defined by

\[W^{4,1}[0,2\pi] = \{ u : [0,2\pi] \to \mathbb{R} \mid u, u', u'', u''' \text{ abs. cts on } [0,2\pi] \},\]

with norm

\[|u|_{W^{4,1}} = \sum_{j=0}^{4} \int_0^{2\pi} |u^{(j)}(x)| \,dx.\]

For \(u \in L^1[0,2\pi]\), let us write

\[\bar{u} = \frac{1}{2\pi} \int_0^{2\pi} u(x)\,dx\text{ and } \bar{u}'(x) = u(x) - \bar{u},\]

so that

\[\int_0^{2\pi} \bar{u}'(x)\,dx = 0.\]

Let \(\tilde{H}^2[0,2\pi] = \{ u \in H^2[0,2\pi] \mid \bar{u} = 0 \}.\)

The following lemmas are proved in Gupta [2].

**Lemma 1:** Let \(\Gamma \in L^1[0,2\pi]\) be such that for a.e. \(x \in [0,2\pi]\),

\[\Gamma(x) \leq 1,\]

with strict inequality holding on a subset of \([0,2\pi]\) of positive measure. Then there exists
a \delta = \delta(\Gamma) > 0 such that for all \( \bar{u} \in \tilde{H}^2[0,2\pi] \) with \( \bar{u}(0) - \bar{u}(2\pi) = \bar{u}'(0) - \bar{u}'(2\pi) = 0 \),

\[
B_1(\bar{u}) = \frac{1}{2\pi} \int_0^{2\pi} ((\bar{u}'(x))^2 - \Gamma(x)\bar{u}'(x))^2 dx \geq \delta \| \bar{u} \|_{H^2}^2.
\]

Lemma 2: Let \( \Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty \) where \( \Gamma_\infty \in L^{\infty}[0,2\pi] \), \( \Gamma_1 \in L^1[0,2\pi] \) and \( \Gamma_0 \in L^1[0,2\pi] \) is such that \( \Gamma_0(x) \leq 1 \) for a.e. \( x \in [0,2\pi] \) with strict inequality holding on a subset of \([0,2\pi]\) of positive measure. Let \( \delta(\Gamma_0) > 0 \) be given by Lemma 1. Then for every \( \bar{u} \in \tilde{H}^2[0,2\pi] \) with \( \bar{u}(0) - \bar{u}(2\pi) = \bar{u}'(0) - \bar{u}'(2\pi) = 0 \),

\[
B_1(\bar{u}) \geq (\delta(\Gamma_0) - \frac{\pi^2}{3} \| \Gamma_1 \|_{L^1} - \| \Gamma_\infty \|_{L^\infty}) \| \bar{u} \|_{H^2}.
\]

Lemma 3: Let \( \gamma \in L^1[0,2\pi] \), \( \Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty \) be as in Lemma 2 and \( \delta(\Gamma_0) \) be given by Lemma 1. Then for all measurable functions \( p(x) \) on \([0,2\pi]\) with \( \gamma \leq \bar{p} \), \( p(x) \leq \Gamma(x) \) for a.e. \( x \in [0,2\pi] \), all continuous functions \( f : \mathbb{R} \to \mathbb{R} \) and all \( u \in W^{4,1}[0,2\pi] \) with \( u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0 \), we have

\[
\frac{1}{2\pi} \int_0^{2\pi} (\bar{u} - \bar{u}(x)) \left[ -\frac{d^4u}{dx^4} + f(u(x))u'(x) + p(x)u(x) \right] dx \\
\geq \gamma \| \bar{u} \|^2 + (\delta(\Gamma_0) - \frac{\pi^2}{3} \| \Gamma_1 \|_{L^1} - \| \Gamma_\infty \|_{L^\infty}) \| \bar{u} \|_{H^2}^2
\]

2. Main Results.

Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous and let \( g : [0,2\pi] \times \mathbb{R} \to \mathbb{R} \) be a function satisfying Carathéodory's conditions, namely,

(i) for each \( u \in \mathbb{R} \), the function \( x \in [0,2\pi] \to g(x, u) \in \mathbb{R} \) is measurable on \([0,2\pi]\),

(ii) for a.e. \( x \in [0,2\pi] \), the function \( u \in \mathbb{R} \to g(x, u) \in \mathbb{R} \) is continuous on \( \mathbb{R} \), and

(iii) for each \( r > 0 \), there exists a function \( \alpha_r(x) \in L^1[0,2\pi] \) such that \( |g(x, u)| \leq \alpha_r(x) \) for a.e. \( x \in [0,2\pi] \) and all \( u \in \mathbb{R} \) with \( |u| \leq r \).
We prove the following existence theorem for the boundary value problem (1.1).

**Theorem 1:** Let \( \gamma \in L^1[0,2\pi] \) with \( \overline{\gamma} = 0 \) and let \( \Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty \) with \( \Gamma_1 \in L^1[0,2\pi], \Gamma_\infty \in L^\infty[0,2\pi], \Gamma_0 \) measurable on \([0,2\pi]\), \( \Gamma_0(x) \leq 1 \) for a.e. \( x \in [0,2\pi] \) with strict inequality holding on a subset of \([0,2\pi]\) of positive measure and

\[
\frac{\pi^2}{3} |\Gamma_1| + |\Gamma_\infty| < \delta(\Gamma_0),
\]

where \( \delta(\Gamma_0) \) is given by Lemma 1. Assume that the inequalities

\[
\gamma(x) \leq \liminf_{|u| \to \infty} u^{-1}g(x,u) \leq \limsup_{|u| \to \infty} u^{-1}g(x,u) \leq \Gamma(x),
\]

(2.1)

hold uniformly for a.e. \( x \in [0,2\pi] \).

Also assume that there exists a \( \rho > 0 \) and a \( C^2 \)-function \( m: \mathbb{R} \to \mathbb{R} \) with \( m(u) > 0, m'(u) \leq 0, m''(u) \geq 0 \), such that

\[
\int_0^{2\pi} g(x,u(x))m(u(x))dx \geq 0
\]

(2.2)

for all \( C^2 \)-real-valued functions \( u(x) \) on \([0,2\pi]\) with \( u''' \) absolutely-continuous on \([0,2\pi]\), \( u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0 \), and \( \min_{\pi \in [0,2\pi]} |u(x)| \geq \rho \).

Then for every given \( e(x) \in L^1[0,2\pi] \) with

\[
\int_0^{2\pi} e(x)m(u(x))dx \leq 0,
\]

(2.3)

for all \( C^2 \)-real-valued functions \( u(x) \) on \([0,2\pi]\) as above, the boundary-value problem (1.1) has at least one solution.

**Proof:** Let \( \eta_0 = \frac{1}{2} \delta(\Gamma_0) - \frac{\pi^2}{3} |\Gamma_1|_{L^1} - |\Gamma_\infty|_{L^\infty} > 0 \). Then, for each \( \eta, 0 < \eta \leq \eta_0 \), we can find \( r(\eta) > 0 \) such that for a.e. \( x \in [0,2\pi] \) and all \( u \) with \( |u| \geq r(\eta) \), we have
\[ \gamma(x) - \eta \leq u^{-1}g(x,u) \leq \Gamma(x) + \eta. \]

Define, \( g_\eta: [0,2\pi] \times \mathbb{R} \rightarrow \mathbb{R} \) by \( g_\eta(x,u) = \gamma_\eta(x,u)u \), where,

\[
\gamma_\eta(x,u) = \begin{cases} 
  u^{-1}g(x,u) & \text{if } |u| \geq r(\eta), \\
  [r(\eta)]^{-1}g(x,r(\eta))\left[\frac{x}{r(\eta)}\right] + (1 - \frac{x}{r(\eta)})\Gamma(x) & \text{if } 0 \leq x < r(\eta) \\
  [r(\eta)]^{-1}g(x,-r(\eta))\left[\frac{x}{r(\eta)}\right] + (1 + \frac{x}{r(\eta)})\Gamma(x) & \text{if } -r(\eta) < x < 0,
\end{cases}
\]

so that \( g_\eta \) and \( \gamma_\eta \) satisfy Caratheodory's conditions and

\[
\gamma(x) - \eta \leq \gamma(x,u) \leq \Gamma(x) + \eta,
\]

for a.e. \( x \in [0,2\pi] \) and all \( u \in \mathbb{R} \). Let us, next, define \( h_\eta: [0,2\pi] \times \mathbb{R} \rightarrow \mathbb{R} \) by

\[
h_\eta(x,u) = g(x,u) - g_\eta(x,u).
\]

Then, there exists \( \alpha_\eta(x) \in L^1[0,2\pi] \), depending only on \( \gamma, \Gamma \), and \( \alpha_r(\eta) \), such that for a.e. \( x \in [0,2\pi] \) and all \( u \in \mathbb{R} \), we have

\[
|h_\eta(x,u)| \leq \alpha_\eta(x).
\]

Now, the equation in (1.1) is equivalent to

\[
- \frac{d^4u}{dx^4} + f(u(x))u'(x) + \gamma_\eta(x,u(x))u(x) + h_\eta(x,u(x)) = e(x)
\]

We shall use the same degree arguments as the ones used in [2,3] to prove our theorem. Accordingly, it suffices to show that the set of all possible solutions of the family of equations

\[
- \frac{d^4u}{dx^4} + \lambda f(u(x))u'(x) + ((1-\lambda)(\Gamma(x)+\eta) + \lambda \gamma_\eta(x,u(x)))u(x) + \lambda h_\eta(x,u(x)) = \lambda e(x)
\]

\[
+ \lambda h_\eta(x,u(x)) - \lambda e(x) = 0,
\]

\[
u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,
\]
is, a priori, bounded in $C^1[0,2\pi]$ independently of $\lambda \in [0,1]$. Let, now, $u(x)$ be a possible solution of (2.5) for some $\lambda \in [0,1]$. Multiplying the equation in (2.5) by $(\bar{u} - \bar{u}(x))$ we obtain, on integrating the resulting equation on $[0,2\pi]$ and using (2.4) along with Lemma 3, with $\Gamma_\infty$ replaced by $\Gamma_\infty + \eta$ and $\gamma$ replaced by $\gamma - \eta$,

\[
0 = \frac{1}{2\pi} \int_0^{2\pi} \left[ \bar{u} - \bar{u}(x) \right] \left\{ -\frac{d^4 u}{dx^4} + \lambda \int (u(x))u'(x) + \left[ (1-\lambda)(\Gamma(x) + \eta) + \lambda\gamma(x,u(x)) \right] u(x) \right. \\
+ \left. \lambda h(x,u(x)) - \lambda e(x) \right\} dx \\
= \frac{1}{2\pi} \int_0^{2\pi} \left\{ (\bar{u}'(x))^2 - \left[ (1-\lambda)(\Gamma(x) + \eta) + \lambda\gamma(x,u(x)) \right] \bar{u}'(x) \right. \\
+ \left. \left[ (1-\lambda)(\Gamma(x) + \eta) + \lambda\gamma(x,u(x)) \right] \bar{u}^2 \right\} dx \\
\geq \eta_0 \left\{ \bar{u}^2 + \eta \bar{u}'^2 - (|\alpha_\eta|_{H^1} + |e|_{H^1}) (|\bar{u}|_{L^1} + |\bar{u}|_{L^\infty}) \right\} dx \\
\geq \eta_0 \left\{ \bar{u}^2 + \eta \bar{u}'^2 - \beta_\eta (|\bar{u}|_{H^2} + |\bar{u}|_{H^2}) \right\},
\]

where $\beta_\eta > 0$, is a constant depending on $\eta$ and independent of $\lambda \in [0,1]$.

We next show that there exists a $\tau \in [0,2\pi]$ such that $|u(\tau)| < \rho$. Suppose, on the other hand, that $|u(x)| \geq \rho$ for all $x \in [0,2\pi]$. Integrating the equation in (2.4) over $[0,2\pi]$ after multiplying it by $m(u(x))$, we obtain, after noticing that

\[
-\int_0^{2\pi} \frac{d^4 u}{dx^4} m(u(x)) dx = -\int_0^{2\pi} m'(u(x))(u'(x))^2 dx + \int_0^{2\pi} m'''(u(x)) \frac{(u'(x))^4}{3} dx \geq 0, \\
(1-\lambda) \int_0^{2\pi} \left( \Gamma(x) + \eta \right) u(x) m(u(x)) dx + \lambda \int_0^{2\pi} g(x,u(x)) m(u(x)) dx - \lambda \int_0^{2\pi} e(x) m(u(x)) dx \leq 0,
\]

which is impossible, because the first term is positive, the second one is non-negative and the third non-positive, in view of our assumptions. Thus $|u(\tau)| < \rho$ for some
\( \tau \in [0,2\pi] \), and if \( \xi \in [0,2\pi] \) is such that \( \overline{u} = u(\xi) \), we obtain
\[
|\overline{u}| = |u(\xi)| = |u(r) + \int_r^\xi u'(x) \, dx | \\
\leq \rho + \sqrt{2\pi} \left( \int_0^{2\pi} |u'(x)|^2 \, dx \right)^{1/2} \\
\leq \rho + 2\pi |\overline{u}|_{H^2} .
\]
(2.7)

Combining (2.6) and (2.7) with sufficiently small \( \eta > 0 \), we obtain that there is a constant \( C \), independent of \( \lambda \in [0,1] \), such that
\[
|u|_{H^2} \leq C,
\]
which implies that \( |u|_{C_{[0,2\pi]}} \leq C_1 \), for some constant \( C_1 \), independent of \( \lambda \in [0,1] \).

This completes the proof of the theorem. //

We present a few corollaries to Theorem 1 in the following.

**Corollary 1:** Let the function \( m : \mathbb{R} \to [-\rho, \rho] \to \mathbb{R} \) in Theorem 1 be given by
\[
m(u) = \text{sgn } u = \frac{u}{|u|},
\]
(2.8)
for \( u \in \mathbb{R} \), \( |u| \geq \rho \).

Then for every \( e \in L^1[0,2\pi] \) with \( \bar{e} = 0 \), the boundary value problem (1.1) has at least one solution.

**Proof:** The corollary is immediate as it is easy to see that all the assumptions of Theorem 1 remain valid. //

The following Corollary gives a necessary and sufficient condition for the boundary-value problem (1.1) to have a solution for a given \( e \in L^1[0,2\pi] \) with \( \bar{e} = 0 \).

**Corollary 2:** Assume that \( g(x,\cdot) : \mathbb{R} \to \mathbb{R} \) is non-decreasing for a.e. \( x \in [0,2\pi] \) and that
condition (2.1) hold with $\gamma$ and $\Gamma$ as in Theorem 1.

Then for any given $c \in L^1[0, 2\pi]$ with $\bar{c} = 0$, the boundary value problem (1.1) has a solution if and only if there exists a $y \in L^\infty[0, 2\pi]$ such that

$$\int_0^{2\pi} g(x, y(x))\, dx = 0. \quad (2.9)$$

**Proof:** The necessity is immediate if we take for $y$ a solution $u(x)$ and integrate the equation over $[0, 2\pi]$. For sufficiency, let $\rho = |y|_{L^\infty}$ and $m(u) = \frac{u}{|u|}$. Then, if $u \in C^3[0, 2\pi]$ with $u'''$ absolutely continuous and

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = u''(0) - u''(2\pi) = u'''(0) - u'''(2\pi) = 0,$$

is such that

$$\min_{u \in [0, 2\pi]} |u(x)| > \rho = |y|_{L^\infty} \quad (2.10)$$

we have, by monotonicity

$$\frac{u(x)}{|u(x)|} g(x, u(x)) \geq \frac{u(x)}{|u(x)|} g(x, y(x))$$

for a.e. $x \in [0, 2\pi]$, and hence, as $\frac{u(x)}{|u(x)|}$ is a constant when (2.10) holds, we get using (2.9) that

$$\int_0^{2\pi} g(x, u(x)) \frac{u(x)}{|u(x)|} \, dx \geq 0,$$

for those $u$, and the result follows from Corollary 1.//

The following Corollary replaces the assumption (2.2) by an integral type asymptotic sign condition on $g(x, u)$.

**Corollary 3:** Assume that the assumptions of Theorem 1 hold except that (2.2) is
replaced by the existence of \( m : \mathbb{R} \{0\} \rightarrow \mathbb{R} \) of class \( C^3 \) with \( u \ m(u) > 0 \), \( m'(u) \leq 0 \), \( m'''(u) \geq 0 \) and of a function \( \mu(x) \in L^1[0,2\pi] \) such that

\[
\lim_{|u| \to \infty} \inf_{x,u} g(x,u)m(u) \geq \mu(x), \quad (2.11)
\]

uniformly a.e. in \( x \in [0,2\pi] \) and

\[
\int_0^{2\pi} \mu(x)dx > 0.
\]

Then for any given \( e \in L^1[0,2\pi] \) such that there is a \( \rho > 0 \) with

\[
\int_0^{2\pi} e(x)m(u(x))dx \leq 0, \quad (2.12)
\]

for all \( u \in C^3[0,2\pi] \) with \( u''' \) absolutely-continuous on \( [0,2\pi] \), \( u(0) = u(2\pi) = u'(0) - u'(2\pi) = u'''(0) - u'''(2\pi) = 0 \) and \( \min_{x \in [0,2\pi]} |u(x)| \geq \rho \); the boundary value problem (1.1) has at least one solution.

**Proof:** It suffices to check that (2.2) of Theorem 1 holds. This is identical to the proof of Corollary 1 of [3].//

**Remark 1:** We remark that we have treated the boundary value problem (1.1) with non-zero righthand side, unlike the problem in [3] where the corresponding second order problem is treated with zero right-hand side.

**Remark 2:** We note that the generality of the integral type asymptotic sign condition (2.2) on \( g(x,u) \) imposed by the function \( m(u) \) of (2.8), slightly limits the class of \( e \in L^1[0,2\pi] \) for which (1.1) is solvable when compared to the class of \( e \) allowed by Theorem 2 of [2] under pointwise asymptotic sign condition on \( g(x,u) \).
Bibliography


2. Gupta, C.P.: Asymptotic Conditions For the Solvability of a Fourth Order Boundary Value Problem With Periodic Boundary Value Problem With Periodic Boundary Condition (Submitted)