

**THE RUNGE-KUTTA LOCAL PROJECTION
 P^1 -DISCONTINUOUS-GALERKIN FINITE ELEMENT
METHOD FOR SCALAR CONSERVATION LAWS**

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Bernardo Cockburn

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Chi-Wang Shu

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**THE RUNGE-KUTTA LOCAL PROJECTION
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BERNARDO COCKBURN[†] AND CHI-WANG SHU[‡]

Abstract. In this work we introduce and analyze a new explicit method for solving numerically scalar conservation laws. The time-discretization of the method is based on a second order accurate TVD Runge-Kutta technique (used recently by Osher and Shu [12] to solve scalar conservation laws in the framework of finite difference schemes), while the space-discretization is based on a discontinuous finite element method for which the approximate solution is taken to be piecewise linear in space (i.e., the local projection P^0P^1 -discontinuous Galerkin method introduced recently by Chavent and Cockburn [3]). The resulting scheme is TVBM (total variation bounded in the means), converges to a weak solution, and is formally second order accurate in time and space for $cfl \in [0, 1/3]$. We give extensive numerical evidence that the scheme does converge to the entropy solution, and that the order of convergence away from singularities is optimal; i.e., equal to 2 in the norm of $L^\infty(L^\infty_{loc})$.

Key words. TVD, TVB, Runge-Kutta, discontinuous finite elements, conservation law

AMS(MOS) subject classifications. 65M60, 65 N30, 35L65

1. Introduction. In this work we introduce and analyze a new explicit method, called the Runge-Kutta Local-Projection Discontinuous-Galerkin P^1 ($RK\Lambda\Pi P^1-$) method, for solving numerically the scalar conservation law:

$$(1.1) \quad \begin{aligned} \partial_t u + \partial_x f(u) &= 0, & \text{in } (0, T) \times \mathcal{C}, \\ u(t = 0) &= u_0, & \text{on } \mathcal{C}, \end{aligned}$$

where \mathcal{C} is a circle, the nonlinear function f is C^1 , and the initial data u_0 belongs to the space $BV(\mathcal{C})$, the space of real-valued functions with finite total variation on \mathcal{C} . We have chosen to work in a periodic setting in order to avoid the problem of the numerical treatment of the boundary conditions. The case of non-periodic boundary conditions will be considered elsewhere.

The $RK\Lambda\Pi P^1-$ method is an explicit conservative scheme that displays a convenient local maximum principle and has the property of being total variation bounded in the means (TVBM). At the same time, it is linearly stable under the mild condition $cfl \leq 1/3$, and formally second order accurate in the $L^\infty(0, T; L^\infty_{loc})$ -norm even in the presence of extrema and sonic points. We prove the convergence in $L^\infty(0, T; L^1(\mathcal{C}))$ of a subsequence of the sequence of approximate solutions generated by the method to a weak solution of (1.1).

[†]IMA, University of Minnesota, 514 Vincent Hall, Minneapolis, Minnesota 55455. Present address: School of Mathematics, University of Minnesota, 127 Vincent Hall, Minneapolis, Minnesota 55455.

[‡]IMA, University of Minnesota, 514 Vincent Hall, Minneapolis, Minnesota 55455. Present address: Division of Applied Mathematics, Brown University, Providence, RI 02912.

We also give extensive numerical evidence that the scheme does converge to the entropy solution– the only weak solution with physically relevant meaning– and that the order of convergence away from singularities is optimal; i.e., equal to 2 in the $L^\infty(0, T; L_{loc}^\infty)$ -norm.

Historically, the $RK\Lambda\Pi P^1$ – method has been originated from two sources. On the one hand, recently, Chavent and Cockburn [3] introduced and analyzed what they called the $\Lambda\Pi P^0 P^1$ – scheme for scalar conservation laws. In this finite element method the approximate solution is piecewise-constant in time and piecewise-linear in space. It is determined by using the weak formulation of the explicit Discontinuous Galerkin Method, introduced by Chavent and Salzano in [4], and a local projection based on the monotonicity-preserving local projections introduced by Van Leer [10]. This local projection, the $\Lambda\Pi_h$ – projection, guarantees a local maximum principle and the TVDM (total variation diminishing in the means) property. It was proven that these properties, together with the conservativity of the scheme, ensure the convergence of a subsequence to a weak solution, and extensive numerical evidence showing that the scheme converges to the entropy solution at a rate of $O(h)$ in the norm of $L^\infty(0, T; L^1(\mathcal{C}))$ (even in the presence of discontinuities) was given. Unfortunately, any attempt to obtain higher order accurate extensions of the $\Lambda\Pi P^0 P^1$ – scheme staying in the framework of this finite element technique leads naturally to implicit schemes. And it is very well known that implicit schemes do not perform as well as explicit ones for this kind of problems (on this respect we want to bring the reader’s attention to the explicit-implicit version of the PPM-scheme; see [7]).

On the other hand, also very recently, Shu and Osher [12] showed how to use efficiently Runge-Kutta techniques in order to obtain total variation diminishing (TVD) high order accurate time-discretizations of scalar conservation laws. They use the method of lines; i.e., first, they discretize the equation in space by using a finite difference non-oscillatory technique. The equation satisfied by the approximate solution can be then written in ODE form: $\frac{d}{dt} u_h = L_h(u_h)$. Then, the latter equation is discretized in time by using a suitably chosen Runge-Kutta technique. This is done in such a way that the local truncation error of the resulting scheme is formally $O(\Delta^r t + h^r)$ whenever $L_h(u)$ approximates $-\partial_x f(u)$ with an $O(h^r)$ error; see [12]. These Runge-Kutta techniques are essentially ODE-discretization techniques which do not increase the total variation of the spatial part, and so are absolutely independent of the type of discretization used in space as well as of the dimension of the space variables. As long as the spatial dimension is bigger than one, finite difference approximations of L_h are difficult to obtain when the domain has a complicated geometry. However, this is not the case if a finite element discretization technique is used! It is then very natural to combine the above mentioned Runge-Kutta technique with the finite element space discretization of the $P^0 P^1$ – scheme. The $RK\Lambda\Pi P^1$ –scheme is a realization of this idea.

The $RK\Lambda\Pi P^1$ – method is obtained in three steps:

- (1) the conservation law is discretized in space by using the explicit discontinuous-Galerkin finite element used by Chavent and Salzano [4]. An ODE of the form

$\frac{d}{dt} u_h = L_h(u_h)$ is thus obtained;

- (2) the latter ODE is discretized in time by using a suitable chosen TVD Runge-Kutta technique as indicated by Shu and Osher [12];
- (3) a TVBM extension of the TVDM $\Lambda\Pi_h$ – projection introduced by Chavent and Cockburn [3] is then used adequately in order to render the scheme stable and the sequence of approximate solutions compact in $L^\infty(0, T; L^1(\mathcal{C}))$, without compromising the accuracy of the method.

Higher order accurate versions of this scheme, as well as extensions to the nonperiodic case will be considered in a forthcoming paper. Also, n-dimensional extensions of it will be developed elsewhere.

An outline of the paper follows. In Section 2 the discretization of the scalar conservation law in space is obtained. In Section 3 the time discretization is constructed. In Section 4 the $\Lambda\Pi_h$ – projection is studied. In Section 5 the $RK\Lambda\Pi P^1$ – method is defined and its stability, formal accuracy and convergence properties are proven. Finally, in Section 6 our numerical results are shown.

2. The space-discretization.

2.1 Preliminaires. First, let us introduce some notations. As usual, the set $\{x_{i+1/2}\}_{i=0,\dots,nx-1}$ is a partition of the circle \mathcal{C} . For commodity, we define $x_{j+1/2}$ to be $x_{i+1/2}$ if $j = i \bmod(nx)$, in this way we can write for example $x_{-1/2} = x_{nx-1/2}$, $x_{nx+1/2} = x_{1/2}$, $x_{nx+3/2} = x_{3/2}$. We define $\Delta x_i = x_{i+1/2} - x_{i-1/2}$, denote by I_i the interval $(x_{i-1/2}, x_{i+1/2})$, and set $h = \sup_i \{\Delta x_i\}$. By $I(a_1, \dots, a_m)$ we shall denote the closed interval $[\min\{a_1, \dots, a_m\}, \max\{a_1, \dots, a_m\}]$.

The weak formulation on which the discretization in space of the explicit Discontinuous Galerkin method is based, see [3,4], on the definition of the Godunov numerical flux, and on a formal integration by parts formula that we next recall.

The Godunov flux associated to the function h , h^G , is a consistent two-point numerical flux – $h^G(w, w) = h(w)$ – defined to be

$$(2.1) \quad \begin{aligned} h^G(w, v) &= h(\xi), \quad \text{with } \xi \in I(w, v) : \\ (h(\xi) - h(c)) \cdot \text{sign}(w - v) &\leq 0, \quad \forall c \in I(w, v). \end{aligned}$$

See Osher [9], and Brenier and Osher [1] for further details.

The equality:

$$(2.2) \quad \begin{aligned} &\int_{I_i} \partial_t u \cdot \varphi \\ &- \int_{I_i} f(u) \cdot \partial_x \varphi + \int_{\partial I_i} f(u) \cdot \varphi \cdot n_x = 0, \quad \forall \varphi \in C^1(I_i), \end{aligned}$$

where n_x is the outward unit normal to ∂I_i , is obtained by simply multiplying (1.1) by φ , integrating over the domain I_i , and formally integrating by parts. In order to compute numerically integrals over I_i we shall use a quadrature rule:

$$(2.3) \quad \int_{I_i} \psi \sim \sum_{l=1}^L \omega_l \psi(x_{i,l}) \Delta x_i,$$

where $x_{i,l} = x_i + \theta_l \Delta x_i$, $l = 1, \dots, L$.

2.2 Space discretization by the discontinuous Galerkin method.

We proceed in two steps. First, we introduce the finite dimensional space V_h . A function u_h is said to belong to $V_h(\mathcal{C})$ if $u_h \in BV(\mathcal{C}) \cup L^1(\mathcal{C})$ and:

(2.4a) In each element $I_i \subset \mathcal{C}$, u_h is linear: $u_h|_{I_i} \in P^1(I_i)$; i.e.,

$$u_h(x) = \bar{u}_i^n \varphi^0(s) + \tilde{u}_i^n \varphi^1(s), \quad x \in I_i,$$

where $\varphi^0(s) = 1$, and $\varphi^1(s) = 2s$, $\forall s = (x - x_i)/\Delta x_i \in (-\frac{1}{2}, \frac{1}{2})$.

(2.4b) The *trace* of u_h in each ∂I_i is chosen as follows:

$$\begin{aligned} u_h(x_{i+1/2}) &= \xi_{h,i+1/2}, \\ f(\xi_{h,i+1/2}) &= f^G(u_h(x_{i+1/2} + 0), u_h(x_{i+1/2} - 0)), \end{aligned}$$

where f^G denotes the Godunov flux (2.1) associated to the function f .

Now, set $\mathbf{V}_h(\mathcal{C}) = \{v_h : [0, T] \times \mathcal{C} \rightarrow \mathbb{R} \mid v_h(t) \in V_h(\mathcal{C}), \forall t \in [0, T]\}$. The approximate solution, u_h , will be taken in the space $\mathbf{V}_h(\mathcal{C})$, and will be determined as the unique solution of:

$$\begin{aligned} \forall t \in (0, T), \quad \forall \varphi \in P^1(I_i) : \\ \int_{I_i} \partial_t u_h \cdot \varphi \\ - \sum_{l=1}^L \omega_l \cdot f(u_h((x_{i,l})) \cdot \partial_x \varphi(x_{i,l}) \Delta x_i, \quad + \int_{\partial I_i} f(\xi_h) \cdot \varphi \cdot n_x = 0, \end{aligned}$$

satisfying the initial condition $u_h(t = 0) = \mathbf{P}_h(u_h)$, where \mathbf{P}_h is the L^2 - projection into the space $V_h(\mathcal{C})$.

Roughly speaking, by using the variational formulation based on (2.2) we force the approximate solution u_h to be an approximation to a weak solution of (1.1); and by using the definition of its trace as in (2.4b) we are forcing u_h to behave like the entropy solution of (1.1).

We can easily rewrite this as the following initial value problem:

$$(2.5a) \quad \begin{aligned} \frac{d}{dt}u_h(t) &= L_h(u_h(t)), & \text{in } (0, T), \\ u_h(t=0) &= u_{o,h} \end{aligned}$$

where the operator L_h is given by

$$(2.5b) \quad \begin{aligned} L_h(u) : BV(\mathcal{C}) \cup L^1(\mathcal{C}) &\longrightarrow V_h(\mathcal{C}), \\ u &\longmapsto w_h, \end{aligned}$$

and the degrees of freedom of w_h are given by

$$(2.5c) \quad \begin{aligned} \bar{w}_i &= - (f^G(u(x_{i+1/2}^+), u(x_{i+1/2}^-)) - f^G(u(x_{i-1/2}^+), u(x_{i-1/2}^-)))/\Delta x_i, \\ \tilde{w}_i &= - 3(f^G(u(x_{i+1/2}^+), u(x_{i+1/2}^-)) + f^G(u(x_{i-1/2}^+), u(x_{i-1/2}^-))) \\ &\quad + 2\left\{ \sum_{l=1}^L \omega_l \cdot f(u(x_{i,l})) \right\} / \Delta x_i. \end{aligned}$$

Compare this with equations (2.7) in [3].

2.3 The accuracy of the space discretization and the effect of the numerical integration.

The operator $L_h(\cdot)$ is the discretization by the discontinuous Galerkin method under consideration of the nonlinear operator $-\partial_x f(\cdot)$. In fact, if the function u is continuous – and the quadrature rule (2.3) used in (2.5c) is exact – then from (2.5) it is easy to see that $L_h(u)$ is nothing but $\mathbf{P}_h(-\partial_x f(u))$ – the L^2 -projection of $-\partial_x f(u)$ on the finite element space $V_h(\mathcal{C})$. In the following result we make precise this assertion.

PROPOSITION 2.1. *Assume that u is an element of $W^{3,\infty}(\mathcal{C})$, that f' belongs to $W^{2,\infty}(C(u))$, where $C(u)$ denotes the range of u , and that the quadrature rule (2.3) is exact for polynomials of degree two. Then, there is a constant C such that*

$$\| -\partial_x f(u) - L_h(u) \|_{L^\infty(\mathcal{C})} \leq Ch^2 \| \partial_x^3 f(u) \|_{L^\infty(\mathcal{C})}.$$

Proof. Set $E_i(\psi) = \int_{I_i} \psi - \sum_{l=1}^L \omega_l \psi(x_{i,l}) \Delta x_i$. For $x \in I_i$ we have

$$-\partial_x f(u(x)) - L_h(u)(x) = e_1(x) + e_2(x),$$

where

$$\begin{aligned} e_1(x) &= -\partial_x f(u(x)) - \mathbf{P}_h(-\partial_x f(u))(x), \\ e_2(x) &= \mathbf{P}_h(-\partial_x f(u))(x) - L_h(u)(x) \\ &= -12(x - x_i) E_i(f(u)) / (\Delta x_i)^3, \text{ by the definition of } L_h \text{ (2.5)}. \end{aligned}$$

Now, using the theory of interpolation, see for example Ciarlet [5], we obtain easily

$$\begin{aligned}\|e_1\|_{L^\infty(I_i)} &\leq C(\Delta x_i)^2 \|\partial_x^3 f(u)\|_{L^\infty(I_i)}, \\ \|e_2\|_{L^\infty(I_i)} &\leq 6(\Delta x_i)^{-2} \|E(f(u))\|_{L^\infty(I_i)} \\ &\leq C(\Delta x_i)^k \|\partial_x^{k+1} f(u)\|_{L^\infty(I_i)},\end{aligned}$$

assuming that the quadrature rule (2.3) is exact for polynomials of degree k . By hypothesis we can take $k = 2$. This proves the result.

Note that if the quadrature formula (2.3) is exact only for polynomials of degree one $L_h(u)$ becomes only a first order accurate approximation of $-\partial_x f(u)$, unless, of course, if $f' \equiv \text{constant}$.

3. The time-discretization.

3.1 The Runge-Kutta technique. Let $\{t^n\}_{n=1,\dots,nt}$ be a partition of $[0, T]$, set $\Delta t^n = t^{n+1} - t^n$, and let the *cfl*-number be defined by

$$cfl = \sup_{i \in \mathbf{Z}; n=1,\dots,N} \frac{\Delta t^n}{\Delta x_i} \cdot \|f'\|_{L^\infty(C(u_0))}$$

where $C(u_0)$ denotes the convex hull of the range of the initial data u_0 .

Let us introduce the operator H_h^n :

$$(3.1) \quad \begin{aligned} H_h^n : BV(\mathcal{C}) \cap L^1(\mathcal{C}) &\longrightarrow V_h(\mathcal{C}), \\ u &\longmapsto \mathbf{P}_h(u) + \Delta t^n \cdot L_h(u). \end{aligned}$$

We now discretize the ODE (2.3) as indicated by Shu and Osher in [12]:

$$(3.2a) \quad \text{Set } u_h(t=0) = \mathbf{P}_h(u_o);$$

$$(3.2b) \quad \text{For } n = 0, \dots, nt - 1 \text{ obtain } u_h(t^{n+1}) \text{ from } u_h(t^n) \text{ as follows:}$$

$$(1) \quad \text{Compute } w_h(t^{n+1}) = H_h^n(u_h(t^n));$$

$$(2) \quad \text{Set } u_h(t^{n+1}) = \frac{1}{2}u_h(t^n) + \frac{1}{2}H_h^n(w_h(t^{n+1}));$$

Note that the time step sizes are not necessarily equal. Note also that at each time step, $u_h(t^{n+1})$ is obtained from $u_h(t^n)$ by simply applying twice the operator H_h^n . This makes the algorithm very easy to code.

As it is very well known, this time discretization of the ODE (2.3) is formally second order accurate. This fact, together with Proposition 2.1, allows us to say that the method (2.3) is formally second order accurate.

3.2 The linear stability. In this Subsection we assume that $\Delta x_i \equiv h$, that $\Delta t^n \equiv \Delta t$, and that $f'(u) = \text{constant}$. In this case the operator H_h^n , in this Subsection written simply H_h , is linear. The L^2 -stability of the iterative procedure defined by (3.2), $u_h(t^{n+1}) = \frac{1}{2}[Id + H_h \circ H_h](u_h(t^n))$, is a necessary condition for its stability. Indeed, if the method is L^2 -unstable it is then automatically L^p -unstable, $\forall p \in [1, \infty]$. The reciprocal of this statement is not necessarily true; see Geveci [6]. In what follows we display the necessary and sufficient condition under which L^2 -stability is achieved.

An iterative procedure $u_h(t^{n+1}) = A(u_h(t^n))$, $u_h(0) = \mathbb{P}_h(u_o)$ is said to be L^2 -stable if there exists a constant C independent of the discretization parameters and the initial data u_0 such that

$$\|u_h(t^n)\|_{L^2(\mathcal{C})} \leq C \cdot \|u_0\|_{L^2(\mathcal{C})},$$

for every $t^n \in [0, T]$. The iterative procedure $u_n(t^{n+1}) = H_h(u_h(t^n))$ was proved to be L^2 -stable if and only if the extremely restrictive condition $cfl = O(h^{1/2})$ is satisfied; see [2,3]. In our case we have a much more satisfactory result:

PROPOSITION 3.1. *Let f be an affine function on u . Then, the method (3.2) is L^2 -stable if and only if $cfl \leq 1/3$.*

See [2,3] for details on the proof for a simpler case. It is interesting to note that the application of the Runge-Kutta technique has a stabilizing effect. We can interpret this by pointing out that in the P^0P^1 -scheme the precision in space was higher than the one in time, and this was responsible for the restrictive cfl -condition for the scheme, whereas now both the precision in space and the one in time are the same! Another interpretation is that if A denotes the symbol of the operator H_h , and Γ denotes the curve in the complex plane of its eigenvalues, the condition $cfl = O(h^{1/2})$ reflects the fact that there is a part of Γ that lies outside the unit disk and that its farthest point is at a distance of $O(cfl^3)$ from it. This part of Γ lies in a neighborhood of $z = 1$ (the fact that the point $z = 1$ lies in Γ reflects the fact that the scheme is consistent). In our case $\mathcal{A} = F(A) = \frac{1}{2}[Id + A^2]$ is the symbol of the new operator $\frac{1}{2}[Id + H_h \circ H_h]$ and the curve of its eigenvalues is $F(\Gamma) = \frac{1}{2}[1 + \Gamma^2]$. Note that the transformation F leaves the point $z = 1$ invariant, and that, by the preceding result, for $cfl \leq 1/3$, $F(\Gamma)$ lies entirely in the unit disk. Roughly speaking F pushes Γ into the unit circle.

Although we have $L^2(\mathcal{C})$ -stability of the method under a very mild cfl condition in the linear case, this does not ensure its $L^2(\mathcal{C})$ -stability in the nonlinear case. In order to render the scheme not only $L^2(\mathcal{C})$ -stable but $L^\infty(\mathcal{C}) \cup BV(\mathcal{C})$ -stable, without compromising its formal second order accuracy, we are going to use the $\Lambda\Pi_h$ -projection, that we study in detail in the next Section.

4. The Local Projection $\Lambda\Pi_h$.

4.1 Definition of the operator $\Lambda\Pi_h$. The $\Lambda\Pi_h$ - operator we are going to use is a simplified TVBM version – inspired on the TVB-technique introduced by Shu, [11]– of the $\Lambda\Pi_h$ - projection used by Chavent and Cockburn, [3]. We shall define the operators $\Lambda\Pi_h$ in three steps.

Let us first introduce the following projection operator:

$$(4.1a) \quad \mathbf{P}_{[a,b]}(c) = \begin{cases} b, & \text{if } c > b, \\ c, & \text{if } c \in [a, b], \\ a, & \text{if } c < a. \end{cases}$$

Next, let M be a nonnegative real number that is going to be a parameter whose purpose is to prevent the $\Lambda\Pi_h$ - projection of destroying the second order accuracy of the scheme. This parameter is strongly related to the second derivative of the initial data u_o , and will be estimated later. It was set equal to 0 for the $\Lambda\Pi P^0 P^1$ - scheme; see [3]. Let w_h be an element $V_h(\mathcal{C})$. We associate to M , and w_h the set of intervals $\{\mathfrak{l}_{M,i}\}_{i=1,\dots,nx}$ defined as follows

$$(4.1b) \quad \begin{aligned} \mathfrak{l}_{M,i} &= [I(0, \eta_{i-1/2}^+) \cap I(0, \eta_{i+1/2}^-)] \cup [-M \cdot h^2, M \cdot h^2], \\ \eta_{i-1/2}^+ &= \bar{w}_i - \bar{w}_{i-1}, \\ \eta_{i+1/2}^- &= \bar{w}_{i+1} - \bar{w}_i. \end{aligned}$$

Finally, we define the operator $\Lambda\Pi_h$ as follows:

$$(4.1c) \quad \begin{aligned} \Lambda\Pi_h : V_h(\mathcal{C}) &\longrightarrow V_h(\mathcal{C}), \\ w_h &\longmapsto w_h^*, \end{aligned}$$

where, the degrees of freedom of w_h^* are defined by:

$$(4.1d) \quad \begin{aligned} \bar{w}_i^* &= \bar{w}_i, \\ \tilde{w}_i^* &= \mathbf{P}_{\mathfrak{l}_{M,i}}(\tilde{w}_i). \end{aligned}$$

4.2 Properties of the operator $\Lambda\Pi_h$. The following result is a straightforward consequence of the definition of the $\Lambda\Pi_h$ - operator, (4.1). We use the notation introduced above.

LEMMA 4.1. *The $\Lambda\Pi_h$ - operator is a projection*

$$(1) \quad \Lambda\Pi_h(w_h^*) = w_h^*;$$

satisfying the following global properties:

- (2) $\bar{w}^*_h = \bar{w}_h$;
- (3) $\int_{\mathcal{C}} w^*_h = \int_{\mathcal{C}} w_h$;
- (4) $\|w^*_h\|_{L^\infty(\mathcal{C})} \leq \|\bar{w}_h\|_{L^\infty(\mathcal{C})} + M \cdot h^2$;
- (5) $\|\tilde{w}^*_h\|_{L^1(\mathcal{C})} \leq \frac{1}{2}h \cdot \|\bar{w}_h\|_{BV(\mathcal{C})} + M \cdot |\mathcal{C}| \cdot h^2$.

Thus, the operator $\Lambda\Pi_h -$ is a conservative, (3), (local) projection, (1), that leaves invariant the means, (2). Properties (4) and (5) allow us to control $\Lambda\Pi_h(w_h)$ only in terms of \bar{w}_h . This is the key fact that will allow us to obtain the compactness of the sequence of approximate solutions generated by the $RK\Lambda\Pi P^1 -$ scheme, as we shall see in the next Section.

Next, we show that when the solution of the conservation law is smooth enough, the application of the $\Lambda\Pi_h -$ projection does not destroy the already achieved accuracy of the scheme. We state this property in the following way:

LEMMA 4.2. *Let u be an element of $C^2(\mathcal{C})$. Let \mathbf{P}_h be the $L^2 -$ projection into $V_h(\mathcal{C})$. Then, there exists an $h_o = h_o(u) > 0$ and an $M_o = M_o(u)$ such that if $M \geq M_o$:*

$$\Lambda\Pi_h(\mathbf{P}_h(u)) = \mathbf{P}_h(u), \quad \forall h \leq h_o.$$

Proof. Let us assume that $u \in C^3(\mathcal{C})$, the extension of the proof to the case under consideration is straightforward. Let x be an arbitrary point of \mathcal{C} , set $w_h = \mathbf{P}_h(u)$ and $w^*_h = \Lambda\Pi_h(w_h)$. By using Taylor expansion we can easily find that

$$\begin{aligned} \bar{w}_{i+1} - \bar{w}_i &= \left[\frac{\Delta x_{i+1} + \Delta x_i}{2} \right] \cdot \partial_x u(x) + O(h^2), \\ \tilde{w}_i &= \left[\frac{\Delta x_i}{2} \right] \cdot \partial_x u(x) + \\ &\quad \left[\frac{\Delta x_i}{2} (x_i - x) \right] \partial_{xx} u(x) + O(h^3). \end{aligned}$$

If $\partial_x u(x) \neq 0$ it is clear that for h small enough \tilde{w}_i will lie in the interval $I(0, \bar{w}_i - \bar{w}_{i-1}) \cap I(0, \bar{w}_{i+1} - \bar{w}_i)$ for $i = 1, \dots, nx$. In this case we shall have $\tilde{w}^* = \tilde{w}$; see (4.1).

Now let us consider the case $\partial_x u(x) = 0$. Set $\mathcal{X} = \{x \in \mathcal{C} : \partial_x u(x) = 0, \partial_{xx} u(x) \neq 0\}$ and $\mathcal{J}_h^\nu = \{j : \text{the interval } I(x_j, x) \text{ is covered by at most } \nu \text{ intervals } I_i, \text{ where } x \in \mathcal{X}\}$. From the above expressions we see that for h small there is a constant independent of h , c_o , such that $\max_{\{j \in \mathcal{J}_h^\nu\}} |\tilde{w}_j| \leq c_o \frac{\nu}{2} h^2 \sup_{\{x \in \mathcal{X}\}} |\partial_{xx} u(x)|$. Thus, in order to have $\tilde{w}^*_j = \tilde{w}_j$ for $j \in \mathcal{J}_h^\nu$ it is enough to take $M_o = c_o \frac{\nu}{2} \sup_{\{x \in \mathcal{X}\}} |\partial_{xx} u(x)|$. This proves the result.

4.3 Estimating $M \cdot h^2$. From the last proof we can obtain an algorithm for obtaining the quantity $\mathcal{M}_o = M_o h^2$ directly from $\mathbf{P}_h(u)$. First, note that

$$\tilde{w}_{i+1} - \tilde{w}_i = \left[\frac{1}{2} (\Delta x_{i+1} (x_{i+1} - x) - \Delta x_i (x_i - x)) \right] \cdot \partial_{xx} u(x) + O(h^3)$$

In particular, when $\Delta x_i \equiv h$ then $\tilde{w}_{i+1} - \tilde{w}_i = \frac{1}{2} h^2 \cdot \partial_{xx} u(x) + O(h^3)$. The interest of this expression is that in fact it does not depend on x as $h \downarrow 0$. Next, note that if u has a local maximum or minimum at x and $x \in I_i$ then for h small enough at x the function \bar{w}_h , where $w_h = \mathbf{P}_h u$, has also a local maximum or minimum, respectively. These remarks motivate the following algorithm for estimating numerically $\mathcal{M}_o = M_o h^2$.

(4.2a) Compute $w_h = \mathbf{P}_h u$;

(4.2b) Set $\mathcal{I}_h = \{j : (\bar{w}_{j+1} - \bar{w}_j) \cdot (\bar{w}_{j-1} - \bar{w}_j) \geq 0 \text{ and } u_o \text{ has no jumps in } I_j\}$;

(4.3b) Set $\mathcal{M}_o = 2 \max_{\{j \in \mathcal{I}_h\}} \{|\tilde{w}_{j+1} - \tilde{w}_j|, |\tilde{w}_{j-1} - \tilde{w}_j|\}$.

The set \mathcal{I}_h is an approximation of \mathcal{J}_h^1 . Note that when $\Delta x_i \equiv h$, then $\mathcal{M}_o = h^2 \cdot \sup_{\{x \in \mathcal{X}\}} |\partial_{xx} u(x)| + O(h^3)$. Compare this with the $M_o \cdot h^2$ obtained in the proof of Lemma 4.2.

We end this Section by noting that if the solution of (1.1) is always smooth, the absolute values of its second derivative at its extrema never increase. In this way if $M_o = M_o(u_o)$ is such that $\Lambda \Pi_h(u_{o,h}) = u_{o,h}$ it is reasonable to expect that $\Lambda \Pi_h(u_h(t^n)) = u_h(t^n)$ for $n = 1, \dots, nt$.

5. The $RK\Lambda\Pi P^1$ – method.

5.1 Definition. We can now define the $RK\Lambda\Pi P^1$ – method as follows:

(5.1a) Set $u_h(t = 0) = \mathbf{P}_h(u_0)$;

(5.1b) For $n = 0, \dots, nt - 1$ the approximate solution $u_h(t^{n+1})$ is obtained from u_h^n as follows:

(1) Compute $w_h(t^{n+1}) = \Lambda \Pi_h(H_h^n(u_h(t^n)))$;

(2) Set $u_h(t^{n+1}) = \Lambda \Pi_h(\frac{1}{2} u_h(t^n) + \frac{1}{2} H_h^n(w_h(t^{n+1})))$;

Compare with algorithm (3.2).

5.2 Convergence properties of the $RK\Lambda\Pi P^1$ – method. We need the following result.

LEMMA 5.1. *Let u_h be any element of the space $C_h(\mathcal{C}) = \Lambda \Pi_h(V_h(\mathcal{C}))$, and suppose that $h \cdot nx \leq C_o |\mathcal{C}|$. Then, for $cfl \in [0, 1/2]$:*

$$\|\overline{H_h^n}(u_h)\|_{BV(\mathcal{C})} \leq \|\bar{u}_h\|_{BV(\mathcal{C})} + C \cdot h,$$

where $C = 8C_o \cdot cfl \cdot M \cdot |\mathcal{C}|$.

Proof. Set $w_h = H_h^n(u_h)$. The means of w_h are given by:

$$\bar{w}_i = \bar{u}_i - (\Delta t^n / \Delta x_i)(f^G_{i+1/2} - f^G_{i-1/2}).$$

In order to rewrite this equation adequately, note that as $u_h \in C_h(\mathcal{C}) = \Lambda \Pi_h(V_h(\mathcal{C}))$ we have that $\tilde{u}_i = \mathbf{P}_{\mathbf{1}_{M,i}}(\tilde{u}_i)$; see (4.1c). Then, if we set

$$\begin{aligned} \tilde{w}_i &= \mathbf{P}_{\mathbf{1}_{(M=0),i}}(\tilde{u}_i), \\ r_i &= \tilde{u}_i - \tilde{v}_i, \end{aligned}$$

we have that $|r_i| \leq h^2 M$. Now, we rewrite this equation as follows:

$$\bar{w}_i = \bar{u}_i + C_i^n(\bar{u}_{i+1} - \bar{u}_i) + D_i^n(\bar{u}_{i-1} - \bar{u}_i) + A_i^n(-r_{i+1} + r_i) + B_i^n(r_{i-1} - r_i),$$

where

$$\begin{aligned} A_i^n &= -\frac{\Delta t^n}{\Delta x_i} \cdot \frac{f^G(\bar{u}_{i+1} - \tilde{u}_{i+1}, \bar{u}_i + \tilde{u}_i) - f^G(\bar{u}_i - \tilde{u}_i, \bar{u}_i + \tilde{u}_i)}{(\bar{u}_{i+1} - \tilde{u}_{i+1}) - (\bar{u}_i - \tilde{u}_i)}, \\ B_i^n &= \frac{\Delta t^n}{\Delta x_i} \cdot \frac{f^G(\bar{u}_i - \tilde{u}_i, \bar{u}_i + \tilde{u}_i) - f^G(\bar{u}_i - \tilde{u}_i, \bar{u}_{i-1} + \tilde{u}_{i-1})}{(\bar{u}_{i-1} + \tilde{u}_{i-1}) - (\bar{u}_i + \tilde{u}_i)}. \end{aligned}$$

and

$$\begin{aligned} C_i^n &= \left(1 - \frac{\tilde{v}_{i+1}^n - \tilde{v}_i^n}{\bar{u}_{i+1}^n - \bar{u}_i^n}\right) \cdot A_i^n, \\ D_i^n &= \left(1 + \frac{\tilde{v}_{i-1}^n - \tilde{v}_i^n}{\bar{u}_{i-1}^n - \bar{u}_i^n}\right) \cdot B_i^n. \end{aligned}$$

We only have to follow [3] in order to obtain

$$\|\bar{w}_h\|_{BV(\mathcal{C})} \leq \|\bar{u}_h\|_{BV(\mathcal{C})} + 8 \max_i \{A_i^n, B_i^n\} \cdot \max_i \{|r_i|\} nx,$$

for $cfl \in [0, 1/2]$. Finally, as $\max_i \{A_i^n, B_i^n\} \leq cfl$, $\max_i \{|r_i|\} \leq h^2 M$, and $nx \leq C_o |\mathcal{C}| h^{-1}$ the result follows.

Let $\{u_h(t^n)\}_{n=0, \dots, nt}$ be the sequence generated by the $RK\Lambda\Pi P^1$ - method (5.1). Let us define its Q^1 - interpolate, denoted again by u_h , as follows:

$$u_h(t) = \left[\frac{t - t^n}{\Delta t^n}\right] u_h(t^{n+1}) + \left[\frac{t^{n+1} - t}{\Delta t^n}\right] u_h(t^n), \forall t \in [t^n, t^{n+1}].$$

THEOREM 5.2. *Suppose that $cfl \in [0, 1/2]$, that $nx \cdot h \leq C_o |\mathcal{C}|$, and that $nt \cdot h \leq C_1 T$. Let $\{u_h\}_{h \downarrow 0}$ be the sequence of Q^1 - interpolates of the approximate solution defined by the $RK\Lambda\Pi P^1$ - method (5.1). Then*

$$\begin{aligned} \|\bar{u}_h(t)\|_{BV(\mathcal{C})} &\leq \|u_o\|_{BV(\mathcal{C})} + C, \\ \|\tilde{u}_h(t)\|_{L^1(\mathcal{C})} &\leq h \cdot \left\{\frac{1}{2}\|u_o\|_{BV(\mathcal{C})} + C\right\}, \forall t \in [0, T], \end{aligned}$$

where $C = \max\{8C_o C_1 \cdot cfl \cdot M \cdot T \cdot |\mathcal{C}|, 2M \cdot |\mathcal{C}| \cdot h\}$. In other words, the scheme is TVB.

Proof. It is enough to prove these inequalities for $t = t^n$. Noting that $u_h(t^{n+1}) = w^*_h = \Lambda \Pi_h(w_h)$, with $w_h = \frac{1}{2}u_h(t^n) + \frac{1}{2}H_h^n(\Lambda \Pi_h(H_h^n(u_h(t^n))))$, the second inequality is obtained easily from the first one:

$$\begin{aligned} \|\tilde{u}_h(t^{n+1})\|_{L^1(\mathcal{C})} &= \|\tilde{w}^*_h\|_{L^1(\mathcal{C})} \\ &\leq \frac{1}{2}h \cdot \|\bar{w}_h\|_{BV(\mathcal{C})} + M \cdot |\mathcal{C}| \cdot h^2, \text{ by (5), Lemma 4.1,} \\ &= \frac{1}{2}h \cdot \|\bar{w}^*_h\|_{BV(\mathcal{C})} + M \cdot |\mathcal{C}| \cdot h^2, \text{ by (2), Lemma 4.1,} \\ &\leq \frac{1}{2}h \cdot \|\bar{u}_h(t^{n+1})\|_{BV(\mathcal{C})} + \frac{1}{2}\mathcal{C}. \end{aligned}$$

The first one is obtained as follows:

$$\begin{aligned} \|\bar{u}_h(t^{n+1})\|_{BV(\mathcal{C})} &= \|\bar{w}^*_h\|_{BV(\mathcal{C})} \\ &= \left\| \frac{1}{2}\bar{u}_h(t^n) + \frac{1}{2}\bar{H}_h^n(\Lambda \Pi_h(H_h^n(u_h(t^n)))) \right\|_{BV(\mathcal{C})} \\ &\leq \frac{1}{2}[\|\bar{u}_h(t^n)\|_{BV(\mathcal{C})} + \|\bar{H}_h^n(\Lambda \Pi_h(H_h^n(u_h(t^n))))\|_{BV(\mathcal{C})}] \\ &\leq \frac{1}{2}[\|\bar{u}_h(t^n)\|_{BV(\mathcal{C})} + \|\bar{H}_h^n(u_h(t^n))\|_{BV(\mathcal{C})} + C' \cdot h], \\ &\text{by Lemma 5.1, and (2), Lemma 4.1,} \\ &\leq \frac{1}{2}[\|\bar{u}_h(t^n)\|_{BV(\mathcal{C})} + \|\bar{u}_h(t^n)\|_{BV(\mathcal{C})} + 2C' \cdot h], \\ &\text{again by Lemma 5.1,} \\ &\leq \|\bar{u}_h(t^n)\|_{BV(\mathcal{C})} + C' \cdot h, \\ &\leq \|\bar{u}_{o,h}\|_{BV(\mathcal{C})} + C' \cdot nt \cdot h, \\ &\leq \|u_o\|_{BV(\mathcal{C})} + C' \cdot C_1 \cdot T, \text{ by hypothesis.} \end{aligned}$$

The result follows from the fact that $C' = 8C_o \cdot cfl \cdot M \cdot |\mathcal{C}|$; see Lemma 5.1.

COROLLARY 5.3. *Suppose the hypothesis of the preceding Theorem are verified. Then, there is a subsequence of the sequence $\{u_h\}_{h \downarrow 0}$ generated by the RK Λ IP¹– method (5.1) that converges to a weak solution of (1.1).*

Proof. By Theorem 5.2 there is a subsequence $\{u_{h'}\}_{h' \downarrow 0}$ that converges in $L^\infty(0, T; L^1(\mathcal{C}))$ to a function $u^* \in L^\infty(0, T; L^1(\mathcal{C}) \cap BV(\mathcal{C}))$. Let us think of the functions \tilde{u}_h as already known parameters that tend to 0 in $L^\infty(0, T; L^1(\mathcal{C}))$, and let us consider the scheme (5.1) as a scheme only for the means \bar{u}_h . It can be easily seen that this scheme is a conservative scheme whose numerical flux is consistent with f . Thus, the limit u^* must be a weak solution of (1.1) by the well known Lax-Wendroff Theorem, [8]. This proves the result.

6. Numerical results.

6.1 The test problems. In this Subsection we test the $RK\Lambda\Pi P^1$ – method in six different problems (1.1) for which we can calculate the exact solution. Our test problems are defined by giving the circle \mathcal{C} – that we identify with the intervall $[0, l)$, the final time T , the nonlinearity f , and the initial data u_0 on Ω ; see the table 6.1. Their corresponding exact solutions are displayed on Figures 6.1. On table 6.2 we define the sets \mathcal{C}' on which the entropy solution u can be considered smooth. They have been obtained from \mathcal{C} by subtracting subsets of it that contained discontinuities of either u or $\partial_x u$.

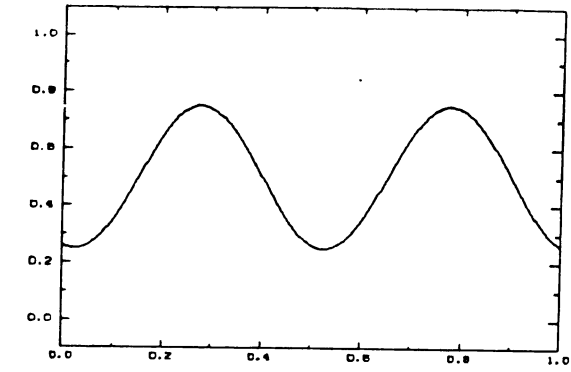
Table 6.1 . Definition of the test problems.

problem	\mathcal{C}	T	$f(u)$	$u_0(x)$
1	$[0,1)$	0.15	u	$\frac{1}{2}(1 + \frac{1}{2}\sin(4\pi x))$
2a	$[0,1)$	$\left\{ \begin{array}{l} 0.15 \\ 1/\pi \\ 0.55 \end{array} \right\}$	$u^2/2$	$\frac{1}{2}(\frac{1}{2} + \sin(2\pi x))$
2b				
2c				
3	$[0,1)$	0.1	$\frac{1}{2} \frac{u^2}{u^2+(1-u)^2}$	$\frac{1}{2}(1 + \frac{1}{2}\sin(4\pi x))$
4	$[0,1)$	0.15	u	$\begin{cases} 1, & \text{if } x \in (0.4, 0.6), \\ 0, & \text{otherwise.} \end{cases}$
5	$[0,2)$	0.5	$u(1 - u)$	$\begin{cases} 1, & \text{if } x \in (0.5, 1.5), \\ 0, & \text{otherwise.} \end{cases}$
6	$[0,2)$	0.5	$\frac{1}{2} \frac{u^2}{u^2+(1-u)^2}$	$\begin{cases} 1, & \text{if } x \in (0.5, 1.5), \\ 0, & \text{otherwise.} \end{cases}$

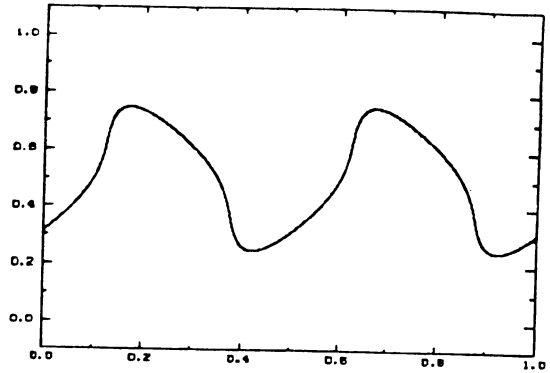
6.2 The results. The quadrature rule (2.3) used in these computations is the three-point Gauss rule. Although Corollary 5.3 ensures the convergence of the method for $cfl \in [0, 1/2]$, Proposition 3.1 guarantees L^2 -stability only for $cfl \in [0, 1/3]$ in the linear case. This is why we are going to perform our computations with $cfl = 1/3$. On table 6.3 we display the $L^1(\mathcal{C}')$ –errors, and on table 6.4 the $L^\infty(\mathcal{C}')$ ones. Details of how the discontinuities are captured are shown on Figures (6.2).

Let us point out that the algorithm (4.2) that allows us to compute $\mathcal{M}_o = M \cdot h^2$, although convenient for the test problems we considered, is not unique. A more complete treatment of the way in which this quantity can be estimated numerically will be considered in a forthcoming paper.

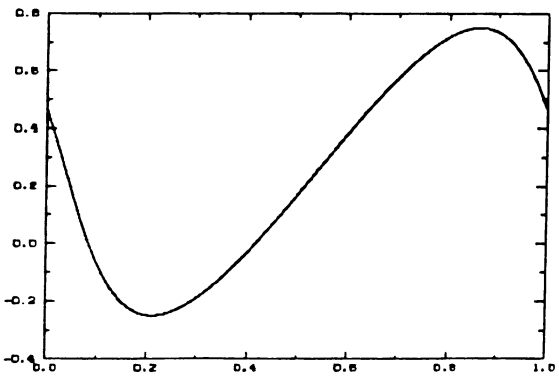
Figures 6.1 . The entropy solutions $u(T)$ of the test problems.



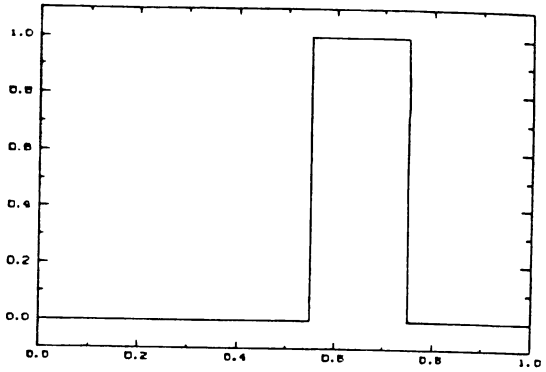
1



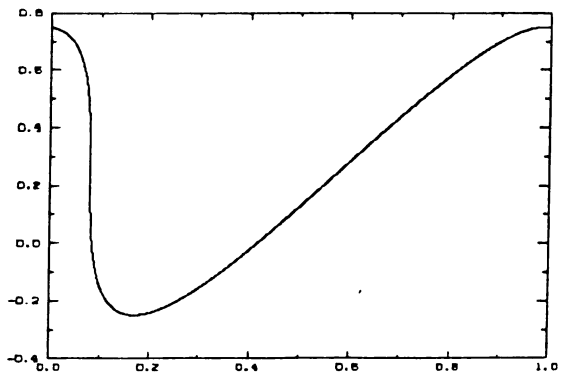
3



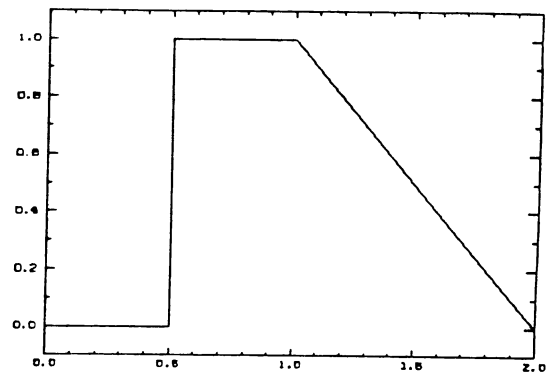
2a



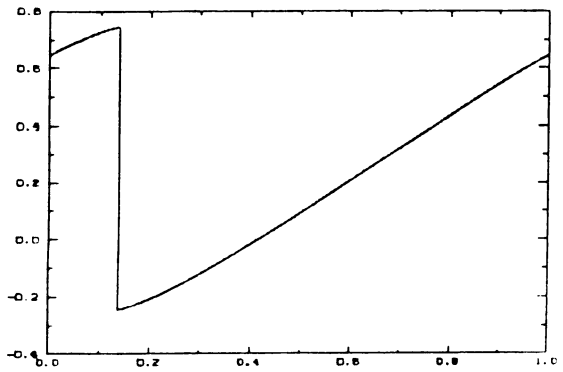
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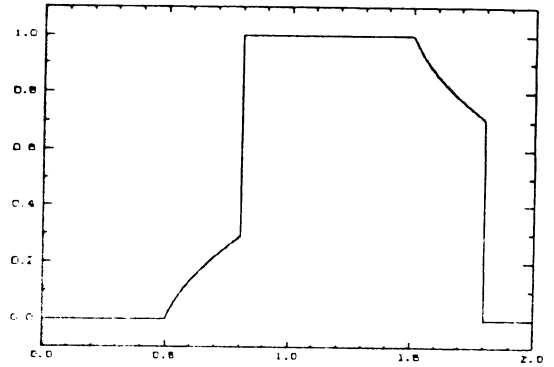
2b



5



2c



6

Table 6.2 . Definition of the domains \mathcal{C}' on which $u(T)$ is smooth.

problem	\mathcal{C}'	singularities of $u(T)$
1	[0,1)	none
2a	[0,1)	none
2b	[0,.04]∪[.14,1)	shock appears near x=0.08
2c	[0,.09]∪[.19,1)	shock near x=0.14
3	[0,1)	none
4	[0,.5]∪[.6,.7]∪[.8,1)	two contact disc. at x=0.55, 0.75
5	[.05,.45]∪[.55,.95]∪[1.05,1.95]	{ stationary shock at x=0.5, two disc. of $\partial_x u$ at x=1, 2.
6	[0,.75]∪[.85,1.75]∪[1.85,2)	two shocks near x=0.8,1.8

Table 6.3 . L^1 -errors and orders of convergence for $cf1 = 1/3$. The quantity e_1 is equal to $\|u(T) - u_h(T)\|_{L^1(\mathcal{C}')}$, and α_1 is the corresponding order of convergence. For all the tests we have taken $\Delta x = \frac{1}{200}$. The sets \mathcal{C}' are defined in table 6.2.

problem	no projection		$M \equiv 0$		$M \cdot h^2 = \mathcal{M}_o$	
	$10^4 \cdot e_1$	α_1	$10^4 \cdot e_1$	α_1	$10^4 \cdot e_1$	α_1
1	0.57	1.97	1.28	2.24	0.57	1.97
2a	0.21	1.96	0.34	2.11	0.21	1.96
2b	0.09	2.01	0.30	2.25	0.09	2.01
2c	0.02	2.00	0.02	2.00	0.02	2.00
3	0.87	1.94	1.25	2.06	0.87	1.94
4	21.4	1.00	0.0004	-	0.0004	-
5	7.20	1.13	6.19	0.99	6.19	0.95
6	248	0.002	0.39	0.99	0.39	0.99

Table 6.4 . L^∞ -errors and orders of convergence for $cfl = 1/3$. The quantity e_∞ is equal to $\|u(T) - u_h(T)\|_{L^\infty(\mathcal{C}')}$, and α_∞ is the corresponding order of convergence. For all the tests we have taken $\Delta x = \frac{1}{200}$. The sets \mathcal{C}' are defined in table 6.2.

	no projection		$M \equiv 0$		$M \cdot h^2 = \mathcal{M}_o$	
problem	$10^4 \cdot e_\infty$	α_∞	$10^4 \cdot e_\infty$	α_∞	$10^4 \cdot e_\infty$	α_∞
1	1.56	1.94	10.1	1.55	1.56	1.94
2a	1.28	1.98	5.04	1.24	1.28	1.98
2b	1.22	2.04	6.51	1.71	1.22	2.04
2c	0.15	1.81	0.15	1.80	0.15	1.81
3	10.5	1.95	10.5	1.95	10.5	1.95
4	199.4	1.16	0.05	-	0.05	-
5	24.15	1.82	16.31	0.96	16.31	0.96
6	1966	0.08	2.63	1.06	2.63	1.06

(Some of the results corresponding to problem 4 have not been included, for in this case a superconvergence, that is far from being typical, is observed. This is due to the facts that, on \mathcal{C}' , we have $\partial_x^n u \equiv 0, \forall n \geq 0$, and that the approximate solution does not oscillate! When the projection is not used, strong oscillations *away* of the discontinuities appear.)

As the preceding results show, L^1 second order accuracy has been obtained for test problems 1, 2, and 3, and this is regardless of the values of M . Uniform second order accuracy away from discontinuities has also been obtained, but this time the influence of the $\Lambda\Pi_h$ - projection is evident, as expected.

For M small enough the $\Lambda\Pi_h$ - projection is not equal to the identity, and its application produces loss of accuracy only near the extrema:

- (1) that the influence of the $\Lambda\Pi_h$ - projection appears only in the presence of extrema can be verified by noting that in problem 2c, for which the entropy solution does not have extrema but two smooth monotone regions between the shocks, the results are independent of the value of M (the value of M for problems 4,5,6 is zero);
- (2) that this lost of accuracy is indeed of a local character can be verified by comparing the corresponding L^1 - and L^∞ - errors on tables 6.3 and 6.4, respectively (consider the problems 1, 2a,b and 3 — that are the only ones for which the entropy solution does have extrema). Note how the lost of accuracy is greater when $M \equiv 0$, and

how in this case the order of convergence is lowered dramatically for the L^∞ norm while the one of the L^1 norm remains the same. (This cannot be seen for problem 3 because the maximum error is attained not at the extrema but at some points between them at which the function $|\partial_x u|$ becomes very big!);

- (3) that for "big" values of M the $\Lambda\Pi_h$ - projection reduces to the identity can be verified by noting that the results with no projection coincide with the ones obtained with M calculated by (4.2).

In problems 4, 5, and 6 for which the initial data presents a discontinuity, we see that the introduction of the $\Lambda\Pi_h$ - projection (i) improves the accuracy of the method, and (ii) enforces the convergence to the entropy solution. The most dramatic case is, of course, problem 6 (the nonlinearity f is nonconvex) for which the scheme without projection does not converge to the entropy solution.

Figure 6.2a. Detail of the convergence in test problem 5: Aproximation to a discontinuity of $\partial_x u$ when the nonlinearity f is concave. Note how the discontinuity of $\partial_x u$ is approximated without any oscillation. The "corner" is very well reproduced by the approximate solution u_h in a single element. However, its position with respect to the "right corner" is correct up to an element. We found that in this case the $L^1(\mathcal{C})$ - error is $O(\Delta x)$. This reflects the fact that the initial data is discontinuous, and indicates that this picture is essentially the same for all the values of Δx . In this case algorithm (4.2) gives $M = 0$, so that the $\Lambda\Pi_h$ - projection maintains the range of the approximate solution in $[0,1]$.

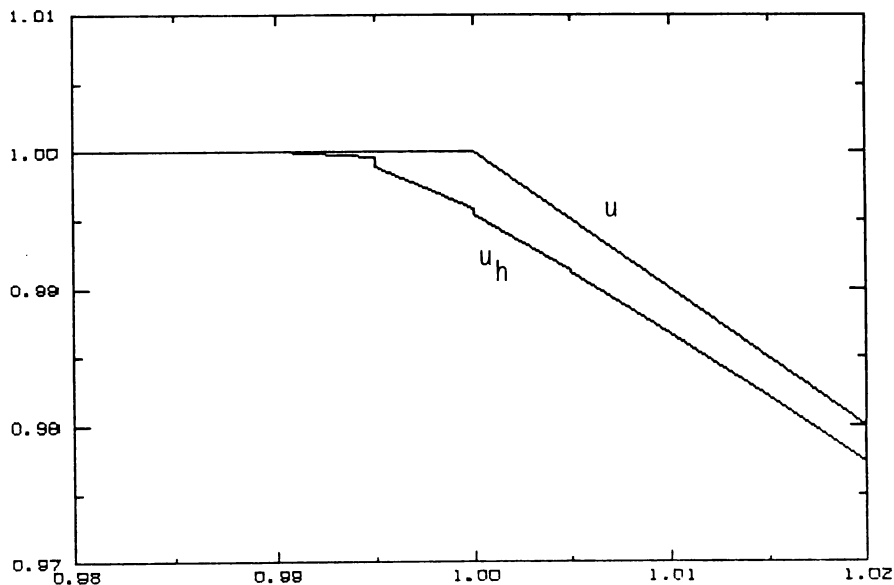


Figure 6.2b. Detail of the convergence in test problem 2c: Approximation to a discontinuity of u when the nonlinearity f is convex. The discontinuity has been captured with a single element. In this case the value of M is different from zero, nevertheless, no visible oscillation has been introduced, as expected.

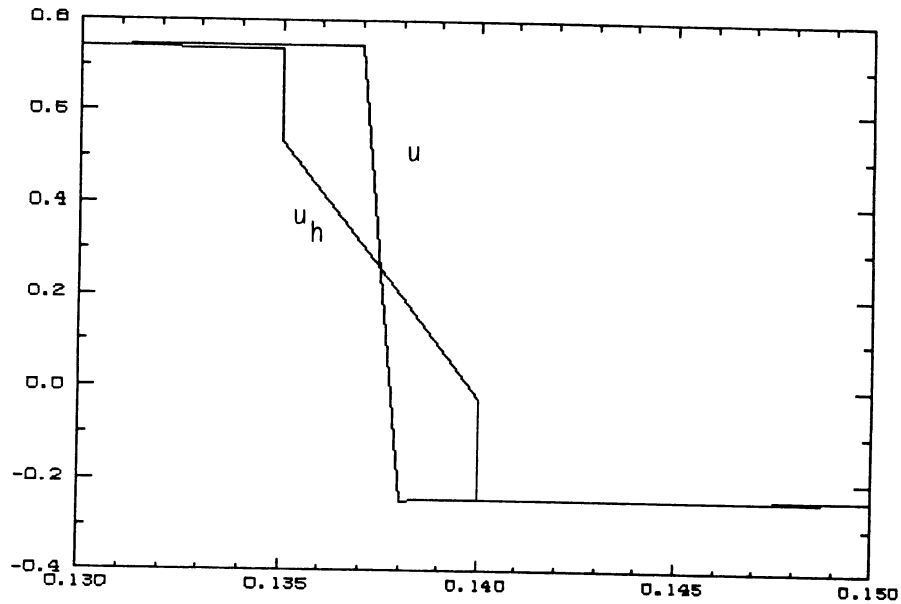
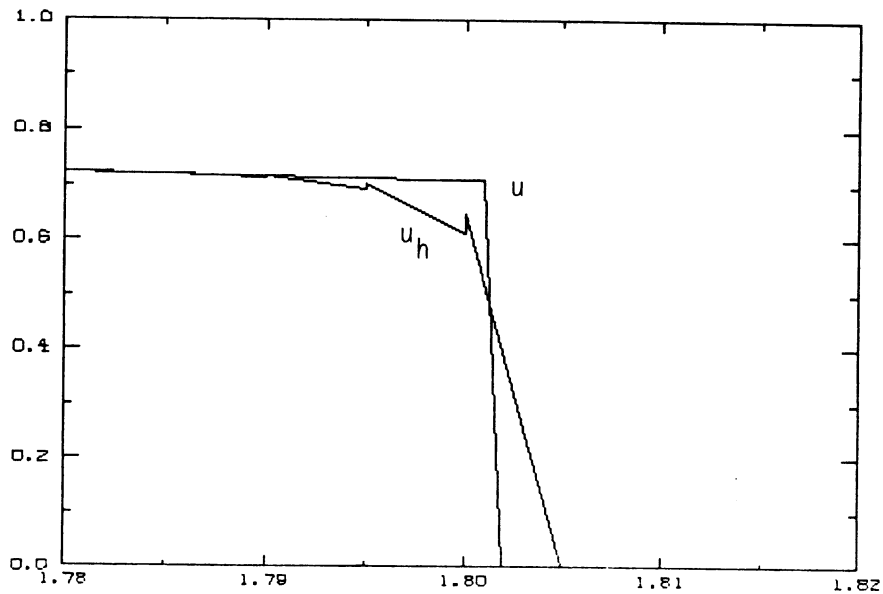


Figure 6.2c. Detail of the convergence in test problem 2c: Approximation to a discontinuity of u when the nonlinearity f is nonconvex. This is the most difficult case. Note, however, how the discontinuity of u is approximated without any oscillation, and how at least 90% of it has been captured with a single element.



All these calculations have been redone for $cfl = 1/6$ with identical results. The influence of this decrease of cfl - number was negligible.

6.3 Conclusion. Our numerical results indicate that the $RK\Lambda\Pi P^1$ - method is a stable method that converges to the entropy solution, even in the presence of a nonconvex nonlinearity f . In smooth regions of the entropy solution the method was found to be uniformly second order accurate away from the discontinuities when the initial data was smooth, and was able to capture shocks, essentially, within a single element.

We want to stress the fact that this method is easy to code, and that its definition does not depend on the type of nonlinearity f under consideration.

Higher order versions of this method, as well as extensions to the nonperiodic case, constitute the object of work in progress.

REFERENCES

- [1] Y. BRENIER AND S. OSHER, *Approximate Riemman Solvers, and Numerical Flux Functions*, ICASE report n^o 84-63 (1984).
- [2] G. CHAVENT AND B. COCKBURN, *Consistance et Stabilité des Schémas LRG pour les Lois de Conservation Scalaires*, INRIA report # 370 (1987).
- [3] G. CHAVENT AND B. COCKBURN, *The Local Projection Discontinuous Galerkin Finite Element Method for Scalar Conservation Laws*, IMA Preprint Series # 341, (1987).
- [4] G. CHAVENT AND G. SALZANO, *A Finite Element Method for the 1D Water Flooding Problem with Gravity*, J. Comp. Phys., 45 (1982), pp. 307-344.
- [5] P.G. CIARLET, *The Finite Element Method for Elliptic Problems*, North Holland, 1975.
- [6] T. GEVECI, *The Significance of the Stability of Difference Schemes in Different l^p -spaces*, SIAM Review, 24 (1982), pp. 413-426.
- [7] B.A. FRYXEL, P.R. WOODWARD, P. COLLELA AND K.H. WINKLER, *An Implicit-Explicit Hybrid Method for Lagrangian Hydrodynamics*, J. Comp. Phys., 63 (1986), pp. 283-310.
- [8] P.D. LAX AND B. WENDROFF, *Systems of Conservation Laws*, Comm. Pure and Appl. Math., 13 (1960), pp. 217-237.
- [9] S. OSHER, *Riemman Solvers, the Entropy Condition, and Difference Approximations*, SIAM J. Numer. Anal., 21 (1984), pp. 217-235.
- [10] B. VAN LEER, *Towards the Ultimate Conservative Scheme, VI. A New Approach to Numerical Convection*, J. Comp. Phys., 23 (1977), pp. 276-299.
- [11] C.W. SHU, *TVB uniformly high order schemes for conservation laws*, Math. of Comp., 49 (1987), pp. 105-121.
- [12] C.W. SHU AND S. OSHER, *Efficient Implementation of Essentially Non-Oscillatory Shock-Capturing Schemes*, ICASE report # 87-33, to appear in J. Comp. Phys..

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