CRYSTAL PRECIPITATION WITH DISCRETE INITIAL DATA

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where μ_m are positive constants and

$$0 < x_1 < x_2 < \cdots < x_m < \infty$$

are given; $\delta(x)$ is the Dirac measure with unit mass at $x = 0$.

This problem was recently studied by Friedman and Ou [2] in case $n_0(x)$ is a continuous nonnegative function with compact support. The system describes a model of crystal precipitation [5]; see also [1] [3] [4]. The initial values of the form (1.6) are of particular interest for the model.

In §2 we establish a solution for (1.1)–(1.5). In §3 we prove that, as $t \to \infty$, the radii of all the crystals that have not disappeared (in finite time) converge to a critical radius ξ, which is either one of the two zeros of the transcendental equation

$$\beta\mu_N\xi^3 + c^* e^{\frac{\Gamma}{\xi}} = c_0 + \beta \sum_{m=1}^{N} \mu_m\, x_m^3 \equiv c_1. \tag{1.7}$$

It follows that for a large class of initial data,

$$c^* < \lim_{t\to\infty} c(t) < c_1 \ ;$$

this is in sharp contrast to the asymptotic behavior in case $n_0(x)$ is a continuous function where (see [2]).

$$\lim_{t\to\infty} c(t) \quad \text{is equal to either } c^* \quad \text{or} \quad c_1.$$

In §4 we analyze for a given c_1 the three sets of initial data (all having the same c_1) for which all the crystals disappear in finite time, converge to ξ_1 as $t \to \infty$ or converge to ξ_2 as $t \to \infty$.

§2. Solution of the crystal precipitation problem. In order to arrive at a natural definition of the solution to the crystal precipitation problem we approximate $n_0(x)$ by smooth initial data $n_{0j}(x)$. Let $\rho_j(x)$ be smooth functions satisfying:

$$\rho_j \geq 0, \quad \rho_j(x) = 0 \quad \text{if} \quad |x| > \frac{1}{j}\,, \int_{-\infty}^{\infty} \rho_j(x)\, dx = 1.$$

Then we take

$$n_{oj}(x) = \sum_{m=0}^{N} \mu_m \rho_j(x - x_m)\,. \tag{2.1}$$

By [2] there exists a solution $(n(x,t), c(t))$ of the crystal growth problem; we denote it by $(n_j(x,t),\ c_j(x,t))$. Then

$$c^* < c_j(t) < c_{1,j} \tag{2.2}$$

where

$$c_{1,j} = c_0 + \beta \sum_{m=1}^{N} \mu_m \int_0^\infty x^3\ \rho_j(x - x_m)\, dx \to c_1 \quad \text{as} \quad j \to \infty\ .$$

Recall the relations

$$c_j(t) = c_{1,j} - \beta \int_0^\infty x^3 n_j(x,t)\, dx\ ,$$

$$\frac{dc_j(t)}{dt} = -3\beta \int_0^\infty x^2 n_j(x,t) \hat{G}_j(x,t)\, dx \tag{2.3}$$

where

$$\hat{G}_j(x,t) = \begin{cases} k_\gamma (c(t) - c^* e^{\Gamma/x})^\gamma & \text{if} \quad x > x_j^*(t) \\ -k_\delta (c^* e^{\Gamma/x} - c(t))^\delta & \text{if} \quad x < x_j^*(t) \end{cases} \tag{2.4}$$

and

$$x_j^*(t) = \frac{\Gamma}{\log \dfrac{c_j(t)}{c}}\ . \tag{2.5}$$

Denote by $x_j(t) \equiv x_j(t;x)$ the solution of

$$\frac{dx_j}{dt} = \hat{G}_j(x,t), \qquad x_j(0) = x\ . \tag{2.6}$$

As in [2],

$$\hat{G}_j(x,t) \le C, \qquad \frac{dx_j}{dt} \le C$$

and thus $x_j(t) \le C(t+1)$ if $0 \le t \le T$. Using these estimates, we deduce from (2.3) that

$$\frac{dc_j}{dt} \ge -3\beta \int_{x_j^*(t)}^{c(T+1)} x^2 n_j(x,t) \hat{G}_j(x,t)\, dx$$

$$\ge -C \int_{x_j^*(t)}^{\infty} n_j(x,t)\, dx \ge -C\ ,$$

since $\int\limits_0^\infty n_j(x,t)\,dx \le \int\limits_0^\infty n_{0j}(x)\,dx \le C$; see [2].

It follows that the functions $c_j(t)+Ct$ are monotone increasing and uniformly bounded for $0 \le t \le T$. Hence, by Helly's theorem, we can extract a subsequence $c_j(t)$ which is pointwise convergent, say to a function $c(t)$. We conclude, by diagonalization, that for some subsequence,

$$c_j(t) \to c(t) \qquad \text{for all} \qquad t > 0\ . \tag{2.7}$$

From (2.3)-(2.7) we thus easily deduce that, for any $\epsilon > 0$,

$$x_j(t) \to x(t)$$

uniformly in bounded t-intervals as long as $x(t) \ge \epsilon$, where

$$\begin{aligned} \frac{dx}{dt} &= \hat{G}(x,t)\ , \\ x(0) &= x\ , \end{aligned} \tag{2.8}$$

and $G(x,t)$ is given by (1.3).

Suppose $x(t;x_0) \ge \epsilon$ for $0 \le t \le T$. Then $x(t;x) \ge \epsilon/2$ if $0 \le t \le T$ and $|x - x_0| \le \epsilon'$ for ϵ' small enough, and $x_j(t;x) \to x(t;x)$ uniformly in $t\epsilon[0,T], x \in [x_0 - \epsilon', x_0 + \epsilon']$. By [2; Lemma 5.1] we get

$$\int\limits_{x_j(t;x_0+\epsilon')}^{x_j(t;x_0+\epsilon')} n_j(x,t)\,dx = \int\limits_{x_0-\epsilon'}^{x_0+\epsilon'} n_j(x)\,dx = \mu_0\ .$$

The same considerations apply to each of the points x_m; further

$$\int\limits_{x(t;\bar{x})}^{x(t;\bar{\bar{x}})} n_j(x,t)\,dx = 0$$

if the interval $\{\bar{x} \le x \le \bar{\bar{x}}\}$ does contain any points x_m and j is large enough. It follows that, for any $0 < a < b < \infty$,

$$\int\limits_a^b n_j(x,t)\,dx \to \int\limits_a^b \sum_{m=1}^N \mu_m \delta(x(t;x_m) - x)\,dx$$

as $j \to \infty$. Thus, in the sense of weak convergence of measures, for any $t > 0$

$$n_j(x,t) \to n(x,t)$$

where, upon setting $x_m(t) = x(t; x_m)$,

$$n(x,t) = \sum_{m=1}^{N} \mu_m \, \delta(x_m(t) - x). \tag{2.9}$$

We also have

$$c(t) = c_1 - \beta \sum_{m=1}^{N} \mu_m x_m^3(t) \tag{2.10}$$

and, if $x^*(t)$ is defined as in (1.4),

$$\begin{aligned} \frac{dx_m}{dt} &= G(x_m, c(t)), \\ x_m(0) &= x_m \end{aligned} \tag{2.11}$$

where

$$G(x_m,\ c(t)) = \begin{cases} k_\gamma(c_1 + \sum_{j=1}^{N} \beta\mu_j x_j^3(t) - c^* e^{\frac{\Gamma}{x_m}})^\gamma & \text{if } x_m > x^*(t), \\ -k_\gamma(c^* e^{\frac{\Gamma}{x_m}} - c_1 - \sum_{j=1}^{N} \beta\mu_j x_j^3(t))^\delta & \text{if } x_m < x^*(t). \end{cases} \tag{2.12}$$

THEOREM 2.1. *There exists a unique solution of the differential system (2.11) with G defined by (2.12).*

Indeed, existence was already proved. To prove uniqueness we suppose that there is another solution with $\tilde{x}_m(t)$, $\tilde{c}(t)$. From (2.12) we then deduce that, for $0 < t < T$,

$$\begin{aligned} &|G(x_m(t), c(t)) - G(\tilde{x}_m(t), \tilde{c}(t))| \\ &\qquad \le C(\|x - \tilde{x}\|_T + \|c - \tilde{c}\|_T) \end{aligned}$$

where $x = (x_1, \dots, x_N)$, $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_N)$, $\|u\|_T = \sup_{0 \le t \le T} |u(t)|$, and from (2.10) we get $\|c - \tilde{c}\|_T \le C\|x - \tilde{x}\|_T$. Thus

$$|G(x_m(t),\ c(t)) - G(\tilde{x}_w(t), \tilde{c}(t))| \le C\|x - \tilde{x}\|_T\ ,$$

and using (2.11) uniqueness easily follows.

From the uniqueness part of Theorem 2.1 it follows that the limit function $n(x,t)$ is independent of the choice of the approximating sequence ρ_j, and that the full sequence n_j is convergent to n. this motivates:

DEFINITION 2.1. The function $n(x,t)$ defined (uniquely) by (2.9) (with (2.10)–(2.12)) is called the solution of the *crystal growth problem*.

§3. **Asymptotic behavior.** Set

$$c_1 = c_0 + \beta \sum_{m=1}^{N} \mu_m x_m^3 \,. \tag{3.1}$$

Since the function

$$\mu_m \beta \xi^3 + c^* \, e^{\Gamma/\xi}$$

is convex and tends to $+\infty$ as $\xi \to 0$ and as $\xi \to \infty$, there exist at most two positive zeros ξ_1, ξ_2 of the equation

$$\mu_m \beta \xi^3 + c^* e^{\Gamma/\xi} = c_1 \,. \tag{3.2}$$

We choose them so that $\xi_1 \le \xi_2$. We shall prove:

THEOREM 3.1. *(a) For any $m < N$, $x_m(t) \to 0$ in finite time; (b) either (i) $x_N(t) \to 0$ in finite time, or (ii) $x_N(t) \to \xi_i$ (for $i = 1$ or $i = 2$) as $t \to \infty$.*

We note that case $b(ii)$ cannot occur if (3.2) has no positive solutions.

Clearly $x_j(t) < x_{j+1}(t)$ for each j as long as $x_j(t)$ remains positive. If all the $x_m(t)$ converge to zero in finite time, then the assertions (a), (b)(i) follow. Thus it remains to consider the case where, for some $k \le N$, $x_k(t) > 0$ for all $t > 0$ whereas $x_{k-1}(t)$ converges to zero in finite time $t = t_0$. For simplicity we take $t_0 = 0$.

We first consider the case $k < N$.

LEMMA 3.2. *For any $m \ge k+1$,*

$$x_m(t) - x_{m-1}(t) \qquad \textit{is strictly increasing in } t, \tag{3.3}$$

and

$$x_N(t) \le \left(\frac{c_1}{\beta \mu_N}\right)^{1/3} \tag{3.4}$$

Proof. Since $G(x, c(t))$ is strictly monotone increasing in x,

$$\frac{dx_m}{dt} = G(x_m(t), c(t)) > G(x_{m-1}(t), \; c(t)) = \frac{d\, x_{m-1}}{dt}$$

and (3.3) follows. Next, since $c(t) > 0$,

$$\beta \mu_N \, x_N^3(t) \le \beta \sum_{m=k}^{N} \mu_m \, x_m^3(t) \le c_1$$

and (3.4) follows.

LEMMA 3.3. *There exists t_0 such that either*

$$x_N(t) > x^*(t) \qquad \text{for all} \quad t > t_0, \tag{3.5}$$

or

$$x_N(t) < x^*(t) \qquad \text{for all} \quad t > t_0. \tag{3.6}$$

Proof. It suffices to show that whenever $x = x_N(t)$ and $x = x^*(t)$ intersect at some time $t = t_1$, then $dx_N(t_1)/dt = 0$ and $dx^*(t_1)/dt < 0$. The equality is obvious. To prove the inequality we use

$$\begin{aligned} \frac{dc(t)}{dt} &= -3\beta \sum_{m=k}^{N} \mu_m \, x_m^2(t) \frac{dx_m(t)}{dt} \\ &= -3\beta \sum_{m=k}^{N} \mu_m \, x_m^2(t) G(x_m(t), c(t)) > 0 \quad \text{at } t = t_1 \end{aligned}$$

since $G(x_N(t_1), c(t_1)) = 0$ whereas $G(x_m(t_1), c(t_1)) < 0$ if $k \le m < N$. Hence, from (1.4), $dx^*(t_1)/dt < 0$.

Consider first the case (3.6). Then

$$\frac{d\, x_N}{dt} = G(x_N(t), c(t)) < 0 \tag{3.7}$$

and similarly $dx_m/dt < 0$ if $k \le m < N$. From (2.10) we then deduce that $dc/dt > 0$. Hence $c(\infty) \equiv \lim_{t\to\infty} c(t)$ exists. From (3.7) we also deduce that $x_N(\infty) \equiv \lim_{t\to\infty} x_N(t)$ exists and

$$G(x_N(\infty), c(\infty)) = 0\ . \tag{3.8}$$

From Lemma 3.2 we see that, for some $\eta_0 > 0$,

$$x_m(t) \le x_N(\infty) - \eta_0 \qquad (k \le m < N) \tag{3.9}$$

for all t large enough and therefore, by (3.8),

$$\frac{dx_m}{dt} = G(x_m(t), c(t)) < -\eta < 0 \tag{3.10}$$

for all such t's. It follows that $x_m(t)$ must converge to zero in finite time, which is a contradiction.

It remains to consider the case (3.5). In this case $dx_N/dt > 0$ for all $t > 0$, and $x_N(\infty) = \lim_{t\to\infty} x_N(t)$ exists and is finite (by (3.6)). Proceeding as in [2; §5] we can prove that $x^*(\infty) = \lim_{t\to\infty} x^*(t)$ exists and $x^*(\infty) = x_N(\infty)$. But then $c(\infty) \equiv \lim_{t-1\infty} c(t)$ exists and therefore (3.8) must hold. Proceeding to argue as in (3.9), (3.10) we again derive a contradiction.

We have thus proved that the case $k < N$ cannot occur. Thus all the $x_m(t)$ with $m < N$ must tend to zero in finite time. If $x = x_N(t)$ intersects $x = x^*(t)$ for some $t = t_0$ then $x_N(t) \equiv x_N(t_0)$ is a solution of the differential equation (2.11) with $c(t) \equiv c(t_0)$ for $t > t_0$. It follows that $\xi = \lim_{t\to\infty} x_N(t)$ satisfies (3.8) with $x^*(\infty) = \xi$; consequently ξ is a solution of (3.2), and (b)(ii) follows.

It remains to consider the case where $x = x_N(t)$ does not intersect $x = x^*(t)$. In this case either (3.5) or (3.6) hold. In both cases we can argue as before and deduce that $x_N(\infty)$ and $x^*(\infty)$ exist and are equal. In view of (3.8), their common value must satisfy (3.2), and this completes the proof of Theorem 3.1.

In §4 we shall show that the case (b)(ii) occurs for a large class of initial data $x_1, \ldots, x_N$.

COROLLARY 3.4. *In case (b)(ii) occurs,*

$$c^* < c(\infty) < c_1 \ . \tag{3.11}$$

This is in sharp contrast to the situation when $n_0(x)$ is a continuous functions; in fact, in that case, as shown in [2], $c(\infty)$ must be equal either to c_1 or to c^*.

§4. Asymptotic behavior (continued). We define

$$c_1' = \min_{x>0}\{c^* e^{\frac{\Gamma}{x}} + \beta\mu_N x^3\} \ , \tag{4.1}$$

$$c_1'' = \min_{x>0}\{c^* e^{\frac{\Gamma}{x}} + \beta(\sum_{m=1}^{N} \mu_m) x^3\} \tag{4.2}$$

and notice that $c_1'' > c_1'$ if $N = 1$.

If $c_1 > c_1'$ then the equation

$$c^* e^{\frac{\Gamma}{x}} - (c_1 - \beta\mu_N x^3) = 0 \tag{4.3}$$

has two roots which we have labeled ξ_1 and ξ_2. For $c_1 < c_1'$ this equation has no roots. If $c_1 > c_1''$ then the equation

$$c^* e^{\frac{\Gamma}{x}} - (c_1 - \beta(\sum_{m=1}^{N} \mu_m) x^3) = 0 \tag{4.4}$$

has two roots which we shall denote by η_1 and $\eta_2, \eta_1 < \eta_2$. For $c_1 < c_1''$ this equation has no roots.

As easily seen from (4.3),

$$\frac{d\xi_1}{d\mu_N} > 0, \qquad \frac{d\xi_2}{d\mu_N} < 0$$

and therefore

$$\xi_1 \le \eta_1 < \eta_2 \le \xi_2 \qquad \text{for} \qquad c_1 > c'' , \tag{4.5}$$

with strict inequalities if $N > 1$.

We define

$$G(x,c) = \begin{cases} k_\gamma (c - c^* e^{\frac{\Gamma}{x}})^\gamma & \text{if} \quad x > x^*(t) \\ -k_\delta (c^* e^{\frac{\Gamma}{x}} - c)^\delta & \text{if} \quad x < x^*(t) \end{cases} \tag{4.6}$$

where $x = x^*(t)$ is defined by $c = c^* e^{\Gamma/x}$. We would like to analyze the asymptotic behavior of the solution $(x_1(t), \dots, x_N(t))$ given initial data $(x_1, \dots, x_N)$ in the set

$$S = \left\{ (x_1, \dots, x_N);\ \beta \sum_{m=1}^{N} \ \mu_m \ x_m^3 = c_1 - c_0 \right\} ;$$

each such data yields the same concentration c_1, and thus, the roots $\xi_1, \xi_2, \eta_1, \eta_2$ are independent of the particular point in S.

Introduce

$$S_0 = \{(x_1, \dots, x_N) \in S;\ x_N(t) \to 0 \quad \text{in finite time} \},$$
$$S_j = \{(x_1, \dots, x_N) \in S;\ x_N(t) \to \xi_j \quad \text{if} \quad t \to \infty\}$$

for $j = 1, 2$. Then, by Theorem 3.1, $S = S_0 \cup S_1 \cup S_2$.

Consider first the case $N = 1$ and assume that $c_1 > c_1'$. Then

$$G(x_N, c) < 0 \quad \text{if} \quad x_N < \xi_1 \quad \text{or} \quad x_N > \xi_2$$

and

$$G(x_N, c) > 0 \quad \text{if} \quad \xi_1 < x_N < \xi_2.$$

Hence $x_N(t) \to 0$ in finite time if $x_N(0) < \xi_1, x_N(t) \to \xi_2$ if $x_N(0) > \xi_1$, and $x_N(t) \equiv \xi_1$ if $x_N(0) = \xi_1$.

For $N > 1$ the situation is much more difficult to analyze. We shall give some partial results.

If $c_1 < c_1'$ then Theorem 3.1 implies that $S = S_0$. We also have, for fixed $c_0 > 0$:

$$S = S_0 \quad \text{if} \quad c_1 - c_0 \quad \text{is sufficiently small.} \tag{4.7}$$

Indeed, $x_N(0)$ is then sufficiently small and, as easily seen, $x_N(t) \downarrow 0$ in finite time.

THEOREM 4.1. *If $c_1 > c_1''$ and*

$$(4.8) \qquad c_1 - c_0 < \beta(\sum_{m=1}^{N} \mu_m)\eta_1^3$$

then $S = S_2$.

Proof. Set

$$(4.9) \qquad \overline{c} = c_1 - \beta(\sum_{m=1}^{N} \mu_m)\eta_1^3 \quad ; \quad \overline{c} < c_0.$$

Since $c_1 - c_0 = \beta\Sigma\mu_m\, x_m^3(0)$, it follows from (4.8) that $x_N(0) > \eta_1$. We claim that

$$(4.10) \qquad x_N(t) > \eta_1 \quad \text{for all} \quad t.$$

Indeed otherwise consider the smallest t such that $x_N(t) = \eta_1$; then $x_N'(t) \leq 0$. Also

$$c(t) = c_1 - \beta\sum_{m=1}^{N} \mu_m\, x_m^3(t) > \overline{c} \quad \text{by (4.9)}$$

(some of the $x_m(t)$ may vanish). Since $G(\eta_1, \overline{c}) = 0$. (by the definitions of η_1 and $\overline{c}$) and G is strictly monotone increasing in c,

$$0 < G(\eta_1, c(t)) = G(x_N(t), c(t))$$

and thus $dx_N(t)/dt > 0$, a contradiction.

The above proof of (4.10) shows also that $x_N(t) \geq \eta_1 + \epsilon$ for some sufficiently small $\epsilon > 0$. It follows that $\lim_{t\to\infty} x_N(t) > \eta_1$ and, in view of (4.5) and Theorem 3.1, $\lim x_N(t) = \xi_2$, i.e., $(x_1, \ldots, x_N) \in S_2$ for any $(x_1, \ldots, x_N)$ in S.

THEOREM 4.2. *The sets S_0 and S_2 are open subsets of S.*

It follows that S_1 is closed in S; further if S_0 and S_2 are both nonempty then any continuous curve in S which connects a point in S_0 to a point in S_2 must intersect S_1. Thus S_1 "separates" S_0 from S_2.

Proof. To prove that S_0 is open, let $(x_1^0, \ldots, x_N^0) \in S_0$ and let $(\tilde{x}_1, \ldots, \tilde{x}_N) \in S$ with $\Sigma|x_i^0 - \tilde{x}_i| < \delta$, δ small. Denote the corresponding solutions by $(x_1^0(t), \ldots, x_N^0(t))$ and $(\tilde{x}_i(t), \ldots, \tilde{x}_N(t))$. Then, for some $t_0 > 0, x_N^0(t) > 0$ for $0 \leq t < t_0$ and $x_N(t_0) = 0$. Further, if $\tilde{x}_N(t) > 0$ for $t \leq t_0$, the

$$\tilde{x}_N(t_0) < \sigma(\delta)$$

where $\sigma(\delta) \to 0$ if $\delta \to 0$ (by continuous dependence of solutions on the initial data). If $\sigma(\delta)$ is small enough then

$$d\tilde{x}_N(t)/dt < 0 \quad for \quad t_0 < t < t_1 \ ,$$

for some t_1, and $\tilde{x}_N(t_1) = 0$. Thus $(\tilde{x}_1, \ldots, \tilde{x}_N) \in S_0$ and, consequently, S_0 is open.

We next prove that S_2 is open. Let $(x_1^0, \ldots, x_N^0) \in S_2$ and $\tilde{x}_1, \ldots, \tilde{x}_N) \in S$ with $\Sigma|x_i^0 - \tilde{x}_i| < \delta$, δ small. Choose T large enough such that

$$x_{n-1}^0(t) = 0, \quad \tilde{x}_{n-1}(t) = 0 \quad \text{if} \quad t \geq T \ .$$

Denote the concentrations $c(t)$ corresponding the solutions $(x_1^0(t), \ldots, x_N^0(t))$ and $(\tilde{x}_1(t), \ldots, \tilde{x}_N(t))$, respectively, by $c_0(t)$ and $\tilde{c}(t)$. Consider the functions

$$I^0(t) = c^0(t) - c^* e^{\Gamma/x_N^0(t)} = (c_0 + \Sigma\beta\mu_m(x_m^0)^3) - \mu\beta(x_N^0(t))^3 - c^* e^{\Gamma/x_N^0(t)}$$
$$\tilde{I}(t) = \tilde{c}(t) - c^* e^{\Gamma/\tilde{x}_N(t)} = (c_0 + \Sigma\beta\mu_m \tilde{x}_m^3) - \mu\beta(\tilde{x}_N(t))^3 - c^* e^{\Gamma/\tilde{x}_N(t)} \ .$$

Since $\lim x_N(t) = \xi_2$, we may assume that

$$\frac{\xi_1 + \xi_2}{2} < x_N(t) \qquad \forall\, t > T \ ;$$

hence $I^0(t) > 0$ and $dx_N^0(t)/dt > 0$ for $t > T$ (as in case $N = 1$) as long as $x_N^0(t) < \xi_2$. Similarly, as long as

$$\frac{\xi_1 + \xi_2}{2} < \tilde{x}_N(t) < \xi_2$$

we have $\tilde{I}(t) > 0$ and $d\tilde{x}_N(t)/dt > 0$. We can choose δ small enough so that $\tilde{x}_N(T) > \dfrac{\xi_1 + \xi_2}{2}$ (by continuous dependence of solutions on the initial data) and, then, by the previous observation, $\tilde{x}_N(t)$ cannot decrease to $(\xi_1 + \xi_2)/2$ for any $t > T$. It follows, by Theorem 3.1, that $\lim \tilde{x}_N(t) = \xi_2$ and thus $(\tilde{x}_1, \ldots, \tilde{x}_N) \in S_2$.

THEOREM 4.3. *If $S_1 \neq \phi$ then $S_1 \cap \overline{S}_0 \neq \phi$ and $S_1 \cap \overline{S}_2 \neq 0$; consequently int $S_1 = \phi$.*

Proof. Let $(x_1^0, \ldots, x_N^0) \in S_1$ and set

$$B_\delta = \{(\tilde{x}_1, \ldots, \tilde{x}_N) \in S \ ; \ \Sigma|\tilde{x}_i - x_i^0|^2 < \delta^2\} \ , \ \delta > 0.$$

Choose T such that $x_{N-1}^0(T) = 0$. For any $\epsilon > 0$ there is $\delta > 0$ such that for each solution with initial data in B_δ

$$\tilde{x}_{N-1}(t) = 0 \quad \text{for all} \quad t \geq T + \epsilon \ .$$

We have, by assumption,

$$x_N^0(t) \to \xi_1 \quad \text{as} \quad t \to \infty . \tag{4.11}$$

We claim that actually

$$x_N^0(t) = \xi_1 \quad \text{for all} \quad t \geq T . \tag{4.12}$$

Indeed, if $x_N^0(T_0) > \xi_1$ for some $T_0 > T$ then, as in the proof of Theorem 4.2,

$$I^0(t) > 0 \quad \text{and} \quad \frac{d\, x_N^0(t)}{dt} > 0 \quad \text{for} \quad t > T_0 ,$$

and thus (4.11) cannot occur. Similarly $x_N^0(T_0) < \xi_1$ implies

$$I^0(t) < 0 \quad \text{and} \quad \frac{d\, x_N^0(t)}{dt} < 0 \quad \text{if} \quad t \geq T_0$$

and (4.11) cannot occur.

From the diffeomorphism

$$(\tilde{x}_1, \ldots, \tilde{x}_N) \to (\tilde{x}_1(t), \ldots, \tilde{x}_N(t))$$

(for t fixed) it follows that for any $0 < \delta' < \delta$ there exists a point $(\tilde{x}_1, \ldots, \tilde{x}_N)$ in B_δ, with

$$\xi_1 < \tilde{x}_N(T + \epsilon) < \xi_2 .$$

Arguing as above we find that

$$\tilde{x}_N(t) \geq \tilde{x}_N(T + \epsilon) > \xi_1 \qquad \text{for all} \quad t \in T + \epsilon ;$$

consequently $(\tilde{x}_1, \ldots, \tilde{x}_N) \in S_2$. Thus $\overline{S}_2 \cap S_1 \neq \phi$. Similarly one can prove that $\overline{S}_1 \cap S_1 \neq \phi$.

From the above proof we get:

COROLLARY 4.4. *If $(x_1, \ldots, x_N) \in S_1$ then $x_N(t) \to \xi_1$ is finite time.*

Set

$$c(x) = c_1 - \beta \sum_{m=1}^{N} \mu_m x_m^3 ,$$
$$\widetilde{S} = \{(x_1, \ldots, x_N); x_j > 0 \quad \forall j, \; c_* < c(x) \leq c_1\}$$

and consider the dynamical system

$$\frac{dx_j}{dt} = G(x_j, c(x)) \qquad (x_1 \le x_2 \le \cdots \le x_N) \tag{4.13}$$

with initial values $(x_1(0), \ldots, x_N(0))$ in $\widetilde{S}$. Then the flow remains in $\widetilde{S}$ (since $G(x_j, c^*) > 0$). The roots ξ_i, η_i can be defined as before (they depend only on c_1), and we also define:

$$\widetilde{S}_0 =: \quad \{\text{all } (x_1, \ldots, x_N) \in \widetilde{S} \quad \text{such that } x_N(t) \to 0 \quad \text{in finite time} ,$$
$$\widetilde{S}_j =: \quad \{\text{all } (x_1, \ldots, x_N) \in \widetilde{S} \quad \text{such that } x_N(t) \to \xi_i \quad \text{as } t \to \infty\}.$$

For each $c_0 \in (0, c_1 - c^*)$ the sets

$$S^{c_0} = \widetilde{S} \cap \{c(x) = c_1 - c_0\}, \qquad S_j^{c_0} = \widetilde{S}_j \cap \{c(x) = c_1 - c_0\}$$

coincide with the sets S, S_j defined above, for the same c_0. The previous analysis can easily be extended to the present sets; in particular:

(4.14) $\quad \widetilde{S}_0$ and $\widetilde{S}_2$ are open subsets of $\widetilde{S}$,

(4.15) $\quad$ int $\widetilde{S}_1 \neq \phi$ if $\widetilde{S}_1 \neq \phi$.

$$\widetilde{S}_0 = [0, -\xi_1),\ \widetilde{S}_1 = \{\xi_1\},\ S_2 = \left(\xi_1, \left(\frac{c_1 - c^*}{\beta \mu_N}\right)^{1/3}\right] \quad \text{if} \quad N = 1 . \tag{4.16}$$

The system (4.13) is autonomous and this suggests that some dynamical system approach may be useful for analyzing the set $\widetilde{S}_1$. It can be shown that $G(c(x), x_j) < 0$ along the set $\widetilde{S}_1$, for all j.

Phase diagrams of the dynamical system is given below for $N = 2$:

$$(a) \quad c_1 < c_1' , \qquad (b) \quad c_1' < c_1 < c_1'' , \qquad (c) \quad c_1'' < c_1 .$$

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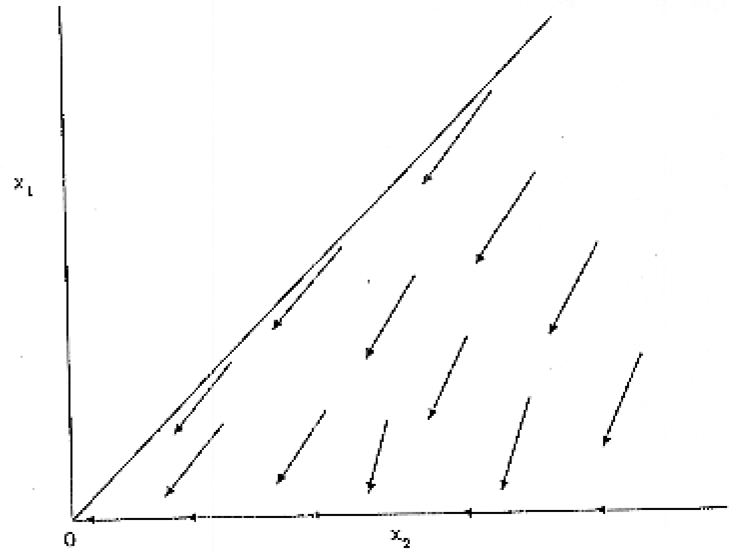
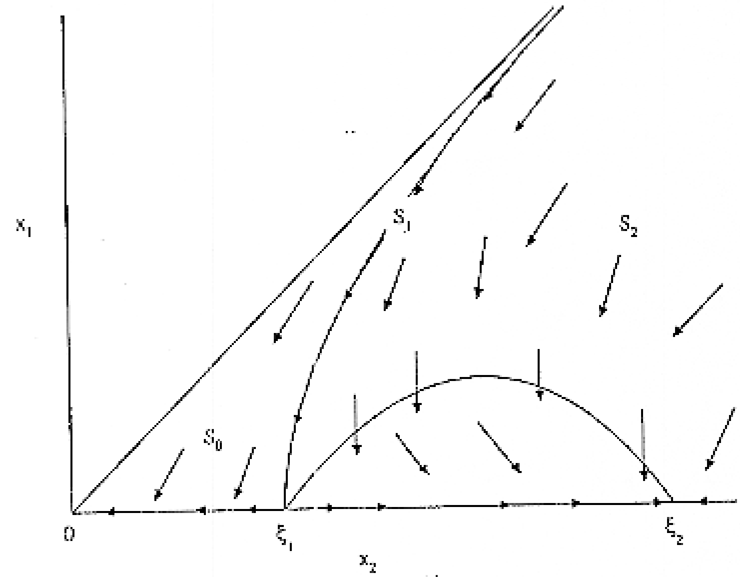
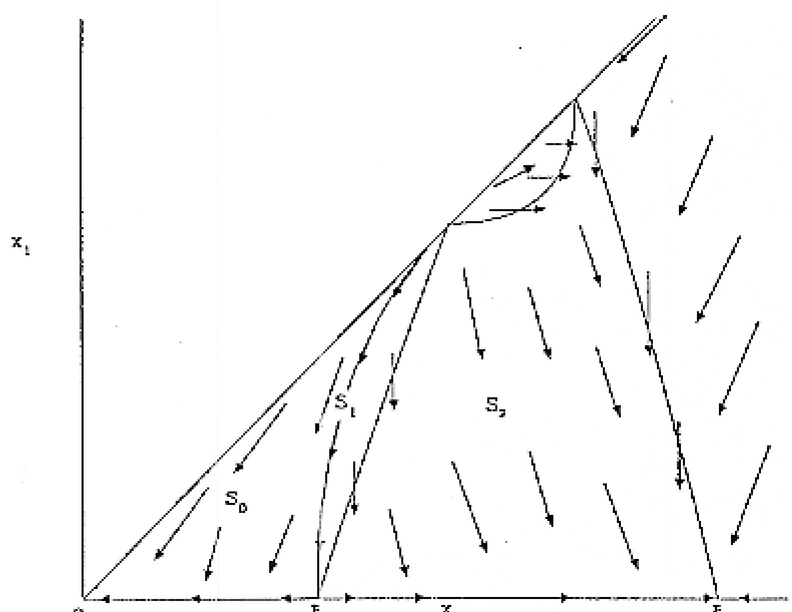


(a)



(b)



(C)