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Abstract

A model is defined to simulate a waterflood in a multidimensional, naturally fractured petroleum reservoir. The imbibition process is correctly modelled as a boundary condition on each matrix block. The model is presented in terms of a saturation and the global pressure of Chavent. The numerical method is based on the use of a mixed finite element method for the pressure and standard Galerkin methods for the saturations in the fractures and the blocks. Optimal order asymptotic convergence of the approximate solution to that of the differential system is established under the assumption of nondegeneracy of the relative permeability functions.

1. Introduction

A new approach to modelling fluid flow in naturally fractured petroleum reservoirs has recently been introduced [1], [2], [11] with the object being to improve the treatment of the interaction between the matrix blocks and the fractures. The flow of a single-phase, compressible liquid of constant compressibility has been considered in a relatively complete fashion; i. e., a model has been defined, shown to be well posed, and discretized in a manner that is both practically feasible and asymptotically convergent to the true solution at an optimal rate [1], [2]. A simple linear waterflood has also been modelled and discretized by means of finite differences [11], though analysis of the discretization has not yet been given.

Here we shall consider the waterflood in a more general setting. We shall derive a model for a two-phase, immiscible, incompressible waterflood in a domain in $\mathbb{R}^d$, $d \le 3$; however, we shall ignore the effects of gravity. The model can be considered to be applicable to a horizontal linear or planar reservoir or to one that is thin in the vertical direction. This is a serious assumption, since gravity affects the imbibition of water into the martrix blocks in an important way. The boundary conditions relating the behavior of the fluids in the fractures to that in the blocks change significantly when gravity is taken into account. Such a model is under development.

As in the case of the linear waterflood model, we shall assume that the blocks are small; as a consequence, we shall neglect the effect of the viscous forces in the fractures in our treatment of the blocks. Thus, we concentrate on modelling the dominant effect of the imbibition process on the blocks. The viscous forces can be treated as they were in [2], but gravity should not be ignored in that case, as its effect exceeds that of the potential drop across a block.

The model to be presented herein will be formulated in terms of a saturation and the global pressure introduced by Chavent [6]. One advantage of this choice is that the differential equations appear in a form quite similar to those for miscible displacement, which has received somewhat greater attention in the mathematical literature in recent years; our numerical model is a variant of techniques first applied to the miscible problem. The second advantage is that, as a consequence of the neglect of the viscous forces in the fractures in the treatment of the blocks, the global pressure will be seen to be a constant over each block at any fixed time, so that the differential system needed to describe the flood in a block will reduce to a single equation for the saturation in the block.

Douglas [7] has described and analyzed finite difference procedures for approximating waterflood problems on standard, unfractured reservoirs; he also used the global pressure in place of a phase pressure. We shall approximate the solution of our problem by adapting a known finite element procedure for miscible displacement problems [9], [10]. We shall combine a mixed finite element method for approximating the pressure and a total flow rate in the fractures with standard Galerkin methods for the saturation both in the fractures and in the individual blocks. This procedure will be analyzed under the same assumptions as were made in [7], where their reasonableness was discussed. In particular, we shall assume that the relative permeabilities remain positive, so that the differential equations stay nondegenerate. Also, we shall take the external flow to be smoothly distributed, instead of being concentrated at wells. These assumptions imply coercivity and regularity for the problem.

An outline of the paper is as follows. In the next section we derive the differential model, and the equations will be put in weak form in the following section. The finite element procedure will be introduced in Section 4 and analyzed in Section 5, which represents about one half of the manuscipt. Finally, in Section 6 a list of otherwise undefined notation is given. We have tried to maintain a consistency in notation here with that used in [7] and [14]. When a quantity exists for both the fracture and matrix block systems, we have chosen to denote the quantity in the fractures by a capital letter and in the blocks by the same symbol in lower case.

2. The Two-Phase, Immiscible Model

Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be the fractured reservoir and let $\Omega_i \subset \Omega$ be the i^{th} matrix block. Let $\Omega_m = \cup_i \Omega_i$. The usual equations describing two-phase, immiscible, incompressible displacement in Ω_i when the effect of gravity is omitted are given by

$$\varphi s_{o,t} - \nabla\cdot(kk_{ro}(s_o)\mu_o^{-1}\nabla p_o) = 0, \qquad (x,t)\epsilon\Omega_i\times J, \tag{2.1}$$

$$\varphi s_{w,t} - \nabla\cdot(kk_{rw}(s_w)\mu_w^{-1}\nabla p_w) = 0, \qquad (x,t)\epsilon\Omega_i\times J, \tag{2.2}$$

where $J = (0,T]$ and the subscript t denotes differentiation with respect to time. Let $s = s_o = 1-s_w$. The pressures in the two phases are related by the capillary pressure

$$p_c = p_c(s) = p_o - p_w, \tag{2.3}$$

and it is the case that $p_c'(s)>0$. Let

$$\lambda(s) = k_{ro}\mu_o^{-1} + k_{rw}\mu_w^{-1}, \qquad \lambda_o(s) = k_{ro}\mu_o^{-1}\lambda^{-1}\ , \qquad \lambda_w(s) = k_{rw}\mu_w^{-1}\lambda^{-1}\ .$$

Chavent's global pressure variable [6] is given by

$$p = \tfrac{1}{2}(p_o+p_w) + \tfrac{1}{2}\int_0^{p_c} (\lambda_o-\lambda_w)(p_c^{-1}(\xi))d\xi. \tag{2.4}$$

As in [7], adding (2.1) and (2.2) gives the pressure equation

$$-\nabla\cdot(k\lambda\nabla p) = 0\ , \tag{2.5}$$

and subtracting (2.2) from (2.1) gives the saturation equation

$$\varphi s_t + \lambda_o' u\cdot\nabla s - \nabla\cdot(k\lambda\lambda_o\lambda_w p_c'\nabla s) = 0, \tag{2.6}$$

where

$$u = -k\lambda\nabla p. \tag{2.7}$$

The initial and boundary conditions for a matrix block are chosen as follows. Let $\{\chi_i\}$ be a partition of unity on Ω such that

$$\Sigma_i \chi_i = 1, \qquad \int_\Omega \chi_i dx = |\Omega_i|, \qquad \chi_i \geq 0,$$

and the support of χ_i is near Ω_i. Let

$$\hat{\psi}_i = |\Omega_i|^{-1} \int_\Omega \psi \chi_i dx.$$

Assume that the pressure variation in the fractures across a block can be ignored. Then the boundary conditions on the block, which reflect continuity of the water and oil pressures, will be taken to be

$$p_o(x,t) = \hat{P}_{oi}(t) \quad \text{and} \quad p_w(x,t) = \hat{P}_{wi}(t), \qquad (x,t) \epsilon \partial\Omega_i \times J. \tag{2.8}$$

With the analogous fracture quantities $S = S_o = 1 - S_w$ and $P_c(S) = P_o - P_w$, these two relations then imply that

$$p_c(s(x,t)) = p_o(x,t) - p_w(x,t) = \hat{P}_{oi}(t) - \hat{P}_{wi}(t) = \hat{P}_c(S(\cdot,t))_i ,$$

so that

$$s(x,t) = p_c^{-1}(\hat{P}_c(S(\cdot,t))_i), \qquad (x,t) \epsilon \partial\Omega_i \times J. \tag{2.9}$$

The physical assumption (2.8) ignores an effect that is of magnitude $O(\mathrm{diam}(\Omega_i))$ and since

$$\hat{P}_c(S(\cdot,t))_i - \hat{P}_c(\hat{S}_i(t)) = O(\mathrm{diam}(\Omega_i))^2,$$

we shall take

$$s(x,t) = p_c^{-1}(P_c(\hat{S}_i(t)), \qquad (x,t) \epsilon \partial\Omega_i \times J, \tag{2.10}$$

in place of (2.9). Below it can be seen that the imposition of continuity of the capillary pressure and the global pressure, rather than the phase pressures, leads to (2.10).

A consequence of the assumption above to ignore the pressure drop in the fractures across a block is that the initial conditions on a block, which reflect initial equilibrium with the surrounding fractures, are given by

$$p_w(x,0) = \hat{P}_{wi}(0) \quad \text{and} \quad p_o(x,0) = \hat{P}_{oi}(0), \qquad x \epsilon \Omega_i , \tag{2.11}$$

so that

$$p_c(s(x,0)) = \hat{P}_c(S(\cdot,0))_i , \qquad x \epsilon \Omega_i . \tag{2.12}$$

Again, instead of (2.12), we shall choose as the block initial saturation the value

$$s(x,0) = p_c^{-1}(\hat{P}_c(\hat{S}_i(0)), \qquad x\epsilon\Omega_i\,. \tag{2.13}$$

Note that from (2.4) and (2.11) it follows for $(x,t)\epsilon\partial\Omega_i\times J$ that

$$p(x,t) = \tfrac{1}{2}(\hat{P}_{oi}(t) + \hat{P}_{wi}(t)) + \tfrac{1}{2}\int_0^{P_c(\hat{S}_i(t))} (\lambda_o - \lambda_w)(p_c^{-1}(\xi))d\xi \tag{2.14}$$

depends only on the time t. Thus, for each time $t\epsilon J$, p is constant on Ω_i by (2.5), and it follows from (2.7) that $u\equiv 0$. Let $\sigma = p_c^{-1}\circ P_c$. Then the differential system for the flow in the matrix block Ω_i becomes the single equation

$$\varphi s_t - \nabla\cdot(k\lambda\lambda_o\lambda_w p_c'\nabla s) = 0, \qquad (x,t)\epsilon\Omega_i\times J, \tag{2.15}$$

with the boundary condition

$$s(x,t) = \sigma(\hat{S}_i(t)), \qquad (x,t)\epsilon\partial\Omega_i\times J, \tag{2.16}$$

and the initial condition

$$s(x,0) = \sigma(\hat{S}_i(0)), \qquad x\epsilon\Omega_i\,. \tag{2.17}$$

Let

$$v_o = -kk_{ro}\mu_o^{-1}\nabla p_o$$

be the Darcy velocity of the oil phase. Then the block Ω_i transmits through its surface $\partial\Omega_i$ a space-averaged flow rate of oil given by

$$Q_{oi}(t) = -|\Omega_i|^{-1}\int_{\partial\Omega_i} v_o\cdot\nu\, da = -|\Omega_i|^{-1}\int_{\Omega_i}\nabla\cdot v_o\, dx = -|\Omega_i|^{-1}\int_{\Omega_i}\varphi s_t dx. \tag{2.18}$$

Thus, we define a total matrix source term by

$$Q_{om}(x,t) = \Sigma_i\, Q_{oi}(t)\chi_i(x)\,, \qquad (x,t)\epsilon\Omega\times J. \tag{2.19}$$

Now consider the differential system for the fractures. Assume that each matrix block interacts only with the fracture system; thus, the blocks do not interact with each other and they do not interact with external sources and sinks (i. e., wells). Then Darcy's law, conservation of mass, the assumed absence of gravitational terms, and the assumption above imply that

$$\Phi S_{ot} - \nabla\cdot(KK_{ro}(S_o)\nabla P_o) = Q_{oe} + Q_{om}\,, \qquad (x,t)\epsilon\Omega\times J, \tag{2.20}$$

$$\Phi S_{wt} - \nabla\cdot(KK_{rw}(S_w)\nabla P_w) = Q_{we} + Q_{wm}, \qquad (x,t)\epsilon\Omega\times J, \tag{2.21}$$

where the assumed incompressibility of the fluids (and of the matrix rock) implies that the matrix water sink term Q_{wm} satisfies the relation

$$Q_{wm} + Q_{om} = 0. \tag{2.22}$$

Let

$$\Lambda(S) = K_{ro}\mu_o^{-1} + K_{rw}\mu_w^{-1}, \qquad \Lambda_o(S) = K_{ro}\mu_o^{-1}\Lambda^{-1}, \qquad \Lambda_w(S) = K_{rw}\mu_w^{-1}\Lambda^{-1}.$$

The global pressure for the fracture system is defined analogously by

$$P = \tfrac{1}{2}(P_o+P_w) + \tfrac{1}{2}\int_0^{P_c} (\Lambda_o-\Lambda_w)(P_c^{-1}(\xi))\,d\xi. \tag{2.23}$$

Again, adding and subtracting (2.20) and (2.21) leads to the system

$$\nabla\cdot U = Q_e, \qquad (x,t)\epsilon\Omega\times J, \tag{2.24}$$

$$U + K\Lambda\nabla P = 0, \qquad (x,t)\epsilon\Omega\times J, \tag{2.25}$$

$$\Phi S_t + \Lambda_o' U\cdot\nabla S - \nabla\cdot(K\Lambda\Lambda_o\Lambda_w P_c'\nabla S) = Q_{om} - \Lambda_o Q_e^+, \qquad (x,t)\epsilon\Omega\times J, \tag{2.26}$$

where $Q_e = Q_{oe} + Q_{we}$ and Q_e^+ is its positive part. To obtain the right-hand side of (2.26) above, we made the assumptions [7] that

$$Q_{we}=Q_e \text{ and } Q_{oe}=0 \text{ if } Q_e\geq 0; \qquad Q_{we}=\Lambda_w Q_e \text{ and } Q_{oe}=\Lambda_o Q_e \text{ if } Q_e<0. \tag{2.27}$$

The boundary conditions for the fracture system will be chosen so as to impose no flow across $\partial\Omega$. If

$$V_o = -K\Lambda\Lambda_o\nabla P_o \qquad \text{and} \qquad V_w = -K\Lambda\Lambda_w\nabla P_w$$

are the Darcy velocities of the oil and water phases, respectively, we ask that

$$V_o\cdot\nu = V_w\cdot\nu = 0, \qquad (x,t)\epsilon\partial\Omega\times J, \tag{2.28}$$

which in turn requires that

$$-K\Lambda\Lambda_o\Lambda_w P_c'\nabla S\cdot\nu = 0, \qquad (x,t)\epsilon\partial\Omega\times J, \tag{2.29}$$

and

$$U\cdot\nu = 0, \qquad (x,t)\epsilon\partial\Omega\times J. \tag{2.30}$$

Finally, we have the initial condition

$$S(x,0) = S^0(x), \qquad x\epsilon\Omega. \tag{2.31}$$

Equations (2.15)-(2.19), (2.24)-(2.26), and (2.29)-(2.31) completely define the behavior of a waterflood in the simplified naturally fractured reservoir that we admit in this study; the pressure P is determined up to an additive constant.

3. A Weak Form of the Problem

Let $H(\mathrm{div};\Omega) = \{v \in L^2(\Omega)^d : \nabla\cdot v \in L^2(\Omega)\}$ and set

$$V = H(\mathrm{div};\Omega) \cap \{v\cdot\nu = 0 \text{ on } \partial\Omega\}, \qquad W = L^2(\Omega)/\{w \equiv \text{constant on } \Omega\}.$$

Let $(\cdot,\cdot)$ denote the $L^2(\Omega)$ or $L^2(\Omega)^d$ inner product and $(\cdot,\cdot)_i$ the corresponding inner products over Ω_i.

Assume boundedness of the coefficients in the differential system and of the components of its solution, both in the fractures and in the blocks. Then, a weak form of the system can be set up as follows. Solving for the global pressure in the fractures is equivalent to finding a map $\{U,P\}:J\to V\times W$ such that

$$([K\Lambda(S)]^{-1}U,v) - (\nabla\cdot v,P) = 0, \qquad v \in V, \tag{3.1}$$

$$(\nabla\cdot U,w) = (Q_e,w), \qquad w \in W. \tag{3.2}$$

To put the saturation equations in weak form first let

$$\overline{\psi}_i = |\Omega_i|^{-1}\int_{\Omega_i}\psi dx, \qquad D(x,S) = K\Lambda\Lambda_o\Lambda_w P_c', \qquad d(x,s) = k\lambda\lambda_o\lambda_w p_c'.$$

The equations (2.15) and (2.26) can be tested against $H^1(\Omega)$ and $H_0^1(\Omega_i)$, respectively, and our weak form of the saturation equations consists of finding the two maps $s:J\to\cup_i\{H_0^1(\Omega_i) + \sigma(\hat{S}_i(t))\}$ and $S:J\to H^1(\Omega)$ such that

$$(\varphi s_t,\zeta)_i + (d(s)\nabla s,\nabla\zeta)_i = 0, \qquad \zeta \in H_0^1(\Omega_i), \tag{3.3}$$

$$(\Phi S_t,\theta) + \Sigma_i\varphi_i\overline{s}_{ti}\hat{\theta}_i|\Omega_i| + (\Lambda_o'(S)U\cdot\nabla S,\theta) + (D(S)\nabla S,\nabla\theta) = -(\Lambda_o(S)Q_e^+,\theta), \quad \theta \in H^1(\Omega), \tag{3.4}$$

where, for simplicity, the porosity of each block has been assumed to be a constant.

Equations (3.1)-(3.4) together with the initial conditions (2.17) and (2.31) and the boundary conditions (2.16) define the weak form of the model. As mentioned in the Introduction we shall make some additional assumptions about the coefficients and the source and sink terms in the convergence analysis. The diffusion coefficients $D(S)$ and $d(s)$ vanish at $S = 1-S_{res,w}$ and $S = S_{res,o}$ and at $s = 1-s_{res,w}$ and $s = s_{res,o}$, respectively; however, they will be assumed to be bounded below and above by positive constants:

$$0 < D_1 \le D(x,S) \le D_2 < \infty, \quad 0 < d_1 \le d(x,s) \le d_2 < \infty. \tag{3.5}$$

Also, the smoothness needed of the various coefficients, the source terms, and the solution itself will be implicit in the argument.

4. A Finite Element Approximation Procedure

For $0<h_f<1$ and $0<h_m<1$, let $\mathcal{T}_{h_f}(\Omega)$ and $\mathcal{T}_{i,h_m}(\Omega_i)$ be quasiregular partitions of Ω and $\{\Omega_i\}$ into simplices or rectangles of diameters bounded by h_f and h_m, respectively. Let $\mathcal{M} = \mathcal{M}_{h_f} \subset H^1(\Omega)$ and $\mathcal{N}_i^0 = \mathcal{N}_{i,h_m}^0 \subset H_0^1(\Omega_i)$ be standard C^0 finite element spaces associated with $\mathcal{T}_{h_f}(\Omega)$ and $\mathcal{T}_{i,h_m}(\Omega_i)$, respectively, such that

$$\inf\{\|u-\theta\|_{W^{j,\rho}(\Omega)} : \theta\in\mathcal{M}\} \le C\|u\|_{W^{q,\rho}(\Omega)} h_f^{q-j}, \qquad 1\le q\le q^*+1, \quad j=0,1, \quad \rho=2,4, \tag{4.1}$$

$$\inf\{\|v-\zeta\|_{j,\Omega_i} : \zeta\in\mathcal{N}_i^0\} \le C\|v\|_{r,\Omega_i} h_m^{r-j}, \qquad 1\le r\le r^*+1, \quad j=0,1, \tag{4.2}$$

for any $u\in W^{q,\rho}(\Omega)$ and $v\in H^r(\Omega_i)\cap H_0^1(\Omega_i)$. Let $\tilde{V}_{h_f}\times\tilde{W}_{h_f}$ denote either the Raviart-Thomas-Nedelec [13], [15], the Brezzi-Douglas-Fortin-Marini [4], the Brezzi-Douglas-Marini [5] (if $d=2$), or the Brezzi-Douglas-Durán-Fortin [3] (if $d=3$) space associated with $\mathcal{T}_{h_f}(\Omega)$ of index such that V is approximated by $\tilde{V}_{h_f}$ to order q^*+1. Set $V_f = V_{h_f} = \{v\in\tilde{V}_{h_f} : v\cdot\nu=0$ on $\partial\Omega\}\subset V$ and $W_f = W_{h_f} \subset W$. It is known that

$$\inf\{\|u-v\|_0 : v\in V_f\} \le C\|u\|_q h_f^q, \qquad 0\le q\le q^*+1, \tag{4.3}$$

$$\inf\{\|\nabla\cdot(u-v)\|_0 : v\in V_f\} \le C\|\nabla\cdot u\|_q h_f^q, \qquad 0\le q\le q^{**}, \tag{4.4}$$

$$\inf\{\|p-w\|_0 : w\in W_f\} \le C\|p\|_q h_f^q, \qquad 0\le q\le q^{**}, \tag{4.5}$$

for all $\{u,p\}\in V\times W$, where $q^{**} = q^*+1$ for the first two spaces ([4], [13], [15]) and $q^{**} = q^*$ for the other two spaces ([3], [5]). Of course, one could take $V_f\times W_f$ over a quasiregular partition $\mathcal{T}_h(\Omega)$ analogous to but different from $\mathcal{T}_{h_f}(\Omega)$. In that case, the analysis of the next section will show that it would be expedient to choose the approximation order R^*+1 in such a way that h^{R^*} and $h_f^{q^*}$ are of the same order.

Let L be a positive integer and $\Delta t = T/L$, and set

$$\psi^n = \psi(n\Delta t), \qquad \partial\psi^n = (\psi^n-\psi^{n-1})/\Delta t$$

for appropriate n.

Following the ideas in [9], [10] and [1], [11], we define a finite element procedure as the solution of the following systems:

i) $\{U_h^n, P_h^n\} \epsilon V_f \times W_f$, $n=0,1,\dots,L$:

$$([K\Lambda(S_h^n)]^{-1} U_h^n, v) - (\nabla\cdot v, P_h^n) = 0, \qquad v \epsilon V_f, \tag{4.6}$$

$$(\nabla\cdot U_h^n, w) = (Q_e^n, w), \qquad w \epsilon W_f \tag{4.7}$$

ii) $S_h^n \epsilon \mathcal{M}$, $n=1,\dots,L$:

$$(\Phi \partial S_h^n, \theta) + \Sigma_i \varphi_i \partial \overline{s}_{hi}^n \hat{\theta}_i |\Omega_i| + (\Lambda_o'(S_h^{n-1}) U_h^{n-1} \cdot \nabla S_h^n, \theta) + (D(S_h^{n-1}) \nabla S_h^n, \nabla\theta)$$
$$= -(\Lambda_o(S_h^{n-1}) Q_e^{+,n}, \theta), \qquad \theta \epsilon \mathcal{M} \tag{4.8}$$

iii) $s_{1h}^n \epsilon \mathcal{N}_i^o + \sigma(\hat{S}_{h,i}^{n-1})$, $n=1,\dots,L$:

$$(\varphi\{s_{1h}^n - s_h^{n-1}\}/\Delta t, \zeta)_i + (d(s_h^{n-1}) \nabla s_{1h}^n, \nabla\zeta)_i = 0, \qquad \zeta \epsilon \mathcal{N}_i^o \tag{4.9}$$

iv) $s_{2h}^n \epsilon \mathcal{N}_i^o + \Delta t$, $n=1,\dots,L$:

$$(\varphi s_{2h}^n/\Delta t, \zeta)_i + (d(s_h^{n-1}) \nabla s_{2h}^n, \nabla\zeta)_i = 0, \qquad \zeta \epsilon \mathcal{N}_i^o. \tag{4.10}$$

where, with the notation $\gamma_i^{n-1} = \sigma'(\hat{S}_{h\,i}^{n-1})$,

v) $s_h^n \epsilon \mathcal{N}_i^o + [\sigma(\hat{S}_{h\,i}^{n-1}) + \gamma_i^{n-1}(\hat{S}_{hi}^n - \hat{S}_{h\,i}^{n-1})]$, $n=1,\dots,L$:

$$s_h^n = s_{1h}^n + \gamma_i^{n-1} \cdot \partial \hat{S}_{hi}^n \cdot s_{2h}^n. \tag{4.11}$$

Note that s_h^n satisfies the equation

$$(\varphi \partial s_h^n, \zeta)_i + (d(s_h^{n-1}) \nabla s_h^n, \nabla\zeta)_i = 0, \qquad \zeta \epsilon \mathcal{N}_i^o. \tag{4.12}$$

To start the procedure, approximate S^0 by $S_h^0 \epsilon \mathcal{M}$ in any fashion such that

$$\|S^0 - S_h^0\|_0 + h_\gamma \|S^0 - S_h^0\|_1 \le C \|S^0\|_q h_\gamma^q, \qquad 1 \le q \le q^* + 1, \tag{4.13}$$

and take

$$s_h^0 = \sigma(\hat{S}_i^0). \tag{4.14}$$

After startup, for $n=1,\dots,L$, the equations are solved in the following order. First, using S_h^{n-1} and (4.6)-(4.7), evaluate $\{U_h^{n-1}, P_h^{n-1}\}$. Next, from S_h^{n-1}, s_h^{n-1}, and (4.9)-(4.10), obtain s_{1h}^n and s_{2h}^n. Now, using U_h^{n-1}, S_h^{n-1}, s_h^{n-1}, s_{1h}^n, s_{2h}^n, (4.11), and (4.8), compute S_h^n. Finally, s_h^n itself is evaluated from s_{1h}^n, s_{2h}^n, S_h^{n-1}, S_h^n, and (4.11).

5. Analysis of the Error

Let $\eta = U-U_h$ and $\pi = P-P_h$. It is known [9] how to estimate the size of these errors in terms of the error $S-S_h$. The estimates (5.4) of [9] can be improved by an application of the ideas in [12] (in particular, the duality lemmas). Such improvements have appeared there and in [3], [4], and [5]. Combining these results with the estimates (6.2) of [9] leads to the refined estimates

$$\|\eta^n\|_0 \le C[\|U^n\|_q h_f^q + \|\tilde{U}^n\|_{L^\infty(\Omega)}\|S^n-S_h^n\|_0], \qquad 1\le q\le q^*+1, \tag{5.1}$$

$$\|\nabla\cdot\eta^n\|_0 \le C\|\nabla\cdot U^n\|_q h_f^q, \qquad 1\le q\le q^{**}, \tag{5.2}$$

$$\|\pi^n\|_0 \le C[\|P^n\|_{\max(2,q)} h_f^q + \|\tilde{U}^n\|_{L^\infty(\Omega)}\|S^n-S_h^n\|_0], \qquad 1\le q\le q^{**}, \tag{5.3}$$

where $\tilde{U}$ is a projection of U that is bounded in $L^\infty(\Omega)$ if u is, and C depends on $\|S^n\|_{W^{1,\infty}(\Omega)}$ (in (5.3)). Next, let us introduce the elliptic projections $\tilde{S}_h : J\to\mathcal{M}$ and $\tilde{s}_h : J\to\mathcal{M}_i^0+\sigma(\hat{S}_i(t))$ defined by

$$(D(S)\nabla\{S-\tilde{S}_h\},\nabla\theta) + (\Lambda_0'(S)U\cdot\nabla\{S-\tilde{S}_h\},\theta) + (\lambda\{S-\tilde{S}_h\},\theta) = 0, \quad \theta\in\mathcal{M}, \tag{5.4}$$

where for coercivity

$$\lambda \ge D_1^{-1}\|\Lambda_0'\|^2_{L^\infty(\Omega)}\|U\|^2_{L^\infty(J;L^\infty(\Omega))},$$

and

$$(d(s)\nabla\{s-\tilde{s}_h\},\nabla\zeta)_i = 0, \qquad \zeta\in\mathcal{M}_i^0, \tag{5.5}$$

$$\tilde{s}_h = s = \sigma(\hat{S}_i(t)) \text{ on } \partial\Omega_i. \tag{5.6}$$

It follows from the usual analyses of Galerkin methods for elliptic problems that

$$\|S-\tilde{S}_h\|_0 + h_f\|S-\tilde{S}_h\|_1 \le C\|S\|_q h_f^q, \qquad 1\le q\le q^*+1, \tag{5.7}$$

$$\|S_t-\tilde{S}_{ht}\|_0 \le C[\|S\|_q + \|S_t\|_q]h_f^q, \qquad 1\le q\le q^*+1, \tag{5.8}$$

$$\|S-\tilde{S}_h\|_{L^4(\Omega)} \le C\|S\|_{W^{q,4}(\Omega)} h_f^q, \qquad 1\le q\le q^*+1, \tag{5.9}$$

$$\|s-\tilde{s}_h\|_{0,\Omega_i} + h_m\|s-\tilde{s}_h\|_{1,\Omega_i} \le C\|s\|_{r,\Omega_i} h_m^r, \qquad 1\le r\le r^*+1, \tag{5.10}$$

$$\|s_t-\tilde{s}_{ht}\|_{0,\Omega_i} \le C[\|s\|_{r,\Omega_i} + \|s_t\|_{r,\Omega_i}]h_m^r, \qquad 1\le r\le r^*+1. \tag{5.11}$$

Thus, the convergence analysis reduces to obtaining optimal order bounds for the errors $\zeta_h = \tilde{S}_h-S_h$ and $\zeta_{mh} = \tilde{s}_h-s_h$.

For simplicity, choose the initial condition

$$S_h^0 = \tilde{S}_h^0.$$

the elliptic projection of the initial condition S^0, so that $\zeta_h^0 = 0$. Then, combine (3.4), (4.8), and (5.4) to obtain the relation

$$\begin{aligned}(\Phi\partial\zeta_h^n,\theta) &+ \Sigma_i\varphi_i\partial\bar{\zeta}_{mhi}^n\hat{\theta}_i|\Omega_i| + (\Lambda_0'(S_h^{n-1})U_h^{n-1}\cdot\nabla\zeta_h^n,\theta) + (D(S_h^{n-1})\nabla\zeta_h^n,\nabla\theta)\\ = &-(\Phi\{S_t^n-\partial\tilde{S}_h^n\},\theta) + (Q_e^{+,n}\{\Lambda_0(S_h^{n-1})-\Lambda_0(S^n)\},\theta) - ([D(S^n)-D(S_h^{n-1})]\nabla\tilde{S}_h^n,\nabla\theta)\\ &+ (\lambda\{S^n-\tilde{S}_h^n\},\theta) - ([\Lambda_0'(S^n)-\Lambda_0'(S_h^{n-1})]U^n\cdot\nabla\tilde{S}_h^n,\theta)\\ &- (\Lambda_0'(S_h^{n-1})\{U^n-U_h^{n-1}\}\cdot\nabla\tilde{S}_h^n,\theta) - \Sigma_i\varphi_i(\bar{s}_{ti}^n - \partial\bar{\tilde{s}}_{hi}^n)\hat{\theta}_i|\Omega_i|, \qquad \theta\in\mathfrak{M},\ 1\le n\le L. \qquad (5.12)\end{aligned}$$

Next, note that (2.17), (4.14), and (5.5)-(5.6) imply that $\zeta_{mh}^0 = 0$. Also, combining (3.3), (4.12), and (5.5) leads to the equation

$$\begin{aligned}(\varphi\partial\zeta_{mh}^n,\zeta)_i &+ (d(s_h^{n-1})\nabla\zeta_{mh}^n,\nabla\zeta)_i\\ &= -(\varphi\{s_t^n-\partial\tilde{s}_h^n\},\zeta)_i - ([d(s^n)-d(s_h^{n-1})]\nabla\tilde{s}_h^n,\nabla\zeta)_i, \qquad \zeta\in\mathfrak{N}_i^0,\ 1\le n\le L. \qquad (5.13)\end{aligned}$$

Also, it follows from (2.16), (4.11), and (5.6) that

$$\zeta_{mh}^n = \sigma(\hat{S}_i^n) - \sigma(\hat{S}_{hi}^{n-1}) - \gamma_i^{n-1}(\hat{S}_{hi}^n-\hat{S}_{hi}^{n-1}), \qquad (x,t)\in\partial\Omega_i\times J. \qquad (5.14)$$

In the analysis that will come it is convenient to rewrite the expression above. Note that

$$\sigma(\hat{S}_i^n) = \sigma(\hat{S}_{hi}^{n-1}) + \gamma_i^{n-1}(\hat{S}_i^n-\hat{S}_{hi}^{n-1}) + \delta_i^n,$$

where

$$\delta_i^n = \int_{\hat{S}_{hi}^{n-1}}^{\hat{S}_i^n} (\hat{S}_i^n - \theta)\,\sigma''(\theta)d\theta, \qquad (5.15)$$

so that (5.14) becomes

$$\begin{aligned}\zeta_{mh}^n &= \gamma_i^{n-1}(\hat{S}_i^n-\hat{S}_{hi}^n) + \delta_i^n\\ &= \gamma_i^{n-1}[\hat{S}_i^n - \hat{\tilde{S}}_{hi}^n] + \gamma_i^{n-1}\hat{\zeta}_{hi}^n + \delta_i^n, \qquad (x,t)\in\partial\Omega_i\times J. \qquad (5.16)\end{aligned}$$

Choose $\theta = \zeta_h^n$ in the error equation (5.12):

$$\begin{aligned}(\Phi\partial\zeta_h^n,\zeta_h^n) &+ \Sigma_i\varphi_i\partial\bar{\zeta}_{mhi}^n\hat{\zeta}_{hi}^n|\Omega_i| + (D(S_h^{n-1})\nabla\zeta_h^n,\nabla\zeta_h^n)\\ = &-(\Phi\{S_t^n-\partial\tilde{S}_h^n\},\zeta_h^n) + (Q_e^{+,n}\{\Lambda_0(S_h^{n-1})-\Lambda_0(S^n)\},\zeta_h^n) - ([D(S^n)-D(S_h^{n-1})]\nabla\tilde{S}_h^n,\nabla\zeta_h^n)\\ &+ (\lambda\{S^n-\tilde{S}_h^n\},\zeta_h^n) - ([\Lambda_0'(S^n)-\Lambda_0'(S_h^{n-1})]U^n\cdot\nabla\tilde{S}_h^n,\zeta_h^n) - (\Lambda_0'(S_h^{n-1})\{U^n-U_h^{n-1}\}\cdot\nabla\tilde{S}_h^n,\zeta_h^n)\\ &- \Sigma_i\varphi_i(\bar{s}_{ti}^n - \partial\bar{\tilde{s}}_{hi}^n)\hat{\zeta}_{hi}^n|\Omega_i| - (\Lambda_0'(S_h^{n-1})U_h^{n-1}\cdot\nabla\zeta_h^n,\zeta_h^n)\\ \equiv\ & T_1^n + \cdots + T_8^n, \qquad 1\le n\le L. \qquad (5.17)\end{aligned}$$

First, the left-hand side of (5.17) is bounded below by

$$(2\Delta t)^{-1}[(\Phi\xi_h^n,\xi_h^n) - (\Phi\xi_h^{n-1},\xi_h^{n-1})] + D_1\|\nabla\xi_h^n\|_0^2 + \Sigma_i\varphi_i\partial\bar{\xi}_{hi}^n\,\hat{\xi}_{hi}^n|\Omega_i|.$$

Then, notice that all of the terms on the right-hand side of (5.17) with the exception of the last two can be bounded using the ideas in [9], [10], so that here we just briefly summarize the bounds for the first six terms. Now, (5.8) shows that

$$\begin{aligned}|T_1^n| &\le C[\|S_t^n-\partial S^n\|_0 + \|\partial(S-\tilde{S}_h)^n\|_0]\|\xi_h^n\|_0\\ &\le C\{\Delta t\|S_{tt}\|^2_{L^2(J_n;L^2(\Omega))}\\ &\qquad + (\Delta t)^{-1}[\|S\|^2_{L^2(J_n;H^q(\Omega))} + \|S_t\|^2_{L^2(J_n;H^q(\Omega))}]h_f^{2q} + \|\xi_h^n\|_0^2\}, \qquad 1\le q\le q^*+1,\end{aligned}$$

where $J_n = ((n-1)\Delta t, n\Delta t]$. With S suitably smooth, known L^∞ estimates for Galerkin methods for elliptic problems imply that $\|\nabla\tilde{S}_h^n\|_{L^\infty(\Omega)} \le C$. Hence, since $S^n - S_h^{n-1} = (S^n - S^{n-1}) + (S^{n-1} - \tilde{S}_h^{n-1}) + \xi_h^{n-1}$, it follows from (5.7) that

$$\begin{aligned}&|T_2^n| + |T_3^n| + |T_4^n| + |T_5^n|\\ &\quad\le C[\|S_h^{n-1} - S^n\|_0\|\xi_h^n\|_0 + \|S^n-S_h^{n-1}\|_0\|\nabla\tilde{S}_h^n\|_{L^\infty(\Omega)}\|\nabla\xi_h^n\|_0\\ &\qquad + \|S^n - \tilde{S}_h^n\|_0\|\xi_h^n\|_0 + \|S^n-S_h^{n-1}\|_0\|U^n\|_{L^\infty(\Omega)}\|\nabla\tilde{S}_h^n\|_{L^\infty(\Omega)}\|\xi_h^n\|_0]\\ &\quad\le C[\Delta t\|S_t\|^2_{L^2(J_n;L^2(\Omega))} + \|S\|^2_{L^\infty(J;H^q(\Omega))}h_f^{2q} + \|\xi_h^{n-1}\|_0^2 + \|\xi_h^n\|_0^2] + \epsilon\|\nabla\xi_h^n\|_0^2,\\ &\qquad\qquad 1\le q\le q^*+1,\end{aligned}$$

for ϵ as small as we please. To estimate T_6^n, write $U^n-U_h^{n-1}$ as $(U^n-U^{n-1}) + \eta^{n-1}$ and apply (5.1) and (5.7):

$$\begin{aligned}|T_6^n| &\le C(\|U^n-U^{n-1}\|_0 + \|\eta^{n-1}\|_0)\|\nabla\tilde{S}_h^n\|_{L^\infty(\Omega)}\|\xi_h^n\|_0\\ &\le C\{\Delta t\|U_t\|^2_{L^2(J_n;L^2(\Omega))} + [\|U\|^2_{L^\infty(J;H^q(\Omega))} + \|S\|^2_{L^\infty(J;H^q(\Omega))}]h_f^{2q}\\ &\qquad + \|\xi_h^{n-1}\|_0^2 + \|\xi_h^n\|_0^2\}, \qquad 1\le q\le q^*+1.\end{aligned}$$

We take up now the estimation of the new terms T_7^n and T_8^n. Using (5.11), we see that

$$\begin{aligned}|T_7^n| &\le C\Sigma_i\{(\bar{s}_{ti}^n - \partial\bar{\tilde{s}}_{hi}^n)^2 + (\hat{\xi}_{hi}^n)^2\}|\Omega_i|\\ &\le C[\|s_t^n-\partial\tilde{s}_h^n\|^2_{0,\Omega_m} + \|\xi_h^n\|_0^2]\\ &\le C\{\Delta t\|s_{tt}\|^2_{L^2(J_n;L^2(\Omega_m))}\\ &\qquad + (\Delta t)^{-1}[\|s\|^2_{L^2(J_n;H^r(\Omega_m))} + \|s_t\|^2_{L^2(J_n;H^r(\Omega_m))}]h_m^{2r} + \|\xi_h^n\|_0^2\}, \qquad 1\le r\le r^*+1.\end{aligned}$$

Let $\Pi:V\to V_{h_f}$ denote the vector projection operator defined in one of [3], [4], [5],

[13], or [15], as indicated by the choice of the mixed finite element space made in setting up our numerical technique. We shall employ some of the approximation and boundedness properties of Π. In particular,

$$\|U - \Pi U\|_0 \le C\|U\|_q h_f^q, \qquad 1 \le q \le q^* + 1,$$

and ΠU is bounded in $L^\infty(\Omega)$. Then,

$$\begin{aligned} |T_8^n| &\le |(\Lambda_0'(S_h^{n-1})\Pi U^{n-1} \cdot \nabla \xi_h^n, \xi_h^n)| + |(\Lambda_0'(S_h^{n-1})(\Pi U^{n-1} - U_h^{n-1}) \cdot \nabla \xi_h^n, \xi_h^n)| \\ &\le C[\|\nabla \xi_h^n\|_0 \|\xi_h^n\|_0 + \|\Pi U^{n-1} - U_h^{n-1}\|_{L^4(\Omega)} \|\nabla \xi_h^n\|_0 \|\xi_h^n\|_{L^4(\Omega)}]. \end{aligned}$$

Quasiregularity of the partition of Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ implies the inverse property

$$\|\Pi U^{n-1} - U_h^{n-1}\|_{L^4(\Omega)} \le C h_f^{-3/4} \|\Pi U^{n-1} - U_h^{n-1}\|_0.$$

So, with (5.1) and (5.7),

$$\begin{aligned} \|\Pi U^{n-1} - U_h^{n-1}\|_{L^4(\Omega)} &\le C h_f^{-3/4} [\|U^{n-1} - \Pi U^{n-1}\|_0 + \|\eta^{n-1}\|_0] \\ &\le C h_f^{-3/4} \{[\|U^{n-1}\|_q + \|S^{n-1}\|_q] h_f^q + \|\xi_h^{n-1}\|_0\}, \qquad 1 \le q \le q^* + 1. \end{aligned}$$

Also, Hölder's inequality and the Sobolev embedding theorem imply that

$$\|\xi_h^n\|_{L^4(\Omega)} \le \|\xi_h^n\|_0^{1/4} (\|\xi_h^n\|_{L^6(\Omega)})^{3/4} \le C \|\xi_h^n\|_0^{1/4} \|\xi_h^n\|_1^{3/4}.$$

Thus,

$$\begin{aligned} |T_8^n| &\le C\{\|\xi_h^n\|_0 \|\nabla \xi_h^n\|_0 + h_f^{-3/4}[(\|U^{n-1}\|_q + \|S^{n-1}\|_q) h_f^q + \|\xi_h^{n-1}\|_0] \|\xi_h^n\|_0^{1/4} \|\xi_h^n\|_1^{7/4}\} \\ &\le C\{1 + \{h_f^{-3/4}[(\|U\|_{L^\infty(J;H^q(\Omega))} + \|S\|_{L^\infty(J;H^q(\Omega))}) h_f^q + \|\xi_h^{n-1}\|_0]\}^8\} \|\xi_h^n\|_0^2 \\ &\qquad + \epsilon \|\nabla \xi_h^n\|_0^2, \qquad\qquad 1 \le q \le q^* + 1. \end{aligned}$$

For convenience in notation set

$$\mathbf{T}_1^2 = \|S_t\|^2_{L^2(J;L^2(\Omega))} + \|S_{tt}\|^2_{L^2(J;L^2(\Omega))} + \|s_{tt}\|^2_{L^2(J;L^2(\Omega_m))} + \|U_t\|^2_{L^2(J;L^2(\Omega))}, \tag{5.18}$$

$$\mathbf{F}_1^2(q) = \|S\|^2_{L^\infty(J;H^q(\Omega))} + \|S_t\|^2_{L^2(J;H^q(\Omega))} + \|U\|^2_{L^\infty(J;H^q(\Omega))}, \tag{5.19}$$

$$\mathbf{M}_1^2(r) = \|s\|^2_{L^2(J;H^r(\Omega_m))} + \|s_t\|^2_{L^2(J;H^r(\Omega_m))}. \tag{5.20}$$

Multiply (5.17) by Δt and sum from $n=1$ to $n=\beta$. Apply the estimates obtained above to see that

$$\tfrac{1}{2}(\Phi\xi_h^\beta,\xi_h^\beta) + (D_1-2\epsilon)\Sigma_{n=1}^\beta\|\nabla\xi_h^n\|_0^2\Delta t + \Sigma_{n=1}^\beta\Sigma_i\varphi_i\partial\bar\xi_{mhi}^n\hat\xi_{hi}^n|\Omega_i|\Delta t$$
$$\le C\{\mathbb{T}_1^2(\Delta t)^2 + \mathbb{F}_1^2(q)h_f^{2q} + \mathbb{M}_1^2(r)h_m^{2r}$$
$$+ \Sigma_{n=1}^\beta\{1 + [h_f^{-3/4}(\mathbb{F}_1(q)h_f^q + \|\xi_h^{n-1}\|_0)]^8\}\|\xi_h^n\|_0^2\Delta t\},\quad 1\le q\le q^*+1,\ \ 1\le r\le r^*+1. \tag{5.21}$$

Next, choose the test function $\zeta = \{\xi_{mh}^n - [\gamma_i^{n-1}(\hat\xi_{hi}^n + (\hat S_i^n-\hat{\tilde S}_{hi}^n)) + \delta_i^n]\}[\gamma_i^{n-1}]^{-1} \epsilon\ \eta_i^0$ in (5.13), multiply by Δt, and sum i and n to obtain the equation

$$\Sigma_{n=1}^\beta\Sigma_i\{([\gamma_i^{n-1}]^{-1}\varphi\partial\xi_{mh}^n,\xi_{mh}^n)_i - \varphi_i\partial\bar\xi_{mhi}^n\hat\xi_{hi}^n|\Omega_i| + ([\gamma_i^{n-1}]^{-1}d(s_h^{n-1})\nabla\xi_{mh}^n,\nabla\xi_{mh}^n)_i\}\Delta t$$
$$= \Sigma_{n=1}^\beta\Sigma_i\{\varphi_i\partial\bar\xi_{m\,hi}^n(\hat S_i^n-\hat{\tilde S}_{hi}^n)|\Omega_i| - (\varphi[\gamma_i^{n-1}]^{-1}\{s_t^n-\partial\tilde s_h^n\},\xi_{mh}^n)_i + \varphi_i(\bar s_{t\,i}^n-\partial\bar{\tilde s}_{hi}^n)\hat\xi_{hi}^n|\Omega_i|$$
$$+ \varphi_i(\bar s_{t\,i}^n-\partial\bar{\tilde s}_{hi}^n)(\hat S_i^n-\hat{\tilde S}_{hi}^n)|\Omega_i| - ([\gamma_i^{n-1}]^{-1}\{d(s^n)-d(s_h^{n-1})\}\nabla\tilde s_h^n,\nabla\xi_{mh}^n)_i$$
$$+ [\gamma_i^{n-1}]^{-1}\varphi_i\partial\bar\xi_{mhi}^n\delta_i^n|\Omega_i| + [\gamma_i^{n-1}]^{-1}\varphi_i(\bar s_{t\,i}^n-\partial\bar{\tilde s}_{hi}^n)\delta_i^n|\Omega_i|\}\Delta t$$
$$\equiv t_1^\beta + \cdots + t_7^\beta . \tag{5.22}$$

Our implicit assumptions on the smoothness of the coefficients must include the boundedness of σ', $[\sigma']^{-1}$, and σ'' in order to deal with γ_i^{n-1} and δ_i^n. These assumptions are consistent with the assumption that the relative permeabilities are bounded below by a positive constant. The left-hand side of (5.22) can then be bounded below by

$$C_1[\|\xi_{mh}^\beta\|_{0,\Omega_m}^2 + \Sigma_{n=1}^\beta\|\nabla\xi_{mh}^n\|_{0,\Omega_m}^2\Delta t] - \Sigma_{n=1}^\beta\Sigma_i\varphi_i\partial\bar\xi_{mhi}^n\hat\xi_{hi}^n\Delta t.$$

Set

$$\mathbb{T}^2 = \mathbb{T}_1^2 + \|s_t\|^2_{L^2(J;L^2(\Omega_m))} + \|S_t\|^4_{L^4(J;L^4(\Omega))}, \tag{5.23}$$
$$\mathbb{F}^2(q) = \mathbb{F}_1^2(q) + \|S\|^4_{L^\infty(J;W^{q,4}(\Omega))}, \tag{5.24}$$
$$\mathbb{M}^2(r) = \mathbb{M}_1^2(r) + \|s\|^2_{L^\infty(J;H^r(\Omega_m))}. \tag{5.25}$$

We shall now bound the terms t_j^β, $1\le j\le 7$.

First, summation by parts on t_1^β gives

$$|t_1^\beta| = |\ \Sigma_i\{\varphi_i\bar\xi_{mhi}^\beta(\hat S_i^\beta-\hat{\tilde S}_{hi}^\beta) - \Sigma_{n=1}^\beta\varphi_i\bar\xi_{mhi}^{n-1}(\partial\hat S_i^n-\partial\hat{\tilde S}_{hi}^n)\Delta t\}|\Omega_i|\ |$$
$$\le \epsilon\|\xi_{mh}^\beta\|_{0,\Omega_m}^2 + C[\mathbb{F}_1^2(q)h_f^{2q} + \Sigma_{n=1}^\beta\|\xi_{mh}^n\|_{0,\Omega_m}^2\Delta t],\qquad 1\le q\le q^*+1,$$

by (5.7) and (5.8). The next three terms are bounded easily with (5.7) and (5.11) by

$$
\begin{aligned}
&|t_2^\beta| + |t_3^\beta| + |t_4^\beta| \\
&\quad \le C\Sigma_{n=1}^{\beta}\Sigma_i\{\|s_t^n-\partial\tilde{s}_h^n\|_{0,\Omega_i}\|\xi_{mh}^n\|_{0,\Omega_i} \\
&\qquad\qquad + |\bar{s}_{t\,i}^n-\partial\bar{\tilde{s}}_{h\,i}^n||\hat{\xi}_{h\,i}^n||\Omega_i| + |\bar{s}_{t\,i}^n-\partial\bar{\tilde{s}}_{h\,i}^n||\hat{S}_i^n-\hat{\tilde{S}}_{h\,i}^n||\Omega_i|\}\Delta t \\
&\quad \le C\Sigma_{n=1}^{\beta}\{\|s_t^n-\partial\tilde{s}_h^n\|_{0,\Omega_m}^2 + \|S^n-\tilde{S}_h^n\|_0^2 + \|\xi_{mh}^n\|_{0,\Omega_m}^2 + \|\xi_h^n\|_0^2\}\Delta t \\
&\quad \le C[\mathbf{T}_1^2(\Delta t)^2 + \mathbf{M}_1^2(r)h_m^{2r} + \mathbf{F}_1^2(q)h_f^{2q} \\
&\qquad\qquad + \Sigma_{n=1}^{\beta}\|\xi_{mh}^n\|_{0,\Omega_m}^2\Delta t + \Sigma_{n=1}^{\beta}\|\xi_h^n\|_0^2\Delta t], \qquad 1\le r\le r^*+1, \quad 1\le q\le q^*+1.
\end{aligned}
$$

In order to estimate the term t_5^β, we need the known L^∞ estimates for Galerkin methods for elliptic problems that imply the bound $\|\nabla\tilde{s}_h^n\|_{L^\infty(\Omega_m))} \le C$. So, (5.10) gives the bound

$$
\begin{aligned}
|t_5^\beta| &\le C\Sigma_{n=1}^{\beta}\Sigma_i\|s^n-s_h^{n-1}\|_{0,\Omega_i}\|\nabla\tilde{s}_h^n\|_{L^\infty(\Omega_i)}\|\nabla\xi_{mh}^n\|_{0,\Omega_i}\Delta t \\
&\le \epsilon\Sigma_{n=1}^{\beta}\|\nabla\xi_{mh}^n\|_{0,\Omega_m}^2\Delta t + C[\mathbf{T}^2(\Delta t)^2 + \mathbf{M}^2(r)h_m^{2r} + \Sigma_{n=1}^{\beta}\|\xi_{mh}^n\|_{0,\Omega_m}^2\Delta t], \quad 1\le r\le r^*+1.
\end{aligned}
$$

The sixth term on the right-hand side of (5.22) must be treated in a different way since we do not have a direct estimate of either $\partial\xi_{mh}^n$ or $\partial\xi_h^n$. We will use the naive estimate

$$
\|\partial\xi_{mh}^n\|_{0,\Omega_m} \le [\|\xi_{mh}^n\|_{0,\Omega_m} + \|\xi_{mh}^{n-1}\|_{0,\Omega_m}](\Delta t)^{-1}.
$$

This loss of a power of Δt will be offset by δ_i^n which, as can be seen from the expression (5.15), is bounded by the square of the error:

$$
\begin{aligned}
\delta_i^n &\le C|\hat{S}_i^n - \hat{S}_{h\,i}^{n-1}|^2 \\
&\le C[|\hat{S}_i^n-\hat{S}_i^{n-1}|^2 + |\hat{S}_i^{n-1}-\hat{\tilde{S}}_{h\,i}^{n-1}|^2 + |\hat{\xi}_{h\,i}^{n-1}|^2] \\
&\le C\{|\Omega_i|^{-1/2}[\|(S^n-S^{n-1})\chi_i^{1/4}\|^2_{L^4(\Omega)} + \|(S^{n-1}-\tilde{S}_{h\,i}^{n-1})\chi_i^{1/4}\|^2_{L^4(\Omega)}] \\
&\qquad + |\Omega_i|^{-1}\|\xi_h^{n-1}\chi_i^{1/2}\|_0^2\}.
\end{aligned}
$$

Unfortunately, the induction argument to be given below will just fail unless we are more careful. Note that, as indicated in [8], there exists $z\in\mathcal{N}_i^0+1$ such that

$$
\|z\|_{0,\Omega_i}^2 \le C|\partial\Omega_i|h_m \qquad \text{and} \qquad \|z\|_{1,\Omega_i}^2 \le C|\partial\Omega_i|h_m^{-1}. \tag{5.26}
$$

Choose $\zeta = [\gamma_i^{n-1}]^{-1}(1-z)\delta_i^n \in \mathcal{N}_i^0$ in (5.13), multiply by Δt, and add over i and n to obtain the relation

$$
\begin{aligned}
t_6^\beta + t_7^\beta &= \Sigma_{n=1}^\beta \Sigma_i [\gamma_i^{n-1}]^{-1} [\, \varphi_i \partial \bar{\xi}_{mhi}^n + \varphi_i (\overline{s}_{t\,i}^{\,n} - \partial \overline{\tilde{s}}_{hi}^{\,n}) \,] \,\delta_i^n |\Omega_i| \Delta t \\
&= \Sigma_{n=1}^\beta \Sigma_i [\gamma_i^{n-1}]^{-1} \{ (\varphi \partial \xi_{mh}^n, z)_i + (d(s_h^{n-1}) \nabla \xi_{mh}^n, \nabla z)_i \\
&\qquad + (\varphi \{ s_t^n - \partial \tilde{s}_h^n \}, z)_i + (\{ d(s^n) - d(s_h^{n-1}) \} \nabla \tilde{s}_h^n, \nabla z)_i \} \,\delta_i^n \Delta t .
\end{aligned}
$$

Hence, with (5.7) and (5.9)-(5.11),

$$
\begin{aligned}
&|t_6^\beta| + |t_7^\beta| \\
&\quad \le C \Sigma_{n=1}^\beta \Sigma_i \{ \| \partial \xi_{mh}^n \|_{0,\Omega_i} \| z \|_{0,\Omega_i} + \| \nabla \xi_{mh}^n \|_{0,\Omega_i} \| \nabla z \|_{0,\Omega_i} + \| s_t^n - \partial \tilde{s}_h^n \|_{0,\Omega_i} \| z \|_{0,\Omega_i} \\
&\qquad\qquad + \| s^n - s_h^{n-1} \|_{0,\Omega_i} \| \nabla s_h^n \|_{L^\infty(\Omega_i)} \| \nabla z \|_{0,\Omega_i} \} \,|\delta_i^n| \Delta t \\
&\quad \le \epsilon \Sigma_{n=1}^\beta [\, \| \xi_{mh}^n \|_{0,\Omega_m}^2 + \| \xi_{mh}^{n-1} \|_{0,\Omega_m}^2 + \| \nabla \xi_{mh}^n \|_{0,\Omega_m}^2 + \| s_t^n - \partial \tilde{s}_h^n \|_{0,\Omega_m}^2 \\
&\qquad\qquad + \| s^n - s_h^{n-1} \|_{0,\Omega_m}^2 \,] \Delta t \\
&\qquad + C \Sigma_{n=1}^\beta \Sigma_i |\partial \Omega_i| \{ h_m [(\Delta t)^{-2} + 1] + h_m^{-1} \} (\delta_i^n)^2 \Delta t \\
&\quad \le C [\, \mathbb{T}^2 (\Delta t)^2 + \mathbb{M}^2(r) h_m^{2r} + \Sigma_{n=1}^\beta \| \xi_{mh}^n \|_{0,\Omega_m}^2 \Delta t \,] + \epsilon \Sigma_{n=1}^\beta \| \nabla \xi_{mh}^n \|_{0,\Omega_m}^2 \Delta t \\
&\qquad + C \Sigma_{n=1}^\beta \Sigma_i |\partial \Omega_i| h_m ((\Delta t)^{-2} + h_m^{-2}) \{ |\Omega_i|^{-1} [\, \| (S^n - S^{n-1}) \chi_i^{1/4} \|_{L^4(\Omega)}^4 \\
&\qquad\qquad + \| (S^{n-1} - \tilde{S}_{hi}^{n-1}) \chi_i^{1/4} \|_{L^4(\Omega)}^4 \,] + |\Omega_i|^{-2} \| \xi_h^{n-1} \|_0^2 \| \xi_h^{n-1} \chi_i^{1/2} \|_0^2 \} \\
&\quad \le C [\, \mathbb{T}^2 (\Delta t)^2 + \mathbb{M}^2(r) h_m^{2r} + \Sigma_{n=1}^\beta \| \xi_{mh}^n \|_{0,\Omega_m}^2 \Delta t \,] + \epsilon \Sigma_{n=1}^\beta \| \nabla \xi_{mh}^n \|_{0,\Omega_m}^2 \Delta t \\
&\qquad + C ((\Delta t)^{-2} + h_m^{-2}) \{ C^{**} [\, \mathbb{T}^2 (\Delta t)^4 + \mathbb{F}^2(q) h_f^{4q} \,] + C^* h_m \Sigma_{n=1}^\beta \| \xi_h^{n-1} \|_0^4 \Delta t \}, \\
&\hspace{20em} 1 \le r \le r^* + 1, \quad 1 \le q \le q^* + 1,
\end{aligned}
$$

where

$$
C^* = \max_i |\partial \Omega_i| |\Omega_i|^{-2}, \qquad C^{**} = h_m \max_i |\partial \Omega_i| |\Omega_i|^{-1} .
$$

The constant C^* is fixed but large; it will not appear in the final error estimates. The constant C^{**} is of moderate size.

Set

$$
\mathbb{E}^2(q,r) = \mathbb{T}^2 (\Delta t)^2 [\, 1 + (\Delta t)^2 h_m^{-2} \,] + \mathbb{F}^2(q) h_f^{2q} [\, 1 + h_f^{2q} ((\Delta t)^{-2} + h_m^{-2}) \,] + \mathbb{M}^2(r) h_m^{2r} .
$$

Then the bounds for t_j^β, $1 \le j \le 7$, applied to (5.22) imply that the following inequality holds:

$$
\begin{aligned}
&(C_1 - 2\epsilon) [\, \| \xi_{mh}^\beta \|_{0,\Omega_m}^2 + \Sigma_{n=1}^\beta \| \nabla \xi_{mh}^n \|_{0,\Omega_m}^2 \Delta t \,] - \Sigma_{n=1}^\beta \Sigma_i \varphi_i \, \partial \bar{\xi}_{mhi}^n \, \hat{\xi}_{hi}^n \, |\Omega_i| \, \Delta t \\
&\quad \le C_2 [\, \mathbb{E}^2(q,r) + \Sigma_{n=1}^\beta (\| \xi_{mh}^n \|_{0,\Omega_m}^2 + \| \xi_h^n \|_0^2) \Delta t \\
&\qquad + h_m C^* ((\Delta t)^{-2} + h_m^{-2}) \Sigma_{n=1}^\beta \| \xi_h^{n-1} \|_0^4 \Delta t \,], \quad 1 \le q \le q^* + 1, \quad 1 \le r \le r^* + 1. \qquad (5.27)
\end{aligned}
$$

Thus, adding (5.21) and (5.27) we see that

$$
\begin{aligned}
&\|\xi_h^\beta\|_0^2 + \|\xi_{mh}^\beta\|_{0,\Omega_m}^2 + \Sigma_{n=1}^\beta \left(\|\nabla\xi_h^n\|_0^2 + \|\nabla\xi_{mh}^n\|_{0,\Omega_m}^2 \right) \Delta t \\
&\qquad \leq C_3 \{ \mathbb{E}^2(q,r) + \Sigma_{n=1}^\beta \left(\|\xi_h^n\|_0^2 + \|\xi_{mh}^n\|_{0,\Omega_m}^2 \right) \Delta t \\
&\qquad\qquad + \Sigma_{n=1}^\beta \left[h_f^{-3/4} (\mathbb{F}_1(q) h_f^q + \|\xi_h^{n-1}\|_0) \right]^8 \|\xi_h^n\|_0^2 \Delta t \\
&\qquad\qquad + h_m C^* [((\Delta t)^{-2} + h_m^{-2}) \; \Sigma_{n=1}^\beta \|\xi_h^{n-1}\|_0^4 \Delta t \} \\
&\qquad \leq C_4 \{ \mathbb{E}^2(q,r) + \Sigma_{n=1}^\beta \|\xi_{mh}^n\|_{0,\Omega_m}^2 \Delta t \\
&\qquad\qquad + \tfrac{1}{2} [1 + (h_f^{-3/2} (\mathbb{F}_1^2(q) h_f^{2q} + \max_{n \leq \beta-1} \|\xi_h^n\|_0^2))^4 \\
&\qquad\qquad\qquad + h_m C^* [((\Delta t)^{-2} + h_m^{-2}) \max_{n \leq \beta-1} \|\xi_h^n\|_0^2] \; \Sigma_{n=1}^\beta \|\xi_h^n\|_0^2 \Delta t \}, \\
&\qquad\qquad\qquad\qquad 1 \leq q \leq q^* + 1, \quad 1 \leq r \leq r^* + 1. \qquad (5.28)
\end{aligned}
$$

Let us make the induction hypothesis that

$$
\begin{aligned}
&\max_{n \leq \beta-1} \left(\|\xi_h^n\|_0^2 + \|\xi_{mh}^n\|_{0,\Omega_m}^2 \right) + \Sigma_{n=1}^{\beta-1} \left(\|\nabla\xi_h^n\|_0^2 + \|\nabla\xi_{mh}^n\|_{0,\Omega_m}^2 \right) \Delta t \leq C_5 \mathbb{E}^2(q,r), \\
&\qquad\qquad\qquad 1 \leq q \leq q^* + 1, \quad 1 \leq r \leq r^* + 1, \qquad (5.29)
\end{aligned}
$$

where C_5 is fixed and defined below. Note that since $\xi_h^0 = \xi_{m\,h}^0 = 0$, (5.29) holds trivially for $\beta = 1$. Using (5.29), (5.28) becomes

$$
\begin{aligned}
&\|\xi_h^\beta\|_0^2 + \|\xi_{mh}^\beta\|_{0,\Omega_m}^2 + \Sigma_{n=1}^\beta \left(\|\nabla\xi_h^n\|_0^2 + \|\nabla\xi_{mh}^n\|_{0,\Omega_m}^2 \right) \Delta t \\
&\qquad \leq C_4 \{ \mathbb{E}^2(q,r) + \Sigma_{n=1}^\beta \|\xi_{mh}^n\|_{0,\Omega_m}^2 \Delta t \\
&\qquad\qquad + \tfrac{1}{2} [1 + (h_f^{-3/2} (\mathbb{F}_1^2(q) h_f^{2q} + C_5 \mathbb{E}^2(q,r)))^4 \\
&\qquad\qquad\qquad + h_m C^* [((\Delta t)^{-2} + h_m^{-2}) C_5 \mathbb{E}^2(q,r)] \; \Sigma_{n=1}^\beta \|\xi_h^n\|_0^2 \Delta t \}, \\
&\qquad\qquad\qquad\qquad 1 \leq q \leq q^* + 1, \quad 1 \leq r \leq r^* + 1.
\end{aligned}
$$

Now let us assume that

$$\{ h_f^{-3/2} + h_m [(\Delta t)^{-2} + h_m^{-2}] \} \mathbb{E}^2(q,r) \longrightarrow 0$$

as Δt, h_f, and $h_m \longrightarrow 0$. This will be satisfied if Δt, h_f^q, and h_m^r are all of the same order as they tend to zero. Choose Δt, h_f, and h_m so small that

$$\{ h_f^{-3/2} [\mathbb{F}_1^2(q) h_f^{2q} + C_5 \mathbb{E}^2(q,r)] \}^4 + h_m C^* [((\Delta t)^{-2} + h_m^{-2}) C_5 \mathbb{E}^2(q,r) \leq 1.$$

Then it follows that

$$\|\xi_h^\beta\|_0^2 + \|\xi_{mh}^\beta\|_{0,\Omega_m}^2 + \Sigma_{n=1}^{\beta} \left(\|\nabla\xi_h^n\|_0^2 + \|\nabla\xi_{mh}^n\|_{0,\Omega_m}^2 \right) \Delta t$$
$$\le C_4 \{ \mathbb{E}^2(q,r) + \Sigma_{n=1}^{\beta} \left(\|\xi_h^n\|_0^2 + \|\xi_{mh}^n\|_{0,\Omega_m}^2 \right) \Delta t \}.$$

Gronwall's inequality applied to this expression implies that

$$\|\xi_h^\beta\|_0^2 + \|\xi_{mh}^\beta\|_{0,\Omega_m}^2 + \Sigma_{n=1}^{\beta} \left(\|\nabla\xi_h^n\|_0^2 + \|\nabla\xi_{mh}^n\|_{0,\Omega_m}^2 \right) \Delta t \le C_6 \mathbb{E}^2(q,r),$$

with

$$C_6 = C_4(1 - C_4\Delta t)^{-T/\Delta t} \le 2C_4 e^{TC_4} \equiv C_5,$$

for Δt not too large, so that the induction argument is complete and (5.29) holds for any $(\beta-1)$ between 1 and L.

The following theorem is a combination of (5.1)-(5.3), (5.7), (5.10), and (5.29).

Theorem. Assuming the nondegeneracy condition (3.5) and sufficient smoothness of the various coefficients, the source terms, and the solution itself, if

$$\Delta t,\ h_f,\ \text{and}\ h_m \longrightarrow 0$$

in such a way that Δt, h_f^q, and h_m^r are all of the same order for some q and r such that $1 \le q \le q^* + 1$ and $1 \le r \le r^* + 1$, then the error generated by the procedure (4.6)-(4.14) satisfies the estimates

$$\max_{0 \le n \le L} [\, \|S^n - S_h^n\|_0 + \|s^n - s_h^n\|_{0,\Omega_m} + \|U^n - U_h^n\|_0 \,] \le C[\, \mathbb{T}\Delta t + \mathbb{F}(q) h_f^q + \mathbb{M}(r) h_m^r \,],$$

$$\left[\sum_{n=0}^{L} \|\nabla(S^n - S_h^n)\|_0^2 \,\Delta t \right]^{1/2} \le C[\, \mathbb{T}\Delta t + \mathbb{F}(q) h_f^{q-1} + \mathbb{M}(r) h_m^r \,],$$

and

$$\left[\sum_{n=0}^{L} \|\nabla(s^n - s_h^n)\|_{0,\Omega_m}^2 \,\Delta t \right]^{1/2} \le C[\, \mathbb{T}\Delta t + \mathbb{F}(q) h_f^q + \mathbb{M}(r) h_m^{r-1} \,],$$

where $\mathbb{T}$, $\mathbb{F}(q)$, and $\mathbb{M}(r)$ are defined in (5.18)-(5.20) and (5.23)-(5.25). Moreover, if $1 \le q \le q^{**}$, then

$$\max_{0 \le n \le L} \|\nabla\cdot(U^n - U_h^n)\|_0 \le C \|\nabla\cdot U\|_{L^\infty(J;H^q(\Omega))} h_f^q$$

and

$$\max_{0 \le n \le L} \|P^n - P_h^n\|_0 \le C[\, \mathbb{T}\Delta t + (\|P\|_{L^\infty(J;L^2(\Omega))} + \mathbb{F}(q)) h_f^q + \mathbb{M}(r) h_m^r \,].$$

By setting $\varphi \equiv 0$, a finite element procedure for an unfractured reservoir Ω is defined in Section 4. This procedure satisfies the above theorem with the matrix quantities deleted from it. Of course, the proof could be substantially simplified in this case.

6. Some Notation

Symbol (fracture)	Symbol (matrix)	Meaning
K	k	permeability
$K_{r\theta}$	$k_{r\theta}$	relative permeability of the θ phase
P	p	Chavent's global pressure
P_c	p_c	capillary pressure (oil minus water pressure)
P_θ	p_θ	pressure of the θ phase
Q_e	—	total external volumetric source
$Q_{\theta e}$	—	external θ source
S	s	oil saturation
$S_{res,\theta}$	$s_{res,\theta}$	residual θ saturation
S_θ	s_θ	saturation of the θ phase
U	$u\ (= 0)$	total flow rate ("global velocity")
V_θ	v_θ	Darcy velocity of the θ phase
θ		oil (o) or water (w)
Φ	φ	porosity ($\varphi = \varphi_i$ on Ω_i is assumed to be a constant)
μ_θ		viscosity of the θ phase

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