

THE TVD-PROJECTION METHOD FOR SOLVING IMPLICIT  
NUMERICAL SCHEMES FOR SCALAR CONSERVATION LAWS:  
A NUMERICAL STUDY OF A SIMPLE CASE

By

A. BOURGEAT

AND

B. COCKBURN

IMA Preprint Series # 311

April 1987

**INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS**  
**UNIVERSITY OF MINNESOTA**  
514 Vincent Hall  
206 Church Street S.E.  
Minneapolis, Minnesota 55455

THE TVD-PROJECTION METHOD FOR SOLVING IMPLICIT NUMERICAL SCHEMES FOR SCALAR  
CONSERVATION LAWS: A NUMERICAL STUDY OF A SIMPLE CASE

A. Bourgeat\* and B. Cockburn\*\*

**Abstract**

The stability of Newton's method applied to implicit numerical schemes for scalar conservation laws requires an upper bound on the size of the time steps. We introduce a simple locally defined projection, the TVD-projection, and show how to use it to obtain an always stable extension of Newton's method. A numerical study of this method applied to the Godunov's implicit scheme is presented.

---

\* Université de Saint-Etienne, Fac. des Sciences et Tech., 23, rue Dr P. Michelon, 42023 St. Etienne Cedex 2, France. This work was done in part when this author was visiting Argonne Laboratory in September'86.

\*\* IMA, University of Minnesota, 514 Vincent Hall, 206 Church Street SE, Minneapolis, MN 55455, USA.

A. Bourgeat and B. Cockburn

**Introduction**

In this paper we consider the problem of actually solving finite difference implicit schemes for the following boundary value problem

$$(1.1a) \quad \partial_t u + \partial_x f(u) = 0 \quad \text{in } (0,T) \times (0,1) \quad ,$$

$$(1.1b) \quad u(x=0) = b_0 \quad \text{on } (0,T) \quad ,$$

$$(1.1c) \quad u(x=1) = b_1 \quad \text{on } (0,T) \quad ,$$

$$(1.1d) \quad u(t=0) = u_0 \quad \text{on } (0,1) \quad ,$$

where  $f \in C^1$ ,  $b_0$  and  $b_1 \in BV(0,T)$ , and  $u_0 \in BV(0,1)$ . The space  $BV(\Omega)$  is the space of functions of finite total variation in  $\Omega$ . It is very well known that the main difficulty concerning the resolution of an implicit scheme is to solve at each time step the nonlinear equation defining the approximate solution. One of the most popular methods to do this is Newton's method. Its quadratic convergence makes it very attractive; however, its lack of global stability makes it useful only if a suitable initial guess can be obtained. A widely used practice is to take the approximate solution at time  $t^n$ ,  $u_h^n$ , as the initial guess for the calculation of  $u_h^{n+1}$ :

$$u_h^{(n)} = u_h^n \quad ,$$

$$u_h^{(m+1)} = \Phi_h^n(u_h^{(m)}) \quad m=1, \dots, M^n \quad .$$

But in this case the sufficient conditions for the convergence of this method given by the now classical theorem of Kantorovich, see the reference in the book of Rall [7], impose an upper bound on the size of the time-step  $\Delta t^n = t^{n+1} - t^n$ .

In this way, the main advantage of using an implicit scheme is lost.

In practice, the number of inner iterations  $M^n$  is usually taken to be equal to 1. This is justified when the time step is small enough; see, for example, the analysis of this newtonian iterations given by Douglas [3] for the case of the long wave regularization implicit finite difference scheme for (1.1) developed by Douglas, Kendall and Wheeler [4]. This choice is also taken because doing  $M_h$  inner iterations with the time step  $\Delta t = \Delta t^h$  cost as much as doing  $M^h$  time iterations of a single inner iteration with  $\Delta t = \Delta t^h/M^h$ ; this strategy has been adopted, for example, for the implicit-explicit version of the PPM scheme, developed by Fryxell, Woodward, Colella and Winkler [5]. Other authors prefer to do a few inner iterations in order to improve the accuracy of the approximate solution; see, for example, the implicit version of Osher upwind scheme developed by Rai and Chakravarthy [9]. But it should be noted that in any case if the time step is too big the stability of the method is lost.

In this paper we introduce a new and very simple method for overcoming this difficulty. It consists in replacing the newtonian iterations by the following modification

$$\begin{aligned} u_h^{(1)} &= u_h^n, \\ u_h^{(m+1)} &= P(u_h^{(m)}) (\Phi_h^n u_h^{(m)}), \quad m=1, \dots, M^n, \end{aligned}$$

where  $\Phi_h^n$  is the same as before, and the operator  $P(v_h)$  is a locally defined projection verifying the following stability properties

$$\begin{aligned} \inf_x P(v_h) w_h &> \inf_x v_h, \\ \sup_x P(v_h) w_h &\leq \sup_x v_h, \\ \|P(v_h) w_h\|_{BV} &\leq \|v_h\|_{BV}. \end{aligned}$$

Because of this last property the operator  $P(\cdot)$  is called the TVD (total variation diminishing) projection. With this new procedure no oscillations due to the linearization leading to the definition of  $\phi_h^n$  appear anymore, and the method is always  $L^\infty \cap BV$  - stable regardless of the size of the time step.

In this work we have tested numerically the proposed method in the case of the Godunov's implicit scheme, the simplest and best known implicit finite difference scheme for (1.1).

An outline of the paper is as follows. In Section 2 Newton's method for solving the implicit Godunov's scheme for (1.1) is written down. In order to have an idea of how the stability properties of these Newtonian iterations depend on  $f'$  and  $f''$  we compute explicitly the eigenvalues of  $\text{grad } \phi_h^n$ . Section 3 is devoted to test the stability of Newton's method in five different test problems. In Section 4 we introduce our TVD-projection, and we show how to modify the newtonian iterations in order to always have  $L^\infty \cap BV$  - stability. In Section 5 we test this method in the same cases at which Newton's method was tested, and we compare the corresponding performances. We end in Section 6 with same concluding remarks.

## 2) Newton's Method for Godunov's implicit scheme

### 2.1) Introduction

In this Section we write down the Godunov's implicit scheme for (1.1), and its Newtonian iterations.

We want to stress the fact that the stability of Newton's iterations depends essentially on the size of  $|f''|$  rather than the one of  $|f'|$  as it is commonly believed. Indeed, if  $f'' \equiv 0$  the Newtonian iterations are trivially stable regardless of the size of  $|f'|$ . Also, it is intuitively clear that the presence of a discontinuity represents a threat to the stability of the method. In Subsection 2.4) we make these statements precise by calculating explicitly the eigenvalues of the operator  $\text{grad } \phi_h^n$  in the simple, but illuminating, case in which  $f' > 0$ . Proof of the results therein are given in Subsection 2.5).

### 2.2) The scheme

As usual, let  $\{t^n\}_0^{nt}$  be a partition of  $[0,T]$ , and  $\{x_{i+1/2}\}_0^{nx}$  one of  $[0,1]$ . Let us set  $\Delta t^n = t^{n+1} - t^n$ , and  $\Delta x_i = x_{i+1/2} - x_{i-1/2}$ . We associate to  $\{t^n\}_0^{nt}$  the space

$$(2.1a) \quad W_{\Delta t} = \{v_{\Delta t} \in BV(0,T) : v_{\Delta t}(x) = v^n, t \in (t^n, t^{n+1}), n=0, \dots, nt-1\},$$

and to the partition  $\{x_{i+1/2}\}_0^{nx}$ , the space

$$(2.1b) \quad V_h = \{v_h \in BV(0,1) : v_h(x_{1/2}) = v_0, v_h(x_{nx+1/2}) = v_{nx+1}, \\ v_h(x) = v_i, x \in (x_{i-1/2}, x_{i+1/2}), i=1, \dots, nx\}.$$

Define  $b_{0\Delta t}$  and  $b_{1\Delta t}$  to be the  $L^2$  projection of  $b_0$  and  $b_1$ , respectively, on the space  $W_{\Delta t}$ ; and let  $u_{0h}$  be the  $L^2$  projection of  $u_0$  on  $V_h$ . Then, Godunov's implicit scheme defines an approximate solution  $u_h \in W_{\Delta t} \times V_h$  in the

following manner:

$$(2.2a) \quad u_i^0 = u_{0i} \quad , \quad i=1, \dots, nx \quad ;$$

$$(2.2b) \quad \text{For } n=1, \dots, nt \quad :$$

$$u_0^n = b_0^n \quad ,$$

$$u_{nx+1}^n = b_1^n \quad ,$$

$$(u_i^n - u_i^{n-1}) / \Delta t^{n-1} + (f_{i+1/2}^{G,n} - f_{i-1/2}^{G,n}) / \Delta x_i = 0 \quad ,$$

$$i=1, \dots, nx \quad ;$$

where the Godunov flux  $f_{i+1/2}^{G,n} = f^G(u_{i-1}^n, u_i^n)$  is defined by

$$(2.2c) \quad f^G(u,v) = f(\xi) \quad ,$$

where  $\xi$  is any point of the interval  $I(u,v) = [\min \{u,v\}, \max \{u,v\}]$  such that

$$(2.2d) \quad (\text{sgn}(u-v)) \cdot (f(\xi) - f(c)) \leq 0 \quad , \quad c \in I((u,v)) \quad .$$

Note that when  $f' > 0$  then  $f_{i+1/2}^{G,n} = f(u_i^n)$ , and when  $f' < 0$ ,  $f_{i+1/2}^{G,n} = f(u_{i+1}^n)$ . Thus,  $f^G$  is a generalization of the classical upwinding flux. Also, among the fluxes of the form  $f_{i+1/2} = f(u_{i+1}, u_i)$  it is the one that produces the smallest amount of viscosity. See Osher [6], and Brenier and Osher [1] for more details.

### 2.3) The Newtonian iterations

It is clear that when  $f' > 0$  or  $f' < 0$  the Godunov flux is  $C^1$ , provided of course that  $f \in C^1$ . However, in the general case this is not true. For example, if  $f(u) = u(1-u)$ , then for  $w > 1-w \neq 1/2$

$$\partial_1 f^{G+}(w, 1-w) = f'(w) \quad ,$$

$$\partial_2 f^{G-}(w, 1-w) = 0 \quad ,$$

and

$$\partial_1 f^{G+}(w, 1-w) = 0 \quad ,$$

$$\partial_2 f^{G-}(w, 1-w) = f'(1-w) \quad .$$

To define our Newtonian iterations we shall use the following definition for  $\partial_1 f^G$  ,  $\partial_2 f^G$  :

$$(2.3a) \quad \begin{aligned} \partial_1 f^G(u, v) &= f'(u) && \text{if } u = \xi \quad , \\ &= 0 && \text{otherwise} \quad ; \end{aligned}$$

$$(2.3b) \quad \begin{aligned} \partial_2 f^G(a, v) &= f'(v) && \text{if } v = \xi \quad , \\ &= 0 && \text{otherwise} \quad , \end{aligned}$$

where  $\xi$  is the same as before.

Thus, we define the operator  $\Phi_h^n$  as follows:

$$(2.4a) \quad \begin{aligned} \Phi_h^n &: V_h \rightarrow V_h \\ &u_h^{(m)} \rightarrow u_h^{(m+1)} \quad , \end{aligned}$$

where

$$(2.4b) \quad \begin{aligned} u_0^{(m+1)} &= b_0^n \quad , \\ u_{nX-1}^{(m+1)} &= b_1^n \quad , \end{aligned}$$

$$(u_i^{(m+1)} - u_i^{n-1}) / \Delta t^{n-1} + (f_{i+1/2}^{NG, m+1/2} - f_{i-1/2}^{NG, M+1/2}) / \Delta x_i = 0 \quad ,$$

$$i=1, \dots, nX \quad ,$$

where the flux  $f_{i+1/2}^{NG,m+1/2}$  is given by

$$(2.4c) \quad f_{i+1/2}^{NG,m+1/2} = f^G(u_{i+1}^{(m)}, u_i^{(m)}) + (u_{i+1}^{(m+1)} - u_{i+1}^{(m)}) \partial_1 f^G(u_{i+1}^{(m)}, u_0^{(m)}) + \\ + (u_i^{(m+1)} - u_i^{(m)}) \partial_2 f^G(u_{i-1}^{(m)}, u_i^{(m)}) \quad ,$$

$\partial_1 f^G$ ,  $\partial_2 f^G$  being defined by (2.3). It is an easy matter to show that this operator is well defined for every partition  $\{t^n\}_0^{nx} \times \{x_{i+1/2}\}_0^{nx}$  of  $(0,T) \times (0,1)$ ; this is guaranteed by the fact that  $f^G(u,v)$  is nonincreasing in  $u$ , and nondecreasing in  $v$ .

Finally, Newton's method can be written as follows:

$$(2.5a) \quad u_0^{(1)} = b_0^n \quad , \\ u_{nx+1}^{(1)} = b_i^n \quad , \\ u_i^{(1)} = u_i^{n-1} \quad , \quad i=1, \dots, nx \quad ;$$

$$(2.5b) \quad \text{For } m=1, \dots, M^n \quad :$$

$$u_h^{(m+1)} = \Phi_h^n(u_h^{(m)}) \quad .$$

#### 2.4) The Eigenvalues of $\Phi_h^n$ for $f' > 0$ .

The convergence of the sequence  $\{u_h^{(m)}\}_{m \geq 1}$  generated by (2.5) with  $M^n \equiv \infty$  depends on the size of the eigenvalues of  $\Phi_h^n, \lambda_i$ . In particular, a sufficient condition for convergence is that the absolute values of  $\lambda_i$  are smaller than one. In what follow we obtain an explicit formula for the  $\lambda_i$ , in the case of  $f' > 0$ . We introduce the following notation:

$$\text{res}_h(w_h)_i = w_i - u_i^{n-1} + \frac{\Delta t^{n-1}}{\Delta x_i} (f(w_i) - f(w_{i-1})) ,$$

$$\gamma_{ij} = [\text{grad res}_h(w_h)]_{ij}^{-1} ,$$

$$\text{del}_h(w_h) = [\text{grad res}_h(w_h)]^{-1} \text{res}_h(w_h) .$$

Note that with this notation  $\Phi_h^n(w_h) = w_h + \text{del}_h(w_h)$  . We have thus the following result.

**Proposition (2.1)** The eigenvalues of the matrix  $\text{grad } \Phi_h^n(w_h)$  are

$$(2.6) \quad \lambda_i = \frac{\frac{\Delta t^{n-1}}{\Delta x_i} f''(w_i)}{(1 + \frac{\Delta t^{n-1}}{\Delta x_i} f'(w_i))} \cdot \text{del}(w_h)_i ,$$

$$i=1, \dots, nx .$$

This result shows that the role played by the size of the second derivate of  $f$  in the stability of (2.5) is essential. It also shows that more stability can be obtained if  $\Delta t^{n-1}$  is smaller. Let us point out that, as  $\Phi_h^n(w_h) = w_h + \text{del}(w_h)$ , the iterations (2.5) converges if and only if  $\{\text{del}(u_h^{(m)})\}_{m \geq 1}$  converges to zero. As (2.6) shows, when (2.5) converges, the eigenvalues also converge to zero.

This fact reflects the quadratic convergence of the method. It is difficult to take into account the dependence of  $\text{del}(w_h)$  on  $\Delta t^{n-1}$ ,  $f'$  and  $f''$ , but a simple expression can be obtained in the important case in which  $w_h$  is a step function with a single discontinuity. We need first the following lemma.

**Lemma (2.2).** The matrix  $\gamma = [\text{grad res}_h(w_h)]^{-1}$  is given by:

$$\begin{aligned} \gamma_{ij} &= 0 && \text{for } i < j , \\ \gamma_{ii} &= 1 - \theta_i \\ \gamma_{ij} &= \theta_j \dots \theta_{i-1} (1 - \theta_i) \frac{\Delta x_j}{\Delta x_i} && \text{for } i > j , \end{aligned}$$

where

$$\theta_i = \frac{(\Delta t^{n-1} / \Delta x_i) f'(w_i)}{1 + \frac{\Delta t^{n-1}}{\Delta x_i} f'(w_i)} .$$

Now we can state the following result, that shows how the presence of a discontinuity influences the eigenvalues of  $\text{grad } \phi_h^n$ .

**Corollary (2.3).** Let  $w_h$  be the following step function:

$$\begin{aligned} w_i &= b_0^n & , & \quad 0 \leq i < i_0 & , \\ &= b_1^n & & \quad i_0 \leq i \leq n_x & . \end{aligned}$$

Then the eigenvalues of  $\text{grad } \phi_h^n$  are

$$\lambda_i = \frac{\frac{\Delta t^{n-1}}{\Delta x_i} f''(w_i)}{\left(1 + \frac{\Delta t^{n-1}}{\Delta x_i} f'(w_i)\right)} = \frac{\Delta t^{n-1}}{\Delta x_{i_0}} \left( \frac{f(b_1^n) - f(b_0^n)}{b_1^n - b_0^n} \right) \gamma_{ii_0} (b_1^n - b_0^n) ,$$

where  $\gamma_{ii_0} = \gamma_{ii_0}(b_0^n, b_i^n)$  is given by Lemma (2.2).

Note that for  $\Delta x_i \equiv \Delta x$ ,  $\lambda_{i+1} = \theta \lambda_i$ ,  $i > i_0$ , where

$$\theta = \frac{\frac{\Delta t^{n-1}}{\Delta x} f'(b_i^n)}{\left(1 + \frac{\Delta t^{n-1}}{\Delta x} f'(b_i^n)\right)} < 1 ,$$

so, the influence of the discontinuity decreases geometrically with the number of elements to it! Note also that if  $f'(b_1^n) = 0$  the only non-zero eigenvalue of  $\text{grad } \phi_h^n$  is

$$\lambda_{i_0} = \frac{\Delta t^{n-1}}{\Delta x_{i_0}} f'(b_1^n) \frac{\Delta t^{u-1}}{\Delta x_{i_0}} \frac{f(b_1^n) - f(b_0^n)}{b_1^n - b_0^n} (b_1^n - b_0^n) .$$

2.5) Proof of Proposition (2.1).

We need the following result.

Lemma (2.4). Let  $\phi_h^n$  be the operator defined by (2.4), then,

$$(\text{grad } \phi_h^n(w_h))_{ij} = - \sum_{\ell, k=1}^{nx} \gamma_{ik} \left( \frac{\partial^2}{\partial w_\ell \partial w_j} \text{res}_h(w_h)_k \right) \text{del}_\ell (w_h)_\ell$$

where  $\gamma_{ij}$  is given by Lemma (2.2).

**Proof.** We use here the standard tensorial notation. By the definition of  $\phi$  and  $\gamma$  we have

$$\phi_i = w_i - \gamma_{im} \text{res}_m \quad .$$

Thus,

$$(\text{grad } \phi)_{ij} = \delta_{ij} - \gamma_{im,j} \text{res}_m - \gamma_{im} \text{res}_{m,j} \quad .$$

From the definition of  $\gamma$  we obtain easily the following expressions:

$$\gamma_{im} \text{res}_{m,j} = \delta_{ij} \quad ,$$

$$\gamma_{im,j} \text{res}_m = - \gamma_{ik} \text{res}_{k,\ell j} \gamma_{\ell m} \text{res}_m \quad ;$$

As  $\text{del}_\ell = \gamma_{\ell m} \text{res}_m$ , the result is obtained by simply inserting these expressions in the equality giving  $(\text{grad } \phi_h^n)$ .

Note that this result is valid regardless of the very form of  $\text{res}_h(w_h)$ . Now we turn to the proof of Proposition (2.1).

From Lemma (2.2)  $\gamma_{ik} = 0$  if  $i < k$ . Thus, by Lemma (2.3)

$$(2.7a) \quad (\text{grad } \phi_h^n)_{ij} = - \sum_{k=1}^i \sum_{\ell=1}^{nx} \gamma_{ik} \left( \frac{\partial^2}{\partial w_\ell \partial w_j} \text{res}_k \right) \text{del}_\ell \quad .$$

But, from the definition of  $\text{res}_h$  :

$$\frac{\partial^2}{\partial w_\ell \partial w_j} \text{res}_k = \frac{\Delta t^{n-1}}{\Delta x_k} (f''(w_k) \delta_{kj} \delta_{k\ell} - f''(w_{k-1}) \delta_{jk-1} \delta_{\ell k-1}) ,$$

and so, if  $k < j$  this quantity is equal to zero.

This implies

$$(2.7b) \quad (\text{grad } \phi_h^n)_{ij} = - \sum_{k=j}^{nx} \sum_{\ell=1}^{nx} \gamma_{ik} \frac{\partial}{\partial w_\ell \partial w_j} \text{res}_h \text{ del}_\ell .$$

From (2.7) it is now clear that for  $i < j$   $(\text{grad } \phi_h^n)_{ij} = 0$  ; i.e., the matrix  $\text{grad } \phi_h^n$  is lower triangular. This implies that its eigenvalues are

$$\begin{aligned} \lambda_i &= (\text{grad } \phi_h^n)_{ii} \\ &= \sum_{\ell=1}^{nx} \gamma_{ii} \frac{\partial}{\partial w_\ell \partial w_i} \text{res}_i \text{ del}_\ell \\ &= \gamma_{ii} \frac{\Delta t^{n-1}}{\Delta x_i} f''(w_i) \text{ del}_i . \end{aligned}$$

Finally, from Lemma (2.2),  $\gamma_{ii} = (1 + \frac{\Delta t^{n-1}}{\Delta x_i} f'(w_i))^{-1}$  , this completes the proof of Proposition (2.1).

### 3) Testing the stability of Newton's method

#### 3.1) Introduction

In this Section we shall test numerically the stability of the Newtonian iterations (2.5) for (2.2) in five different problems of increasing difficulty. All of them are Riemann problems that have a self-similar solution easy to calculate. The first three of them have a non-decreasing nonlinearity; i.e.  $f' \geq 0$ . The last two have a non convex  $f$ , and their solution presents a sonic point; i.e., a point  $u^*$  for which  $f'(u^*) = 0$ . After displaying the test problems, we define a function that measures the stability of (2.2), (2.5), and we evaluate it numerically. We end by a brief discussion of the numerical results that will serve as a reference for evaluating the performance of our proposed method.

#### 3.2) The test problems

The test problems are of the form (1.1) where

$$(3.1) \quad f_i(u) = z_i u^3 (1-r_i(1-u))^3 (z_i u^3 + (1-u)^3)^{-1}, \quad i=1, \dots, 5,$$

$$(3.2a) \quad \begin{array}{llll} u_{0i}(x) = 0 & x \in (0,1) & i=1,2,3 & , \\ b_{0i}(t) = 1 & t \in (0,T=0.3) & i=1,2,3 & , \\ b_{1i}(t) = 0 & t \in (0,T=0.3) & i=1,2,3 & , \end{array}$$

$$(3.2b) \quad \begin{array}{llll} u_{0i}(x) = 0 & x \in (0,0.6) & i=4,5 & , \\ & = 1 & x \in (0.6,1) & , \\ b_{0i}(t) = 0 & t \in (0,T=0.05) & i=4,5 & , \\ b_{1i}(t) = 1 & t \in (0,T=0.05) & i=4,5 & . \end{array}$$

See figures (1),(2) and tables (1) and (2).

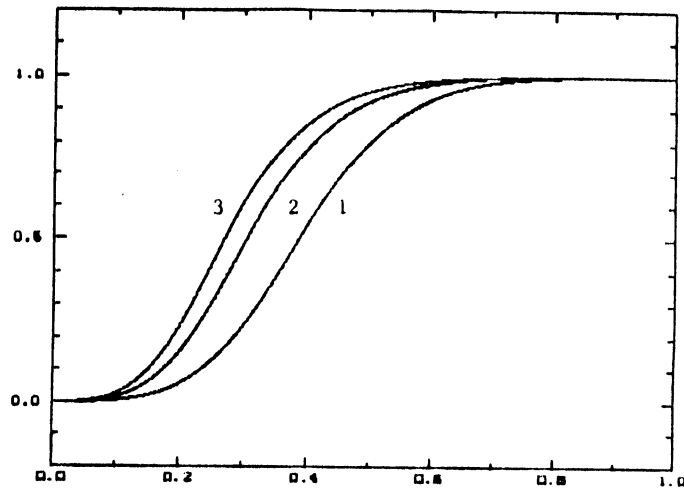


Figure (1.a) The functions  $\mu \mapsto f_i(\mu)$ ,  $i=1,2,3$ .

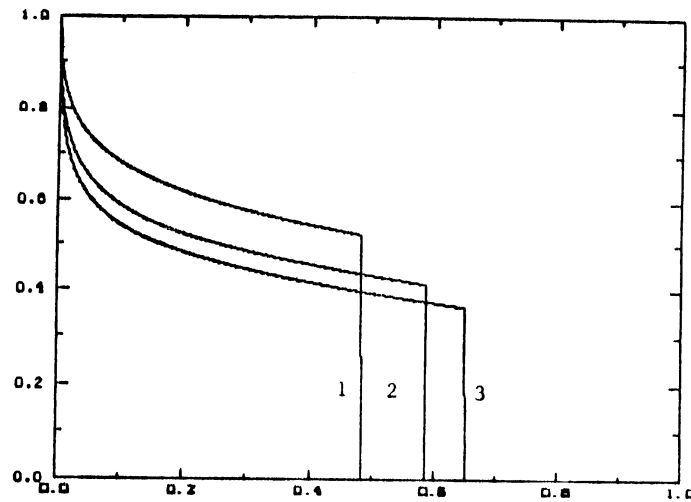


Figure (1.b). The solutions  $x \mapsto u_i(T=0.3, x)$ ,  $i=1,2,3$ .

$i$	$z_i$	$r_i$	$\ f_i^I\ _{L^\infty(0,1)}$	$\ f_i^{II}\ _{L^\infty(0,1)}$	$s_i$
1	4	0	3.18	16.67	1.61
2	12	0	3.61	23.18	1.95
3	20	0	3.91	28.19	2.17

Table (1). The index  $i$  indicates the number of the test problem, and  $s_i$  is the speed of the discontinuity of the solution of problem  $i$ .

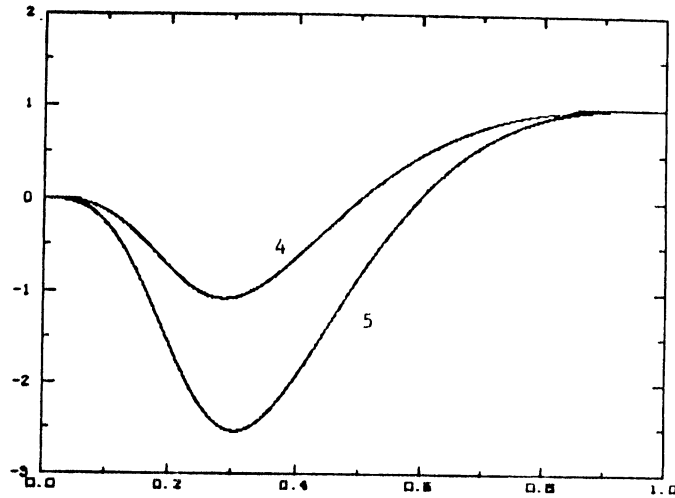


Figure (2.a) - the functions  $u \mapsto f_i(u)$   $i=4,5$ .

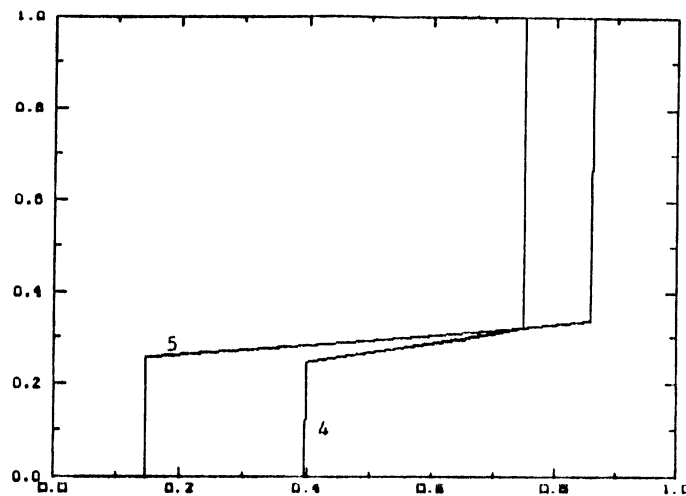


Figure (2.2b) - the solutions  $x \mapsto u_i(T=0.05, x)$ ,  $i=4,5$ .

$i$	$x_i$	$r_i$	$\ f_i^I\ _{L^\infty(0,1)}$	$\ f_i^{II}\ _{L^\infty(0,1)}$	$s_i^+$	$s_i^-$
4	20	8	7.17	99.19	2.98	4.05
5	20	15	16.10	194.99	5.19	9.08

Table (2). As before  $i$  indicates the number of the problem;  $s_i^+$  and  $s_i^-$  are the speeds of the discontinuities (of the solution of problem  $i$ ) traveling to the right and to the left, respectively.

### 3.3) Measuring the stability (and convergence of the method)

To measure the stability properties of (2.2), (2.5) we proceed as follows. First, we fix  $\Delta x$  (all the grids will be taken uniform) small enough in order to ensure that the error already entered in the region of its asymptotic behavior. Then, we define the quantity  $\lambda$  as follows:

$$(3.2a) \quad \lambda = \Delta x^{-1} \sup \{ \Delta t_0 : rc(\Delta t, \Delta x) > 1/2 \quad \forall \Delta t < \Delta t_0 \}$$

where

$$(3.2b) \quad rc(\Delta t, \Delta x) = \frac{\log \left[ \frac{\|u(T) - u_h(\Delta t, \Delta x; T)\|_{L^1(0,1)}}{\|u(T) - u_h(\frac{\Delta t}{2}, \frac{\Delta x}{2}; T)\|_{L^1(0,1)}} \right]}{\log 2}$$

is the estimated rate of convergence. We denoted by  $u_h(\Delta t, \Delta x; T)$  the approximate solution, obtained by (2.2), (2.5) with  $\Delta t^n \equiv \Delta t$  and  $\Delta x_{i+1/2} \equiv \Delta x$ , at time  $T$ ;  $u(T)$  is the exact solution at that time.

We recall the fact that  $1/2$  is the more rate of convergence for the scheme (2.2). (This is also true for 3-point implicit monotone schemes; see Sanders [8]). Thus, we shall consider  $\lambda$  as a measure of the stability of the scheme. The bigger  $\lambda$  is, the more stable is the scheme. We shall take  $\Delta x = 0.001$  in all the examples.

### 3.4) The numerical results.

For each of our five test problems we have calculated  $\lambda(M^n \equiv 1, \equiv 2, \equiv 3)$  (see Table 3), the corresponding total number of time iterations  $nt$  (i.e.,  $M^n$  times the numbers of time steps; see Table 4), the error  $\|u(T) - u_h(T)\|_{L^1(0,1)}$  (see Table 5), and finally, the efficiency of the algorithm (Table 6) that is defined as follows:

(3.4a) 
$$\text{eff} = \{ \|u(T) - u_h(T)\|_{L^1(0,1)} \cdot nt \}^{-1} .$$

test problem	$M^n \equiv 1$	$M^n \equiv 2$	$M^n \equiv 3$
1	0.617	1.327	1.240
2	0.507	0.811	0.777
3	0.457	0.694	0.682
4	0.250	0.595	0.342
5	0.110	0.221	0.160

**Table 3.** Values of  $\lambda$  defined by (3.2) for each of the test problems "i",  $i=1 \rightarrow$  and  $M^n \equiv 1,2,3$  in  $M^n$  is the number of total inner iterations in the Newtonian iterations (2.5).

test problem	$M^n \equiv 1$	$M^n \equiv 2$	$M^n \equiv 3$
1	486	452	726
2	592	740	1158
3	656	864	1320
4	200	168	438
5	452	936	936

**Table 4.** Values of the total number of time iterations  $nt = M^n \times$  number of time steps. Note that  $nt = M^n \times 1000 \times T / \lambda$ .

test problem	$M^n \equiv 1$	$M^n \equiv 2$	$M^n \equiv 3$
1	2.18	62.34	30.50
2	2.24	16.05	12.38
3	2.23	137.82	6.30
4	2.51	31.61	8.46
5	1.41	8.66	4.42

**Table 5.** Values of  $10^3 \times \|u(T) - u_h(T)\|_{L^1(0,1)}$ .

test problem	$M^n \equiv 1$	$M^n \equiv 2$	$M^n \equiv 3$
1	9.44	0.35	0.45
2	7.54	0.84	0.70
3	6.84	0.84	1.20
4	19.92	1.88	2.70
5	15.62	2.55	2.42

**Table 6.** Values of  $10^4 \times \text{eff}$ , defined by (3.3).

From table (3) we see that the stability of (2.5) increases when passing from 1 inner iteration to 2, then it decreases when passing to 3 inner iterations. (For more than 3 inner iterations  $\lambda$  can possibly oscillate slightly around  $\lambda(M^n \equiv 3)$  and then decrease slowly to an asymptotic value; we have not shown these results). It is also interesting to note that for a fixed value of  $M^n$ , the value of  $\lambda$  decreases monotonically with  $i$ , the number of our test problems. This corresponds to the increasing values of  $\|f'_i\|_{L^\infty}$  and  $\|f''_i\|_{L^\infty}$ ; see tables 1 and 2. Though it is not easy to find a simple relation between the values of  $\lambda$  and those of  $f'$ ,  $f''$ , table 6 seems to indicate that the following relationship is verified:

$$\lambda_i(M^n \equiv 1) \cdot \text{Max}\{s_i^+, s_i^-\} = 1 \quad .$$

test problem	$\lambda_i(M^n \equiv 1) \cdot \text{Max}\{s_i^+, s_i^-\}$
1	0.993
2	0.988
3	0.991
4	1.010
5	0.999

**Table 6.** The cf $\ell$ -number  $\lambda_i(M^n \equiv 1) \cdot \text{Max}\{s_i^+, s_i^-\}$ .  
For  $i=1,2,3$   $\text{max}\{s_i^+, s_i^-\}$  is simply  $s_i$  .

This is rather surprising result for, on the one hand, it does not involve the values of  $f''$ , and on the other hand, it is not true when  $f'' \equiv 0$ . It indicates that the stability of the Newtonian iterations (2.5) with  $M^n \equiv 1$  depends only on the cf $\ell$ -number, exactly as if it were an explicit scheme!

Table 4 shows that the amount of work for  $M^n \equiv 1,2,3$  is of the same order of magnitude. Tables 5 and 6 show that with  $M^n \equiv 1$  not only the iterations (5) are more efficient but that they give raise to a better approximate solution.

#### 4) The TVD-projection method

##### 4.1) Introduction

In this section we define and study the stability properties of the so-called TVD (total variation diminishing) projection  $P(w_h)$ . Then, we write down our TVD - projection method and study its stability. We end by some remarks about the relationship the convergence and the conservativity of the method.

##### 4.2) Definition of $P(w_h)$ .

Let  $w_h$  and  $v_h$  be two arbitrary elements of  $V_h$ . To define  $P(w_h)v_h$  we proceed as follows. First, to each point of the partition  $\{x_{i+1/2}\}_0^{n_x}$  we associate a value  $\eta_{i+1/2}$  defined in the following way:

$$(4.1a) \quad \eta_{i+1/2} = w_i \\ \text{if } \operatorname{sgn}(v_{i+1} - w_{i+1}) = \operatorname{sgn}(v_i - w_i) \neq \operatorname{sgn}(w_{i+1} - w_i) ,$$

$$(4.1b) \quad \eta_{i+1/2} = w_{i+1} \\ \text{if } \operatorname{sgn}(v_{i+1} - w_{i+1}) = \operatorname{sgn}(v_i - w_i) = \operatorname{sgn}(w_{i+1} - w_i) ,$$

$$(4.1c) \quad \eta_{i+1/2} = \theta_{i+1/2} w_i + (1 - \theta_{i+1/2}) w_{i+1} \quad \text{otherwise,}$$

where

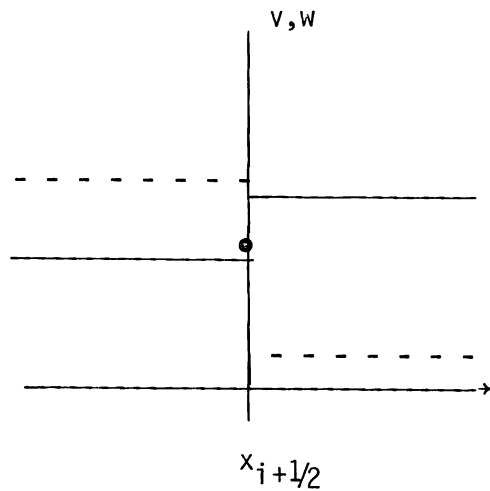
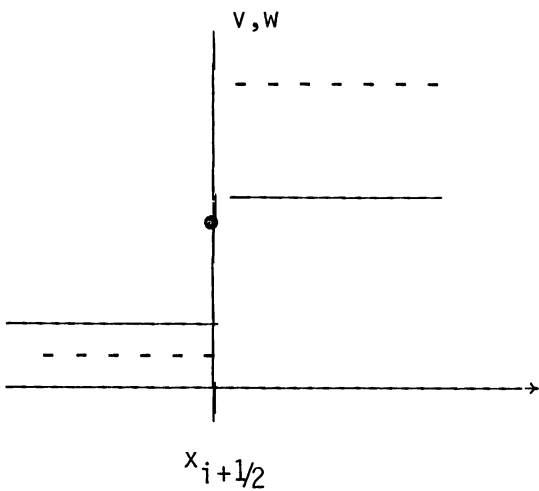
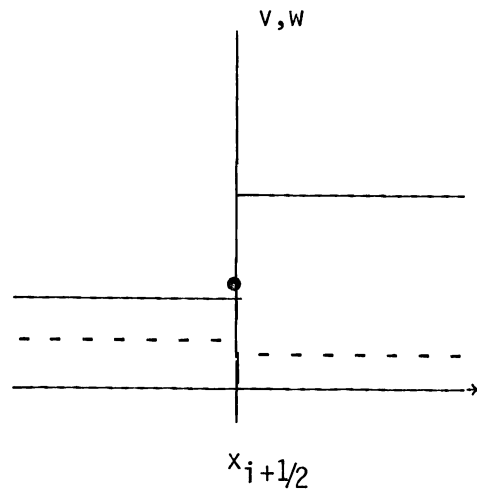
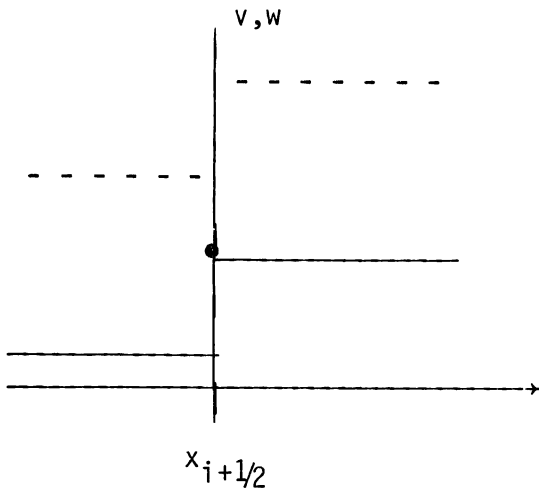
$$(4.1d) \quad \theta_{i+1/2} = \frac{v_{i+1} - w_{i+1}}{(v_{i+1} - w_{i+1}) - (v_i - w_i)} .$$

We also define  $\eta_{-1/2}$  and  $\eta_{n_x+3/2}$  as follows:

$$(4.1e) \quad \eta_{-1/2} = w_0 ,$$

$$(4.1f) \quad \eta_{n_x+3/2} = w_{n_x+1} .$$

The definition of  $\eta_{i+1/2}$  is illustrated in the figures that follow. In there, the continuous lines represent the values of the function  $w_h$ , and the dotted ones the values of the function  $v_h$  to be projected. The bid dot represents the value of  $\eta_{i+1/2}$



Note that  $\eta_{i+1/2}$  always lies in the interval  $I(w_i, w_{i+1})$  regardless of the values  $v_i, v_{i+1}$ . We shall prove this later.

Next, we define the intervals  $I_i$  as follows:

$$I_i = I(w_i, \eta_{i+1/2}) \cup I(w_i, \eta_{i-1/2})$$

$$\text{if } \text{sign} (w_i - \eta_{i+1/2})(w_i - \eta_{i-1/2}) < 0 ,$$

$$= I(w_i, \eta_{i+1/2}) \cap I(w_i, \eta_{i-1/2})$$

$$\text{otherwise; } \quad i = 0, \dots, nx+1 .$$

Now, we introduce the  $L^\infty$ -projection on the interval  $[a,b]$ :

$$(4.2) \quad P_{[a,b]}^\infty (c) = \begin{cases} b & \text{if } c > b , \\ c & \text{if } c \in [a,b] , \\ a & \text{if } c < a . \end{cases}$$

Finally, we define the projection  $P(w_h) v_h$  as follows:

$$(4.3a) \quad P(w_h) : \begin{aligned} V_h &\rightarrow V_h , \\ v_h &\rightarrow u_h , \end{aligned}$$

where

$$(4.3b) \quad u_i = P_{I_i}^\infty (v_i) \quad i=0, \dots, nx+1 ,$$

where the  $\eta_{i+1/2}$  are defined by (4.1). Note that the operator  $P(w_h)$  is a projection for it is easy to check that

$$P(w_h) P(w_h) v_h = P(w_h) v_h .$$

#### 4.3) Stability properties of $P(w_h)$ .

Let us begin with the following result.

Lemma (4.1). Set  $u_h = P(w_h) v_h$ . Then

$$(4.4a) \quad u_i \in I_i \quad i=0, \dots, nx+1,$$

$$(4.4b) \quad \eta_{i+1/2} \in I(w_i, w_{i+1}) \quad i=0, \dots, nx.$$

**Proof.** Property (4.4a) is a direct consequence of (4.2) and (4.3). The second is a consequence of the definition of  $\eta_{i+1/2}$ , (4.1).

We can state now our key result.

Proposition (4.2). Set  $u_h = P(w_h) v_h$ . Then,

$$(4.5a) \quad \inf_x u_h > \inf_x w_h,$$

$$(4.5b) \quad \sup_x u_h < \sup_x w_h,$$

and

$$(4.6) \quad \|u_h\|_{BV} < \|w_h\|_{BV}.$$

**Proof.** Inequalities (4.5) are a corollary of (4.4). To prove (4.6) we proceed as follows:

$$\begin{aligned} \|u_h\|_{BV} &= \sum_{i=0}^{nx} |u_{i+1} - u_i| \\ &< \sum_{i=0}^{nx} (|u_{i+1} - \eta_{i+1/2}| + |\eta_{i+1/2} - u_i|) \\ &= |\eta_{1/2} - u_0| + |\eta_{nx+1/2} - u_{nx+1}| + \\ &\quad \sum_{i=1}^{nx} (|\eta_{i+1/2} - u_i| + |u_i - \eta_{i-1/2}|). \end{aligned}$$

But, from (4.1)  $u_0 = \eta_{-1/2}$ ,  $u_{nx+1} = \eta_{nx+3/2}$ ; and from (4.4a)

$$|\eta_{i+1/2} - u_i| + |u_i - \eta_{i-1/2}| \leq |\eta_{i+1/2} - w_i| + |w_i - \eta_{i-1/2}|$$

Thus,

$$\begin{aligned} \|u_h\|_{BV} &= \leq \sum_{i=0}^{nx+1} (|\eta_{i+1/2} - w_i| + |w_i - \eta_{i-1/2}|) \\ &= |w_0 - \eta_{-1/2}| + |\eta_{nx+3/2} - w_{nx+1}| \\ &\quad + \sum_{i=1}^{nx+1} (|w_i - \eta_{i-1/2}| + |\eta_{i-1/2} - w_{i-1}|) \end{aligned}$$

But again, from (4.1)  $w_0 = \eta_{-1/2}$ ,  $w_{hx+1} = \eta_{nx+3/2}$ ; and from (4.4b)

$$|w_i - \eta_{i-1/2}| + |\eta_{i-1/2} - w_{i-1}| = |w_i - w_{i-1}|$$

Thus,

$$\begin{aligned} \|u_h\|_{BV} &\leq \sum_{i=1}^{nx+1} |w_i - w_{i-1}| \\ &= \|w_h\|_{BV} \end{aligned}$$

This completes the proof.

#### 4.4) The TVD-projection method

With the notation introduced previously, we can write our algorithm for the inner iterations of (2.2) in the following manner:

$$\begin{aligned} (4.7a) \quad u_0^{(1)} &= b_0 \quad , \\ u_{hx+1}^{(1)} &= b_1 \quad , \\ u_i^{(1)} &= u_i^{n-1} \quad , \quad \ell=1, \dots, nx \quad ; \end{aligned}$$

(4.7b) For  $m = 1, \dots, M^n$  :

$$u_h^{(m+1)} = P(u_h^{(m)}) (\Phi_h^n (u_h^{(m)}))$$

Compare this with the Newtonian iterations (2.5). This method has the advantage of being stable regardless the characteristics of the partitions  $\{t^h\}_0^{nt}$  and  $\{x_{i+1/2}\}_0^{nx}$ . Moreover, it also guarantees the compactness of the sequence  $\{u_h\}_{h > 0}$  determined by (2.2), (4.7) in the  $L^\infty(0,T;L^1(0,1))$  - norm. This is a direct consequence of the following result.

**Theorem (4.3).** Let  $u_h$  be the approximate solution obtained by (2.2) and (4.7). Then,

$$\inf_{t,x} u_h > \inf_{t,x} [b_0(t), b_1(t), u_0(x)] \quad ,$$

$$\sup_{t,x} u_h < \sup_{t,x} [b_0(t), b_1(t), u_0(x)] \quad ,$$

$$\|u_h\|_{L^\infty(0,t;BV(0,1))} < \|b_0\|_{BV(0,T)} + \|b_1\|_{BV(0,T)} + \|u_h\|_{BV(0,1)} \quad .$$

The proof is straightforward: it is a direct consequence of Proposition (4.2) and the definition of  $b_{0\Delta t}$ ,  $b_{1\Delta t}$ ,  $u_{0h}$ . Thus, the sequence  $\{u_h\}_{h > 0}$  has a subsequence converging strongly in  $L^\infty(0,T;L^1(0,1))$  to a limit. The problem of proving that this limit is indeed the entropy solution of (1.1) is very delicate and will not be considered here.

#### 4.5) The price of stability

From the definition of the Newtonian iterations (2.4), (2.5) we obtain easily the following expression for the conservations of mass:

$$\sum_{i=1}^{nx} u_i^{(m+1)} \Delta x_i = \sum_{i=1}^{nx} u_i^{n-1} \Delta x_i - \Delta t^{n-1} (f_{nx+1/2}^{NG,m+1/2} - f_{1/2}^{NG,m+1/2}) ,$$

where  $u_h^{(m+1)} = \Phi_h^n(u_h^{(m)})$ . On the other hand, by (2.2) we have for the approximate solution of the Godunov scheme

$$\sum_{i=1}^{nx} u_i^n \Delta x_i = \sum_{i=1}^{nx} u_i^{n-1} \Delta x_i - \Delta t^{n-1} (f_{nx+1/2}^{G,n} - f_{i/2}^{G,n}) ,$$

Thus, the error in the total amount of mass at the time step  $(t^{n-1}, t^n)$  introduced by Newton's method is simply

$$(4.8a) \quad - \Delta t^{n-1} ((f_{nx+1/2}^{G,n} - f_{nx+1/2}^{NG,m+1/2}) - (f_{1/2}^{G,n} - f_{1/2}^{NG,m+1/2})) ,$$

with  $m = M^h$ .

In our method the error in the total amount of mass has a rather different form, for it cannot be written in conservation form like the Newtonian iterations

(2.4), (2.5). Set  $w_h = \Phi_h^n(u_h^{(m)})$ , and  $u_h^{(m+1)} = P(u_h^{(m)}) w_h$ , then

$$\begin{aligned} \sum_{i=1}^{nx} u_i^{(m+1)} \Delta x_i &= \sum_{i=1}^{nx} w_i \Delta x_i + \sum_{i=1}^{nx} (u_i^{(m+1)} - w_i) \Delta x_i \\ &= \sum_{i=1}^{nx} u_i^{n-1} \Delta x_i + \\ &\quad - \Delta t^{n-1} (f_{nx+1/2}^{NG,m+1/2} - f_{1/2}^{NG,m+1/2}) + \\ &\quad + \sum_{i=1}^{nx} (u_i^{(m+1)} - w_i) \Delta x_i , \end{aligned}$$

where, of course  $f^{NG} = f^{NG}(u_h^{(m)}, w_h)$ . This implies that the error in the total amount of mass produced by our method is the sum of a term similar to (4.8a) and the following one

$$(4.8b) \quad \sum_{i=1}^{nx} (u_i^{(m+1)} - w_i) \Delta x_i = \int_0^1 (P(u_h^{(m)}) - Id) w_h(x) dx .$$

The term (4.8a) is due to the linearization procedure, and the term (4.8b) is only due to the projection  $P(u_h^{(m)})$ . In other words, the price that we have to pay for stability is an increase of the loss of mass!

Nevertheless, we have to point out that if  $w_h = u_h^{(m)}$  then  $P(u_h^{(m)}) u_h^{(m)} = u_h^{(m)}$  and the term (4.8b) is equal to zero. Thus, if  $\text{del}(u_h^{(m)}) = \Phi_h^n(u_h^{(m)}) - u_h^{(m)} = w_h - u_h^{(m)}$  is small enough then  $P(u_h^{(m)}) w_h = w_h$  and (4.8b) vanishes. If this is the case, the inner iteration of our method reduces simply to a Newtonian one. In fact, this happens when  $\Delta t^{n-1}$  is small enough. Thus, our method can be considered as an extension of Newton's method. When  $\Delta t^{n-1}$  is big, the term (4.8b) is not anymore equal to zero, and indeed it can be very big. If our method converges we expect the term (4.8b) to go to zero with the number of iterations  $m$ . Our numerical experience indicates that this is always the case, and that once (4.8b) becomes zero it stays equal to zero, so that the method coincides with Newton's method after a certain number of iterations. Once this happens, convergence is achieved very quickly, but it is extremely inefficient to ask the term (4.8b) to be equal to zero in order to stop the iterations. It suffices to stop the inner iterations when (4.8b) is "small enough". We consider this problem in next Section.

## 5) Testing the stability of the TVD-projection method

### 5.1) Introduction

In this Section we test numerically the stability properties of our method on the same six test problems on which the method (2.2), (2.5) was already tested. Our stopping criterion is based on a control of the amount of mass lost by the use of our projection. After defining it we calculate the quantities  $\lambda$  defined by (3.2), and we compare them with the  $\lambda$  obtained for Newton's method. The same  $\Delta x$  has been taken.

### 5.2) The stopping criterion

To define a stopping criterion for (4.7) is to define  $M^n$ . We define such a quantity as follows:

$$(5.1a) \quad M^n = \min \{m : LM(u_h^{(m)}) > \epsilon \} ,$$

where the loss of mass  $LM$  is defined by

$$(5.1b) \quad LM(u_h^{(m)}) = \sum_{i=1}^{nx} |(P(u_h^{(m)}) - I_d) \phi_h^n(m_h^{(m)})|_i \Delta x_i .$$

In our numerical experiences we shall take  $\epsilon$  as follows:

$$(5.2) \quad \epsilon = \Delta t \cdot (\Delta x)^{j/2} \quad j = 1,2,3,4 .$$

Roughly speaking at each time step we are allowing ourselves to lose an amount of mass equal to  $\epsilon$ ; thus, in  $nt$  iterations we allowed ourselves to lose a total amount of mass equal to  $T \cdot (\Delta x)^{j/2}$ . Taking into account that the error behaves like  $C_0(\Delta t)^\alpha$  for  $\alpha \in (0.5,1)$ , and that  $\Delta t \gg \Delta x$ , we think that this is a reasonable criterion.

5.3) The numerical results

Our numerical results are shown in the tables 8 to 11.

test problem	j = 1	j = 2	j = 3	j = 4
1	75	75	150	150
2	50	75	150	150
3	50	50	75	75
4	25	25	25	25
5	25	25	25	25

Table 8. Values of  $\lambda$  for the TVD-projection method. The value  $j$  is associated to the stopping criterion (5.1) by (5.2).

test problem	j = 1	j = 2	j = 3	j = 4
1	583	618	727	728
2	648	769	926	927
3	761	780	889	889
4	196	394	645	645
5	365	369	1465	1465

Table 9. The total number of iterations  $nt$ :  

$$nt = \sum_{\text{time steps}} (\text{number of inner iterations}).$$

test problem	j = 1	j = 2	j = 3	j = 4
1	64.29	71.75	115.69	115.69
2	50.43	73.71	122.14	122.14
3	50.16	48.33	80.06	80.06
4	38.54	55.26	49.69	49.69
5	96.95	96.42	93.00	93.00

Table 10. The values of  $10^3 \|u(T) - u_h(T)\|_{L^1(0,1)}$

test problem	j = 1	j = 2	j = 3	j = 4
1	0.27	0.23	0.12	0.12
2	0.31	0.18	0.09	0.09
3	0.26	0.27	0.14	0.14
4	1.32	0.46	0.31	0.31
5	0.28	0.28	0.07	0.07

Table 11. The values of  $10^4 \text{ eff}$ .

Note that the the minimum value for the number of time steps is 2, see the definition of  $\lambda$  (3.2), and so we have:

$$\lambda_{\max} = \frac{T}{2} \cdot 1000 = 500 T \quad .$$

As  $T = 0.3$  for problems 1,2,3 and  $T = 0.05$  for problems 4,5 the maximum value for  $\lambda$  is 150 for problems 1,2,3 and 25 for problems 4,5.

From these results we can conclude that the stopping criterion (5.1) works well, and works better for  $j = 1$  ; see the errors and efficiency in Tables 10 and 11, respectively. Note that the results obtained with  $j = 3$  and 4 are essentially the same.

By comparing the values of  $\lambda^N$  for the Newtonian iterations (2.5), table 3, and those for our method, table 8, we see that our  $\lambda$  are at least two orders of magnitude bigger than those of the Newtonian iteration. Thus, we can say that the use of the TVD-projection enhances enormously the stability of Newton's iteration.

## 6) Conclusion

In this paper we have introduced a very simple and locally defined projection, the TVD (total variation diminishing) - projection, and we have shown how to use it to define an always-stable extension of Newton's method, applied to solve implicit schemes for the scalar conservation law (1.1). After proving that the TVD-projection makes the inner iterations  $C^0 \cap BV$  - stable, we tested our method on Godunov's implicit scheme (1.1). Our numerical results show that the time step sizes allowed by our method are at least two orders of magnitude bigger than those allowed by the standard newtonian iterations.

As first-order schemes are not very efficient, we are planning to apply this technique for solving more accurate implicit schemes (...with variable space-discretization). In particular, we are interested in solving the implicit finite element quasi-monotones schemes introduced recently by one of the authors; see [2]. These schemes are TVDM (total variation diminishing in the means), and seem to be a suitable application of our TVD-projection method.

## References

- [1] Y. Brenier and S. Osher, Approximate Riemann Solvers and Numerical Flux Functions, ICASE report n<sup>o</sup> 84-63, 1984.
- [2] B. Cockburn, The Quasi-Monotone Schemes for Scalar Conservation Laws, Parts I,II,III; IMA Preprint Series #'s 263,268,277, (1986).
- [3] J. Douglas, Jr. Simulation of a Linear Waterflood, in Free Boundary Problems, Proceedings of a Seminar held in Pavia, Sept.-Oct. 1979, Vol. II, Instituto Nazionale de Alta Matematica Francesco Severi, Rome, 1980.
- [4] J. Douglas, Jr., R.P. Kendall, and M.F. Wheeler, Long Wave Regularization of One-Dimensional, Two-Phase, Inmiscible Flow in Porous Media, Finite Element Methods for Convection Dominated Flows, AMD-Vol. 34, ASME, New York, 1979, pp. 201-211.
- [5] B.A. Fryxell, P.R. Woodward, P. Collella, K.H. Winkler, An Implicit-Explicit Hybrid Method for Lagrangian Hydrodynamics, Journal of Comp. Phys., Vol. 63, 1986, pp. 283-310.
- [6] S. Osher, Riemann Solvers, The Entropy Condition, and Difference Approximations, SIAM J. Numer. Anal., Vol. 21, 1984, pp. 217-235.
- [7] L.B. Rall, Computational Solution of Nonlinear Operator Equations, John Wiley, 1969.
- [8] R. Sanders, On Convergence of Monotone Finite Difference Schemes with Variable Spatial Differencing, Math. of Comp., Vol. 40, 1983, pp. 91-106.
- [9] M.M. Rai, and S.R. Chakravarthy, An Implicit Form for the Osher Upwind Scheme, AIAA journal, Vol. 24, n<sup>o</sup> 5, 1986, pp. 735-743.

- Recent IMA Preprints (continued)
- | #   | Author(s)   | Title   |
|-----|---|---|
| 165 | D. Arnold and R.S. Falk,                          | Continuous Dependence on the Elastic Coefficients for a Class of Anisotropic Materials  |
| 166 | I.J. Bakelman,                                    | The Boundary Value Problems for Non-linear Elliptic Equation and the Maximum Principle for Euler-Lagrange Equations               |
| 167 | Ingo Müller,                                      | Gases and Rubbers   |
| 168 | Ingo Müller,                                      | Pseudoelasticity in Shape Memory Alloys - an Extreme Case of Thermoelasticity   |
| 169 | Luis Magalhães,                                   | Persistence and Smoothness of Hyperbolic Invariant Manifolds for Functional Differential Equations                                |
| 170 | A. Damiani and M. Vogelius,                       | Homogenization Limits of the Equations of Elasticity in Thin Domains  |
| 171 | H.C. Simpson and S.J. Spector,                    | On Hadamard Stability in Finite Elasticity  |
| 172 | J.L. Vazquez and C. Yauri,                        | Isolated Singularities of the Solutions of the Schrödinger Equation with a Radial Potential                                       |
| 173 | G. Dal Maso and U. Mosco,                         | Wiener's Criterion and T-Convergence  |
| 174 | John H. Maddocks,                                 | Stability and Folds   |
| 175 | R. Hardt and D. Kinderlehrer,                     | Existence and Partial Regularity of Static Liquid Crystal Configurations  |
| 176 | M. Norrkar,                                       | Construction of Smooth Ergodic Cocycles for Systems with Fast Periodic Approximations   |
| 177 | J.L. Ericksen,                                    | Stable Equilibrium Configurations of Elastic Crystals   |
| 178 | Patricio Aviles,                                  | Local Behavior of Solutions of Some Elliptic Equations  |
| 179 | S.-N. Chow and R. Lauterbach,                     | A Bifurcation Theorem for Critical Points of Variational Problems   |
| 180 | R. Pego,  | Phase Transitions: Stability and Admissibility in One Dimensional Nonlinear Viscoelasticity                                       |
| 181 | Mariano Giaquinta,                                | Quadratic Functions and Partial Regularity  |
| 182 | J. Bona,  | Fully Discrete Galerkin Methods for the Korteweg De Vries Equation  |
| 183 | J. Maddocks and J. Keller,                        | Mechanics of Robes  |
| 184 | F. Bernis,  | Qualitative Properties for some nonlinear higher order Nonlinear Higher Order Parabolic Equations with Absorption                 |
| 185 | F. Bernis,  | Finite Speed of Propagation and Asymptotic Rates for some Nonlinear Higher Order Parabolic Equations with Minimal Strategy Spaces |
| 186 | S. Reichelstein and S. Reiter,                    | Game Forms with Minimal Strategy Spaces   |
| 187 | An Answer to Littlewood's Problem on Boundedness  |   |
| 188 | J. Rubinstein and R. Mauri,                       | Dispersion and Convexity in Periodic Media  |
| 189 | W.H. Fleming and P.E. Souganidis,                 | Asymptotic Series and the Method of Vanishing Viscosity   |
| 90  | H. Beltrao Da Veiga,                              | Existence and Asymptotic Behavior for Strong Solutions of Navier-Stokes Equations in the Whole Space                              |
| 91  | L.A. Caffarelli, J.L. Vazquez, and N.I. Wolanski, | Lipschitz Continuity of Solutions and Interfaces of the N-Dimensional Porous Medium Equation                                      |
| 92  | R. Johnson,                                       | m-Functions and Floquet Exponents for Linear Differential Systems   |
| 93  | F.V. Atkinson and L.A. Peletier,                  | Ground States and Dirichlet Problems for $-\Delta = F(U)$ in $R^N$  |
| 94  | G. Dal Maso, U. Mosco,                            | The Wiener Modulus of a Radial Measure  |
| 95  | H. A. Levine and H.F. Weinberger,                 | Inequalities between Dirichlet and Neumann Eigenvalues  |
| 96  | J. Rubinstein,                                    | On the Macroscopic Description of Slow Viscous Flow Past a Particle   |
| 97  | G. Dal Maso and U. Mosco,                         | Wiener Criterion and Energy Decay for Relaxed Dirichlet Problems  |
| 98  | V. Ofikar and P. Waltman,                         | On the Monge-Ampere Equation Arising in the Reflect Mapping Problem   |
| 99  | M. Chipot, D. Kinderlehrer and L. Caffarelli,     | Some Smoothness Properties of Linear Laminae  |
| 100 | Y.liga and R. Kohn,                               | Characterizing Blow-up Using Similarity Variables   |
| 101 | P. Cannarsa and H. M. Soner,                      | On the Singularities of the Viscosity Solutions to Hamilton-Jacobi-Bellman Equations  |
| 102 | Andrew Majda,                                     | Nonlinear Geometric Optics for Hyperbolic Systems of Conservation Laws  |
| 203 | G. Buttazzo, G. Dal Maso and U. Mosco,            | A Derivation Theorem for Capacities with Respect to a Radon Measure   |
| 204 | S. Cowin, M. Mehrabadi,                           | On the Identification of Material Symmetry for Anisotropic Elastic Materials  |
| 205 | R.W.R. Darling,                                   | Constructing Nonhomeomorphic Stochastic Flows.  |
| 206 | M. Chipot,  | On the Reynolds Lubrication Equation  |
| 207 | R.V. Kohn and G.W. Milton,                        | On Bounding the Effective Conductivity of Anisotropic Composites  |
| 208 | I.J. Bakelman,                                    | Notes Concerning the Torsion of Hardening Rods and Its N-Dimensional Generalizations  |
| 209 | I.J. Bakelman,                                    | The Boundary Value Problems for Non-linear Elliptic Equations   |
| 210 | Gianglu Gong & Mingping Qian,                     | On the Large Deviation Functions of Markov Chains   |
| 211 | Arje Leizarowitz,                                 | Control Problems with Random and Progressively Known Targets  |
| 212 | R.W.R. Darling,                                   | Ergodicity of a Measure-Valued Markov Chain Induced by Random Transformations   |
| 213 | G. Gong, M. Qian & Zhongxin Zhao,                 | Killed Diffusions and Its Conditioning  |
| 214 | Arje Leizarowitz,                                 | Controlling Diffusion Processes on Infinite Horizon with the Over-taking Criterion  |
| 215 | Willard Beatty,                                   | The Polisson Function of Finite Elasticity  |
| 216 | David Terman,                                     | Traveling Wave Solutions Arising From a Combustion Model  |
| 217 | Yuh-Jia Lee,                                      | Sharp Inequalities and Regularity of Heat Semi-Group on Infinite Dimensional Spaces   |
| 218 | D. Stroock,                                       | Lecture Notes   |
| 219 | Claudio Canuto,                                   | Spectral Methods and Maximum Principle  |
| 220 | Thomas O'Brien,                                   | A Two Parameter Family of Pension Contribution Functions and Stochastic Optimization  |
| 221 | Takeyuki Hida,                                    | Analysis of Brownian Functionals  |
| 222 | Leonid Hurwicz,                                   | On Informational Decentralization and Efficiency of Resource Allocation Mechanisms  |
| 223 | E.B. Fabes and D.W. Stroock,                      | A New Proof of Moser's Parabolic Harnack Inequality via the Old Ideas of Nash   |
| 224 | Minoru Murata,                                    | Structure of Positive Solution to $(-\Delta+V)u = 0$ in $R^N$   |
| 225 | Paul Dupuis,                                      | Large Deviations Analysis of Reflected Diffusion  |
| 226 | F. Bernis,  | Constrained Stochastic Approximation Algorithms in Convex Sets  |
| 227 | S. Orey and S. Pellikan,                          | Existence Results for Doubly Nonlinear Higher Order Parabolic Equations on Unbounded Domains.                                     |
| 228 | R. Gulliver and S. Hildebrandt,                   | Boundary Configurations Spanning Continua of Minimal Surfaces.  |
| 229 | J. Baxter, G. Dal Maso & U. Mosco,                | Stopping Times and T-Convergence.   |
| 230 | Juilio Boulliet,                                  | Self-Similar Solutions, Having Jumps and Intervals of Constancy of a Diffusion-heat Conduction Equation                           |
| 231 | R. Hardt, D. Kinderlehrer & F.-H. Lin,            | A Remark About the Stability of Smooth Equilibrium Configurations of Static Liquid Crystal  |
| 232 | M. Chipot and M. Luskin,                          | The Compressible Reynolds Lubrication Equation  |
| 233 | J.H. Maddocks,                                    | A Model for Disclinations in Nematic Liquid Crystal   |
| 234 | C. Fofas, G.R. Sell and R. Temam,                 | Inertial Manifolds for Nonlinear Evolutionary Equations   |
| 235 | P.L. Chow,  | Expectation Functionals Associated with Some Stochastic Evolution   |
| 236 | Giuseppe Buttazzo,                                | Reinforcement by a Thin Layer with Oscillating Thickness  |
| 237 | W.H. Fleming, S.J. Sheu and H.M. Soner,           | On Existence of the Dominant Eigenfunction and Its Application to the Large Deviation Properties of an Ergodic Markov Process     |
| 238 | R. Jensen and P.E. Souganidis,                    | A Regularity Result for Viscosity Solutions of Hamilton-Jacobi Equations in one Space Dimension                                   |
| 239 | B. Boczar-Karakli, J.L. Bona and D.L. Cohen,      | Interaction of Shallow-Water Waves and Bottom Topography  |
| 240 | F. Colonius and W. Klemann,                       | Infinite Time Optimal Control and Periodicity Dirichlet Problems  |

#	Author(s)	Title	Recent IMA Preprints (continued)
#	Author(s)	Title	#
2241	Harry Kesten	Scaling Relations for 2D-Percolation	277
2242	A. Leizorowitz	Infinite Horizon Optimization for Markov Process with Finite States Spaces	278
2243	Louis H.Y. Chen	The Rate of Convergence in A Central Limit Theorem for Dependent Random Variables with Arbitrary Index Set	279
2244	G. Kallianpur	Stochastic Differential Equations in Duals of Nuclear Spaces with some Applications	280
2245	Tzuu-Shuh Chang, Yunshong Chow and Yuh-Jia Lee	Evaluation of Certain Functional Integrals	281
2246	L. Karp and M. Plinsky	The First Eigenvalue of a Small Geodesic Ball in a Riemannian Manifold	282
2247	Chi-Sing Man	Towards An Acoustoelastic Theory for Measurement of Residual Stress	283
2248	Andreas Stoll	Invariance Principles for Brownian Intersection	284
2249	R.W.R. Darling	Rate of Growth of the Coalescent Set in a Coalescing Stochastic Flow	285
2250	R. Cohen, R. Hardt, D. Kinderlehrer, S.Y. Lin, M. Luskin	Minimum Energy for Liquid Crystals: Computational Results	286
2251	Suzanne M. Lenhart	Viscosity Solutions for Weakly Coupled Systems of First Order PDEs	287
2252	M. Cranston, E. Fabes, Z. Zhao	Condition Gauge and Potential Theory for the Schrödinger Operator	288
2253	H. Brezis, J.M. Coron, E.H. Lieb	Harmonic Maps with Defects	289
2254	A. Carverhill	Flows of Stochastic Dynamical Systems: Nontriviality of the Lyapunov Spectrum	290
2255	A. Carverhill	Conditioning A 'Lifted' Stochastic System in a Product Case	291
2256	R.J. Williams	Local Time and Excursions of Reflected Brownian Motion	292
2257	H. Föllmer, S. Orey	Large Deviations for the Empirical Field of a Gibbs Measure	293
2258	A. Leizorowitz	Characterization of Optimal Trajectories On an Infinite Horizon	294
2259	Y. Giga, T. Miyakawa, H. Osada	Two Dimensional Navier Stokes Flow with Measures As Initial Vorticity	295
2260	M. Chipot, V. Ollitré	Sur une Propriété Des Fonctions Propres De L'Opérateur De Laplace Beltrami	296
2261	V. Perez-Abrau	Decompositions of Semimartingales On Duals of Countably Nuclear Spaces	297
2262	J.M. Ball	Does Rank-One Convexity Imply Quasiconvexity	298
2263	B. Cockburn	The Quasi-Monotone Schemes for Scalar Conservation Laws. Part I	299
2264	K.A. Perlick-Spector	On Radially Symmetric Simple Waves in Elasticity	300
2265	P.N. Shivakumar, Chi-Sing Man, Simon W. Rabkin	Modelling of the Heart and Pericardium at End-Diastole	301
2266	Jose-Luis Menaldi	Probabilistic View of Estimates for Finite Difference Methods	302
2267	Robert Hardt, Harold Rosenberg	Open Book Structures and Unicity of Minimal Submanifolds	303
2268	Bernardo Cockburn	The Quasi-Monotone Schemes for Scalar Conservation Laws	304
2269	H.R. Jauslin, W. Zimmermann, Jr.	4th Dynamics of a Model for an Ac Josephson Effect in Superfluid $^4\text{He}$	305
2270	A.K. Kapila	Introductory Lecture on Reacting Flows	306
2271	J.C. Taylor	Do Minimal Solutions of Heat Equations Characterize Diffusions?	307
2272	J.C. Taylor	The Minimal Eigenfunctions Characterize the Ornstein-Uhlenbeck Process	308
2273	Chi-Sing Man, Quan-Xin Sun	On the Significance of Normal Stress Effects in the Flow of Glaciers	309
2274	Omar Hiljeb	On Partially Observed Control of Markov Processes	310
2275	Lawrence Gray	The Behavior of Processes with Statistical Mechanical Properties	311
2276	R. Hardt, D. Kinderlehrer, M. Luskin	Remarks About the Mathematical Theory of Liquid Crystals	
277	Cockburn, B.	The Quasi-Monotone Schemes for Scalar Conservation Laws Part II	
278	M. Chipot, T. Sideris	On the Abelian Higgs Model	
279	C. Fofas, B. Nicolaenko, G.R. Sell, R. Temam	Inertial Manifolds for the Kuramoto-Sivashinsky Equation and an Estimate of their Lowest Dimension	
280	R. Duran	On the Approximation of Miscible Displacement in Porous Media by a Method of Characteristics Combined with a Mixed Method	
281	H. Axiang, Zhang Bo	The Convergence for Nodal Expansion Method	
282	V. Tversky	Dispersive Bulk Parameters for Coherent Propagation in Correlated Random Distributions	
283	M.Th.F. den Hollander	Mixing Properties for Random Walk in Random Scenery	
284	H.R. Jauslin	Nondifferentiable Potentials for Nonequilibrium Steady States	
285	K. Meyer, G.R. Sell	Homoclinic Orbits and Bernoulli Bundles in Almost Periodic Systems	
286	J. Douglas, Jr., Y. Yuan	Finite Difference Methods for the Transient Behavior of a Semiconductor Device	
287	Li Kaitai, Yan Mingling	The Extrapolation for Boundary Finite Elements	
288	R. Durrett, R.H. Schonmann	Stochastic Growth Models	
289	David Kinderlehrer	Remarks about Equilibrium Configurations of Crystals	
290	D.G. Aronson, J.L. Vazquez	Eventual $C^\infty$ -Regularity and Concavity for Flows in One-Dimensional Porous Media	
291	L.R. Scott, J.M. Boyle, B. Bagheri	Distributed Data Structures for Scientific Computation	
292	J. Douglas, Jr., P.J. Paes Leme, T. Arbogast, T. Schmitt	Simulation of Flow in Naturally Fractured Petroleum Reservoirs	
293	D.G. Aronson, L.A. Caffarelli	Optimal Regularity for One-Dimensional Porous Medium Flow	
294	Haim Brezis	Liquid Crystals and Energy Estimates for $S^2$ -Valued Maps	
295	Todd Arbogast	Analysis of the Simulation of Single Phase Flow Through a Naturally Fractured Reservoir	
296	He Yimin, Li Kaitai	The Coupling Method of Finite Elements and Boundary Elements for Radiation Problems	
297	T. Cazenave, A. Harauz, L. Vazquez, F.B. Weissler	Nonlinear Effects in the Wave Equation with a Cubic Restoring Force	
298	M. Chipot, F.B. Weissler	Some Blow Up Results for a Nonlinear Parabolic Parabolic Equation with a Gradient Term	
299	Li Kaitai	Perturbation Solutions of Simple and Double Bifurcation Problems I	
300	Chen Zhongxin, Li Kaitai	The Convergence on the Multigrid Algorithm for Navier-Stokes Equations	
301	A. Gerardi, G. Nappo	Martingale Approach for Modelling DNA Synthesis	
302	D.N. Arnold, L. Ridgway Scott, Michael Vogelius	Regular Inversion of the Divergence Operator with Dirichlet Boundary Conditions on a Polygon	
303	Duran, Ricardo G.	Error Analysis in $L^p$ , $1 \leq p \leq \infty$ , for Mixed Definite Element Methods for Linear and Quasi-Linear Elliptic Problems	
304	Nochetto, Ricardo H., Verdi, Claudio	An Efficient Linear Scheme to Approximate Parabolic Free Boundary Problems: Error Estimates and Implementation	
305	K.A. Perlick-Spector, Scott J. Spector	Nonuniqueness for a Hyperbolic System	
306	E.G. Kalnins, Willard Miller, Jr.	q-Series and Orthogonal Polynomials Associated with Barnes' First Lemma	
307	D.N. Arnold, Richard S. Falk	A Uniformly Accurate Finite Element Method for Mindlin-Reissner Plate	
308	Chi-Wang Shu	TVD Properties of a Class of ModifiedENO Schemes for Scalar Conservation Laws	
309	Edmund Dikow, Ulrich Horning	A Random Boundary Value Problem Modelling Spatial Variability in Porous Media Flow	
310	Jack K. Hale	Compact Attractors and Singular Perturbations	
311	Bourgeat, A. Cockburn, B.	The TVD-Projection Method for Solving Implicit Numerical Schemes for Scalar Conservation Laws: A Numerical Study of a Simple Case	