CONSTRUCTING UNIFORM ORIENTED MATROIDS
WITHOUT THE ISOTOPY PROPERTY

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CONSTRUCTING UNIFORM ORIENTED MATROIDS WITHOUT THE ISOTOPY PROPERTY*

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Abstract. A simple procedure is given for producing a uniform rank 3 oriented matroid with disconnected realization space from a non-uniform example.

Very recently B. Jaggi and P. Mani-Levitska [4] solved the longstanding isotopy problem for simple line arrangements [7]. Using the well-known correspondence between line arrangements, order types in the projective plane [5] and realizable oriented matroids [2, 3, 6], their main theorem is stated as follows. We write \( R(M) \) for the space of all vector realizations \( (x_1, \ldots, x_n) \in (\mathbb{R}^3)^n \) of a rank 3 oriented matroid \( M \) on \( n \) points. (In other words, \( R(M) \) is the set of \( 3 \times n \)-matrices whose maximal minors have signs given by the alternating map \( M : \{1, 2, \ldots, n\}^3 \to \{-, 0, +\} \).

If \( M \) is uniform (i.e. all minors are non-zero) then \( R(M) \) is an open subset of \( \mathbb{R}^{3n} \). N. White’s earlier paper [8] gives a non-uniform oriented matroid \( M_W \) with \( R(M_W) \) disconnected and \( n = 42 \), while the new uniform oriented matroid \( M_{JM} \) of Jaggi and Mani-Levitska [4] has \( n = 17 \) and \( R(M_{JM}) \) disconnected. It is the objective of the present note to describe an easy general construction for uniform oriented matroids without the isotopy property.

A rank 3 oriented matroid \( M \) is said to be constructible if \((x_1, x_2, x_3, x_4)\) is a projective basis and the point \( x_t \) is incident to at most two lines spanned by \( \{x_1, x_2, \ldots, x_{t-1}\} \) for \( t = 5, 6, \ldots, n \). Using the configuration \( \lambda_1 = \Omega(17, 15, 13)[\lambda_0] \) in [4] or a similar modification of White’s example [8], we easily get a constructible oriented matroid whose realization space has two connected components. For example, the space \( R(\lambda_1) \) modulo the connected group \( PGL(\mathbb{R}^3) \) equals the set of matrices

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 5 & 5 & 0 & 1 & -1 \\
0 & 1 & 0 & 1 & 1 & 0 & -1 & 2 & 2 & 6 & 0 & -1 & t & -t & t \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 6 & 5 & 1 & 0 & t & -1
\end{pmatrix}
\]

with \( \frac{1}{5} < t < \frac{1}{2}(1 - \frac{1}{\sqrt{5}}) \) or \( \frac{1}{2}(1 + \frac{1}{\sqrt{5}}) < t < \frac{4}{5} \). Hence it suffices to prove the following

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**Theorem.** Let $M$ be a constructible rank 3 oriented matroid on $n$ points. Then there exists a uniform rank 3 oriented matroid $\widetilde{M}$ on at most $4(n-3)$ points and a continuous surjective map $\mathcal{R}(\widetilde{M}) \to \mathcal{R}(M)$. Hence $\mathcal{R}(\widetilde{M})$ is disconnected whenever $\mathcal{R}(M)$ is disconnected.

**Proof.** We define a sequence $M =: M_n, M_{n-1}, M_{n-2}, \ldots, M_3, M_4 =: \widetilde{M}$ of oriented matroids and maps between their realization spaces. Let $n \geq t \geq 5$. Then $M_{t-1}$ is constructed from $M_t$ as follows. First suppose that $x_i$ is incident to exactly two lines $x_i \vee x_j$ and $x_k \vee x_l$ with $1 \leq i, j, k, l < t$. Using the notation of Billera & Munson [1], we let $M'_t$ be the oriented matroid obtained from $M_t$ by the four successive principal extensions

$$(1) \quad x_{t,1} := [x_i^+, x_j^+, x_k^+], \quad x_{t,2} := [x_i^+, x_l^+, x_k^-], \quad x_{t,3} := [x_i^+, x_l^-, x_k^-], \quad x_{t,4} := [x_i^+, x_l^-, x_k^+].$$

These extensions can be carried out for every vector realization of $M_t$ by setting $x_{t,1} := x_i + \varepsilon_1 x_i + \varepsilon_2 x_k, x_{t,2} := x_l + \varepsilon_3 x_i - \varepsilon_4 x_k, x_{t,3} := x_l - \varepsilon_5 x_i - \varepsilon_6 x_k, x_{t,4} := x_l - \varepsilon_7 x_i + \varepsilon_8 x_k$ where $1 \gg \varepsilon_1 \gg \varepsilon_2 \gg \ldots \gg \varepsilon_8 > 0$. This implies that the deletion map $\Pi : \mathcal{R}(M'_t) \to \mathcal{R}(M_t)$ is surjective. Geometrically speaking, in every affine realization of $M'_t$, the intersection point $x_t$ is "caught" in the quadrangle $(x_{t,1}, x_{t,2}, x_{t,3}, x_{t,4})$. Define $M_{t-1} := M'_t \setminus x_t$ by deletion of that point, and let $\pi : \mathcal{R}(M'_t) \to \mathcal{R}(M_{t-1})$ denote the corresponding map.

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**Figure.** Illustration of the oriented matroids $M_t, M'_t$ and $M_{t-1}$. 

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Next consider an arbitrary realization $X := (x_1, \ldots, x_{t-1}, x_t, x_t^+, x_t^-, x_{t+3}, x_{t+4}, x_{t+1}, \ldots, x_n)$ of $M_{t-1}$. As a consequence of the principal extension construction used in (1), $x_i \lor x_j$ and $x_k \lor x_l$ are the only lines spanned by $\{x_1, \ldots, x_{t-1}, x_{t+1}, \ldots, x_n\}$ which intersect the quadrangle $(x_{t+1}, x_{t+2}, x_{t+3}, x_{t+4})$. For any other such line the intersection point $x_t := (x_i \lor x_j) \land (x_k \lor x_l)$ is on the same side as $x_{t+1}, \ldots, x_{t+4}$. Therefore $\sigma_t(X) := (x_1, \ldots, x_t, x_{t+1}, \ldots, x_n) \in R(M_t)$.

Hence we have a well-defined continuous map $\sigma_t : R(M_{t-1}) \to R(M_t)$, $X \mapsto \sigma_t(X)$. Moreover, $\sigma_t$ is surjective because $\Pi = \sigma_t \circ \pi$ is surjective.

It remains to define $M_{t-1}$ and $\sigma_t$ when $x_t$ is incident to less than two lines in $M_t$. If $x_t$ is on no such line, then we define $M_{t-1} := M_t$ and $\sigma_t$ as the identity map. Finally, suppose that $x_t$ is on only one line $x_i \lor x_j, 1 \leq i, j < t$. In that case we replace (1) by setting $x_{t,1} := [x_i^+, x_k^+], x_{t,2} := [x_i^+, x_i^-, x_k^-], x_{t,3} := [x_i^+, x_i^-, x_i^-]$ for some $x_k \not\in x_i \lor x_j$, and in the definition of the map $\sigma_t$ we set $x_t := (x_i \lor x_j) \land (x_k \lor x_l)$.

Iterating these constructions resolves all previous dependencies, and we obtain a uniform oriented matroid $\widetilde{M} := M_4$ on $4(n - 3)$ or fewer points. Moreover, we have a continuous surjection $\sigma := \sigma_n \circ \sigma_{n-1} \circ \ldots \circ \sigma_3$ from $R(\widetilde{M})$ onto $R(M)$.

Some remarks.

(1) Using a fairly straightforward procedure for doubling oriented matroids, one gets the following corollary: Given any integer $C$, there exists a uniform rank 3 oriented matroid $\widetilde{M}_C$ with $4(n - 3)C$ points such that $R(\widetilde{M}_C)$ has at least $2^C$ connected components.

(2) The local modification $M_t \mapsto M_{t-1}$ is quite similar to a twofold application of the $\Omega$-"opening" operation of Jaggi and Mani-Levitska which would produce precisely one of the points $x_{t,i}$. The crucial difference: the above opening operation splits each intersection point into four new points and thereby ensures the existence of a well-defined closing map $\sigma_t : R(M_{t-1}) \to R(M_t)$. For the $\Omega$-operation the desired closing map $R(\Omega(i,j,k)(M)) \to R(M)$ does not exist in general.
REFERENCES


