SPANNING TREE EXTENSIONS
OF THE HADAMARD-FISCHER INEQUALITIES

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25	Scott J. Spector	On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension
26	W.A. Coppel	Maps on an Interval
27	James Kirkwood	Phase Transitions in the Ising Model with Traverse Field
28	Luis Magalhaes	The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different
29	Charles Tresser	Homoclinic Orbits for Flow in R^3
30	Charles Tresser	About some Theorems by L.P. Sil'nikov
31	Michael Aizenmann	On the Renormalized Coupling Constant and the Susceptibility in ϕ_4^4 Field Theory and the Ising Model in Four Dimensions
32	C. Eugene Wayne	The KAM Theory of Systems with Short Range Interactions I

(continued on back cover)

SPANNING TREE EXTENSIONS OF THE HADAMARD-FISCHER INEQUALITIES

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SPANNING THREE EXTENSIONS
OF THE HADAMARD-FISCHER INEQUALITIES

Abstract

All possible graph theoretic generalizations of a certain sort for the Hadamard-Fischer determinantal inequalities are determined. These involve ratios of products of principal minors which dominate the determinant. Furthermore, the cases of equality in these inequalities are characterized, and equality is possible for every set of values which can occur for the relevant minors. This relates recent work of the authors on positive definite completions and determinantal identities. When applied to the same collections of principal minors, earlier generalizations give poorer, more difficult to compute bounds than the present inequalities. Thus, this work extends, and in a certain sense completes, a series of generalizations of Hadamard-Fischer begun in the 1960's.

SPANNING TREE EXTENSIONS
OF THE HADAMARD-FISCHER INEQUALITIES

1. Introduction

In [2] we introduced classes of formulae, for the determinant of the n-by-n matrix A in terms of the principal minors of A, which depend upon a certain tree-like decomposition of the graph of A^{-1}. It turns out that, when applied to general A in some positivity class (such as the positive definite Hermitian matrices), these same expressions give determinantal inequalities which generalize the so-called Hadamard-Fischer inequalities. For example, the simplest case of the formulae in [2] is the following: <u>if</u> the n-by-n nonsingular matrix $A = (a_{ij})$ has tridiagonal inverse, then

$$\det A = \frac{\prod_{i=1}^{n-1} \det\begin{bmatrix} a_{ii} & a_{i,i+1} \\ a_{i+1,i} & a_{i+1,i+1} \end{bmatrix}}{\prod_{i=2}^{n-1} a_{ii}} .$$

The analogous inequality, which holds, for example, for <u>any</u> positive definite n-by-n Hermitian $A = (a_{ij})$, is

$$\det A \leqslant \frac{\prod_{i=1}^{n-1} \det\begin{bmatrix} a_{ii} & a_{i,i+1} \\ a_{i+1,i} & a_{i+1,i+1} \end{bmatrix}}{\prod_{i=2}^{n-1} a_{ii}} .$$

This particular inequality may be deduced from the collection of known inequalities sometimes referred to as Hadamard-Fischer and has a connection through [1, 2] with the recent papers [5, 10]. However, our goal here is to give the fullest possible extension of Hadamard-Fischer in a certain direction, which, though motivated by [10], is not so immediate. We also give a complete description of the cases of equality using a technique introduced in [10] and note that the inequalities so generated are always at least as strong and often stronger than previous inequalities utilizing the same information about minors of A.

2. Background

Let $N = \{1, 2, \ldots, n\}$ and let $A = (a_{ij})$ be an n-by-n matrix throughout. For nonempty index sets $\alpha, \beta \subseteq N$, we denote by $A(\alpha, \beta)$ that submatrix of A lying in the rows indicated by α and the columns indicated by β; the principal submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$, and for brevity $\det A(\alpha)$ is denoted by A_α. Hadamard's inequality states that

$$\det A \leqslant \prod_{i=1}^{n} a_{ii} \quad .$$

Fischer's inequality states that

$$\det A \leqslant A_\alpha A_{\alpha'}$$

in which $\alpha \subseteq N$ is an index set and α' is its complement with respect to N. For arbitrary index sets $\alpha, \beta \subset N$, the family of inequalities

$$A_{\alpha \cup \beta} \leqslant \frac{A_\alpha A_\beta}{A_{\alpha \cap \beta}}$$

is often referred to as the Hadamard-Fischer inequalities. We, of course, adopt the convention that $A_\emptyset \equiv 1$. Hadamard's inequality may trivially be deduced from Fischer's which, in turn, in a special case of Hadamard-Fischer. Each is known to hold for matrices in such positivity classes as the positive definite Hermitian matrices, the M-matrices and the totally positive matrices, etc. The Hadamard-Fischer inequalities and their ramifications have been studied, for example, in [3, 4, 6, 7, 8] and [12], partly in response to the unification question raised by Taussky [13].

For completeness we give a perhaps novel, simple proof of Fischer's inequality for positive definite matrices and then note that Hadamard-Fischer may actually be deduced from it. There are analogous proofs for other positivity classes. Let A be positive definite Hermitian and suppose that its triangular factorization is $A = LL^*$ in which L is lower triangular. Assume, without loss of generality, that $\alpha = \{1, \ldots, k\}$, $1 \leqslant k < n$. Then, $A(\alpha) = L(\alpha)L(\alpha)^*$ and $A(\alpha') = L(\alpha')L(\alpha')^* + L(\alpha', \alpha)L(\alpha', \alpha)^*$. Thus, $A_\alpha = \det L(\alpha)L(\alpha)^*$ and $A_{\alpha'} \geqslant \det L(\alpha')L(\alpha')^*$ (since $L(\alpha', \alpha)L(\alpha', \alpha)^*$ is positive semi-definite). However, $\det A = \det LL^* = \det L(\alpha)L(\alpha)^* \cdot \det L(\alpha')L(\alpha')^* \leqslant A_\alpha A_{\alpha'}$, which is Fischer's inequality. Further suppose, without loss of generality, that $\alpha, \beta \subseteq N$ are index sets whose union is N. Then α' and β' do not intersect and Fischer's inequality, applied to the principal submatrix $A^{-1}(\alpha' \cup \beta')$, implies that $A^{-1}_{\alpha' \cup \beta'} \leqslant A^{-1}_{\alpha'} A^{-1}_{\beta'}$. However, Jacobi's formula for minors of the inverse of a matrix, see [2], [7] or [11], specialized to principal minors, states that

$$A^{-1}_{\gamma'} = A_\gamma / \det A \ .$$

Application to each term in $A^{-1}_{\alpha' \cup \beta'} \leqslant A^{-1}_{\alpha'} A^{-1}_{\beta'}$ yields

$$\frac{A_{\alpha \cap \beta}}{\det A} \leqslant \frac{A_\alpha}{\det A} \frac{A_\beta}{\det A} \ , \qquad \text{or} \qquad A_{\alpha \cup \beta} = \det A \leqslant \frac{A_\alpha A_\beta}{A_{\alpha \cap \beta}} \ ,$$

which is Hadamard-Fischer.

3. Example

Before presenting and proving our general results, we wish to illustrate them with a particularly simple example.

The identity

$$\begin{aligned} A_{\{i, i+1, \ldots, j\}} A_{\{i+1, \ldots, j-1\}} &= A_{\{i, \ldots, j-1\}} A_{\{i+1, \ldots, j\}} \\ &\quad - \det A(\{i, \ldots, j-1\}, \{i+1, \ldots, j\}) \det A(\{i+1, \ldots, j\}, \{i, \ldots, j-1\}) \ , \end{aligned} \tag{3.1}$$

a special case of Sylvester's determinantal identity, yields the inequality

$$A_{\{i,i+1,\ldots,j\}} \leqslant \frac{A_{\{i,\ldots,j-1\}}A_{\{i+1,\ldots,j\}}}{A_{\{i+1,\ldots,j-1\}}} \tag{3.2}$$

for positive definite Hermitian A, because the subtracted term on the right hand side of (3.1) is nonnegative for Hermitian A. Of course, (3.2) is also just a special case of the Hadamard-Fischer inequalities.

Repeated application of (3.2) beginning with det A yields the chain of inequalities

$$\det A \leqslant \frac{A_{\{1,\ldots,n-1\}}A_{\{2,\ldots,n\}}}{A_{\{2,\ldots,n-1\}}} \leqslant \frac{A_{\{1,\ldots,n-2\}}A_{\{2,\ldots,n-1\}}A_{\{3,\ldots,n\}}}{A_{\{2,\ldots,n-2\}}A_{\{3,\ldots,n-1\}}}$$

$$\leqslant \cdots \leqslant \frac{A_{\{1,2\}}A_{\{2,3\}}\cdots A_{\{n-1,n\}}}{a_{22}\cdots a_{n-1,n-1}} \leqslant a_{11}a_{22}\cdots a_{nn} \ . \tag{3.3}$$

Ignoring the intermediate terms demonstrates Hadamard's inequality, while focus upon the second to last term yields the inequality mentioned in the Introduction. In general, we have the inequalities

$$\det A \leqslant \frac{\prod_{k=1}^{n-p} A_{\{k,\ldots,k+p\}}}{\prod_{k=1}^{n-p-1} A_{\{k+1,\ldots,k+p\}}} \ , \quad p = 0,1,\ldots,n-1 \ , \tag{3.4}$$

for positive definite Hermitian A. Note that terms in the denominator are based upon index sets which are intersections of those corresponding to successive terms in the numerator, and that the same inequalities hold for a matrix from any class, closed under extraction of principal submatrices, in which the Hadamard-Fischer inequalities hold. The inequalities (3.4) may be demonstrated in other ways, including application of [1] or [2] to [5] or [10].

4. Principal Ideas

Let $V_1, \ldots, V_m \subset N$ be index sets. We assume throughout that

$$\bigcup_{i=1}^{m} V_i = N \quad . \tag{4.1}$$

Let G_I be the intersection graph of the node set $\{V_1, \ldots, V_m\}$, and let G be a spanning subgraph of G_I with edge set $\mathcal{E}(G)$. Recall that H_2 is called a spanning subgraph of the graph H_1 if H_2 has the same node set as H_1 and the edges of H_2 are contained among those of H_1. The graph G is said to satisfy the <u>intersection property</u> if

$$V_i \cap V_j \subseteq V_k \tag{4.2a}$$

whenever V_k lies on a path in G from V_i to V_j and if

$$V_i \cap V_j = \emptyset \tag{4.2b}$$

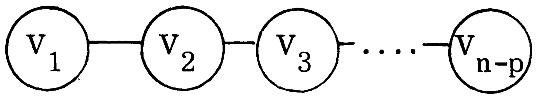
whenever V_i and V_j lie in different connected components of G. The inequality

$$\det A \leqslant \frac{\prod_{k=1}^{m} A_{V_k}}{\prod_{\{V_i, V_j\} \in \mathcal{E}(G)} A_{V_i \cap V_j}} \tag{4.3}$$

is a natural generalization of Hadamard-Fischer, which is simply the special case in which $m = 2$ and G is the tree on two nodes (or consists of two isolated nodes). Our goal is to determine the circumstances under which (4.3) holds for, for example, all positive definite matrices, to determine the cases of equality, and to relate the resulting inequalities to previous generalizations of Hadamard-Fischer.

A few comments illustrate the above concepts and some of the notions to come.

If we choose $V_k = \{k, \ldots, k+p\}$, $k = 1, \ldots, n-p$, as index sets, then the graph G



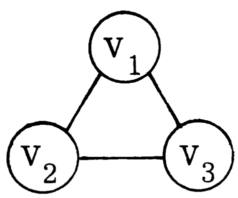
is a spanning tree for the intersection graph G_I of the node set $\{V_1, \ldots, V_{n-p}\}$. This tree satisfies the intersection property (4.2), and, for this tree inequality, (4.3) is just inequality (3.4).

An identity in [1] or [2] shows that equality is attained in (3.4) if A^{-1} is 2p+1-diagonal. The results of [5] or [10] indicate that if entries of an n-by-n Hermitian matrix $A = (a_{ij})$ are specified for $|i-j| \leqslant p$, then the remaining entries of A may be specified so that the resulting Hermitian matrix is positive definite if (and only if) all principal minors within the specified bands are positive. Moreover, among all positive definite completions, there is a unique one with maximum determinant, and it is the unique one whose inverse is 2p+1-banded. This means that equality is attained in (3.4) (if and) only if A^{-1} is 2p+1-diagonal. We will characterize the case of equality when the more general inequality (4.3) holds also in terms of the 0 pattern of A^{-1}.

We finally note an example which suggests that we should not expect inequality (4.3) to hold if there is a circuit in the graph G. The simplest intersection graph containing a circuit results from node sets

$$V_1 = \{1,2\}, \quad V_2 = \{2,3\}, \quad \text{and} \quad V_3 = \{1,3\} .$$

The intersection graph is



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and inequality (4.3) for this graph takes the form

$$\det A \leqslant \frac{A_{\{1,2\}} A_{\{2,3\}} A_{\{1,3\}}}{a_{11} a_{22} a_{33}} \quad . \tag{4.4}$$

However, the positive definite matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

has determinant 4 while the right hand side of (4.4) is $27/8 < 4$. Note that the two sides of (4.4) do enjoy the necessary same degree of homogeneity, but that the graph does not satisfy the intersection property (4.2) because V_2 lies between V_1 and V_3 while $V_1 \cap V_3 = \{1\} \not\subseteq V_2$, for example.

5. Main Results

We now state and prove our main results which are contained in Theorems 1, 2, 3, and 4. Theorem 1 indicates conditions sufficient for Hadamard-Fischer generalizations of the type (4.3) to hold, and Theorem 2 shows that these are the only circumstances in which (4.3) holds for all positive definite Hermitian matrices. Theorem 3 characterizes the cases of equality when (4.3) holds, and Theorem 4 shows that equality may occur for any specification of the right hand side data (from a positive definite matrix) upon which (4.3) is based. In a later section we indicate that (4.3) is generally stronger than previous generalizations of Hadamard-Fischer based upon the same data.

Theorem 1. Let $V_1, \ldots, V_m \subseteq N$ be index sets for which (4.1) holds. If F is a spanning forest of the intersection graph G_I of $V_1, \ldots, V_m$, for which the intersection property (4.2) holds, then the inequality (4.3) holds, with $G = F$, for all n-by-n positive definite Hermitian matrices A.

Proof: First of all, it suffices to consider the case in which F is a single tree, because of Fischer's inequality (since $\det A \leqslant \prod_{q=1}^{p} A_{U_q}$ if F is composed of p trees whose disjoint vertex sets are $U_1, \ldots, U_p$).

The proof of the inequality then proceeds by induction on m. In case $m = 1$, there is nothing to prove, and in case $m = 2$, the inequality is simply

$$\det A \leqslant \frac{A_{V_1} A_{V_2}}{A_{V_1 \cap V_2}} ,$$

in which $V_1 \cup V_2 = N$. This is just the Hadamard-Fischer inequality.

Suppose the inequality has been verified for values up to and including $m-1$. Then suppose, without loss of generality, that V_m is a node of valence 1 of the tree F, with V_{m-1} as its unique neighbor. Applying Hadamard-Fischer, we have

$$\det A \leqslant \frac{A_{(V_1 \cup \ldots \cup V_{m-1})} A_{V_m}}{A_{(V_1 \cup \ldots \cup V_{m-1}) \cap V_m}} .$$

Now, $(V_1 \cup \ldots \cup V_{m-1}) \cap V_m = (V_1 \cap V_m) \cup (V_2 \cap V_m) \cup \ldots \cup (V_{m-1} \cap V_m)$ $= V_{m-1} \cap V_m$. The second equality results from the fact $V_k \cap V_m \subseteq V_{m-1}$, $k = 1, \ldots, m-2$, which holds by the intersection property (4.2) because V_{m-1} lies on a path connecting V_k and V_m. Thus,

$$\det A \leqslant \frac{A_{V_1 \cap \ldots \cap V_{m-1}} A_{V_m}}{A_{V_{m-1} \cap V_m}} .$$

Applying the induction hypothesis to the subtree T on nodes $V_1, \ldots, V_{m-1}$, we have

$$A_{V_1 \cup \ldots \cup V_{m-1}} \leqslant \frac{\prod_{k=1}^{m-1} A_{V_k}}{\prod_{\{V_i, V_j\} \in \xi(T)} A_{V_i \cap V_j}} .$$

Since $\mathcal{E}(F) = \mathcal{E}(T) \cup \{V_{m-1}, V_m\}$, substitution of the latter into the former yields the desired inequality (4.3) and completes the proof. ∎

We note that the same line of proof shows that the inequalities of Theorem 1 also hold for all classes of matrices in which the Hadamard-Fischer inequalities hold, for example, the M-matrices and inverse M-matrices, the totally positive and inverse totally positive, etc. Actually, Theorem 1 may also be demonstrated by properly combining the results of [2] and [10], but such a proof would not conveniently extend to other Hadamard-Fischer classes. See [3] regarding Hadamard-Fischer classes. Note also that assumption (4.1) is inessential and is only for convenience, as the result may be applied to a principal submatrix of a given matrix.

In order that (4.3) hold for all positive definite Hermitian matrices, it is necessary that a certain form of homogeneity be enjoyed by the right hand side of (4.3): each index must appear exactly one more time in the numerator than the denominator.

Observation (Cancellation of Indices). Suppose that (4.3) holds for all n-by-n positive definite Hermitian matrices A. For each $p \in N$, let α_p be the number of node sets V_k containing p and let β_p be the number of $\{V_i, V_j\} \in \mathcal{E}(G)$ for which $V_i \cap V_j$ contains p. Then,

$$\alpha_p - \beta_p = 1 \quad , \qquad p = 1, \ldots, n \; .$$

Proof: Suppose $\alpha_p - \beta_p \neq 1$ for some index p. Let Λ_p be the diagonal matrix with λ in the p-th diagonal position and 1's in the remaining diagonal positions. Then the left hand side of (4.3) is λ, while the right hand side is $\lambda^{\alpha_p - \beta_p}$. Thus, (4.3) would be violated for $0 < \lambda < 1$ if $\alpha_p - \beta_p > 1$ and for $\lambda > 1$ if $\alpha_p - \beta_p < 1$. ∎

Theorem 2. If G is a subgraph of the intersection graph G_I of index sets $V_1, \ldots, V_m \subseteq N$ satisfying (4.1), then (4.3) holds for all n-by-n positive definite matrices A only if G is a spanning forest which satisfies the intersection property (4.2).

Proof: Let G be a subgraph of G_I for which the inequality (4.3) holds for all positive definite Hermitian matrices A.

By considering the matrices A which are the identity except on a single connected component of G, we see that the inequality (4.3) must hold for each connected component of G. Thus we may assume G is connected.

We show that G must be a tree. Let t be the number of edges in G. If G is not a tree, then $t \geqslant m$. For $B = A^{-1}$, application of Jacobi's identity to the right hand side of (4.3) yields

$$\det A \leqslant \frac{(\det B)^{t-m} \prod_{k=1}^{m} B_{W_k}}{\prod_{\{V_i, V_j\} \in \mathcal{E}(G)} B_{W_i \cup W_j}} ,$$

in which $W_k = V_k^C$, $k = 1, \ldots, m$. Replacing $\det A$ with $(\det B)^{-1}$ we may rewrite this as

$$(\det B)^{t-m+1} \geqslant \frac{\prod_{\{V_i, V_j\} \in \mathcal{E}(G)} B_{W_i \cup W_j}}{\prod_{k=1}^{m} B_{W_k}} .$$

Since we have assumed (4.3) to hold for all positive definite Hermitian matrices A, this inequality consequently holds for all positive definite Hermitian B, and, by continuity, for all positive semi-definite Hermitian matrices whose principal minors B_{W_k}, $k = 1, \ldots, m$, are nonzero. Since $V_i \cap V_j \neq \emptyset$ for $\{V_i, V_j\} \in \mathcal{E}(G)$, $W_i \cup W_j$ is a proper subset of N for all $\{V_i, V_j\} \in \mathcal{E}(G)$. Thus, since $t-m+1 \geqslant 1$, this inequality gives a lower bound for the determinant of a positive semi-definite matrix in terms of some proper principal minors. But this is impossible, since there are singular positive semi-definite matrices B all of whose proper principal minors are positive. For example:

$$\begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & 2 & 1 \\ 1 & \cdots & 1 & \frac{n-1}{n} \end{bmatrix} .$$

Thus, G is a tree; call it T.

Finally, we show that T must satisfy the intersection property (4.2) by realizing that if (4.2) fails then $\alpha_p - \beta_p \geqslant 2$ for some $p \in N$, violating the cancellation of indices observation made above. If (4.2) fails, there are node sets V_i, V_j and V_ℓ, with V_ℓ between V_i and V_j and with $V_i \cap V_j \not\subseteq V_\ell$. Pick $p \in N$ such that $p \in V_i \cap V_j$ but $p \notin V_\ell$.

Remove from T the node set V_ℓ and all edges incident with the node V_ℓ (i.e., of the form $\{V_\ell, V_k\}$). Thus, $p \notin V_\ell \cap V_k$ for any edge $\{V_\ell, V_k\}$ which is removed. This leaves two or more subtrees. Let T_1 be the subtree containing V_i, T_2 the one containing V_j, and $T_3, \ldots, T_q$ any remaining subtrees. For each of these trees T_k, let F_k be the forest whose vertex set is the collection of all node sets V_r in T_k which contain p and whose edge set is the collection of all edges $\{V_r, V_s\}$ for which $p \in V_r \cap V_s$. For each nonempty forest F_k,

$$|V(F_k)| \geqslant |\mathcal{E}(F_k)| + 1 \quad ,$$

in which $V(F_k)$ is the vertex set and $\mathcal{E}(F_k)$ is the edge set of F_k. In particular, F_1 contains V_i and F_2 contains V_j, so that this inequality holds at least for $k = 1, 2$. Thus, we have

$$\alpha_p - \beta_p = \sum_{k=1}^{q} \left(|V(F_k)| - |\mathcal{E}(F_k)|\right) \geqslant \sum_{k=1}^{2} \left(|V(F_k)| - |\mathcal{E}(F_k)|\right) \geqslant 2 \quad ,$$

which contradicts the cancellation of indices observation. Thus, the tree T must satisfy the intersection property (4.2), completing the proof. ∎

<u>Theorem 3.</u> Let $V_1, \ldots, V_m$ be index sets for which (4.1) holds and let F be a spanning forest of the intersection graph G_I of $V_1, \ldots, V_m$ for which the intersection property (4.2) holds. Then, equality holds in (4.3) for a positive definite n-by-n Hermitian matrix A if and only if the i, j entry of A^{-1} is 0 whenever $\{i, j\}$ is contained in none of the vertex sets V_k, $k = 1, \ldots, m$.

<u>Proof</u>: For sufficiency, suppose that the i, j entry of A^{-1} is 0 whenever $\{i, j\}$ is contained in none of the vertex sets V_k, $k = 1, \ldots, m$. By (4.2b), $(A^{-1})_{ij} = 0$ if i and j lie in distinct trees of F. It follows that $a_{ij} = 0$ whenever i and j lie in distinct trees of F. Letting $T_1, \ldots, T_r$ be the trees which comprise F, it follows that

$$\det A = \prod_{k=1}^{r} A_{T_k}$$

(here we identify a graph with the indices contained among its node sets) and

$$\frac{\prod_{k=1}^{m} A_{V_k}}{\prod_{\{V_i, V_j\} \in \mathcal{E}(F)} A_{V_i \cap V_j}} = \prod_{k=1}^{r} \frac{\prod_{V_i \in V(T_k)} A_{V_i}}{\prod_{\{V_i, V_j\} \in \mathcal{E}(T_k)} A_{V_i \cap V_j}} .$$

Thus, it suffices to show that equality holds in (4.3) when F is a single tree T.

Now, let G be the usual undirected graph of A^{-1}. (Please distinguish the graph G, whose vertices are indices, from the graph F, of this theorem, whose vertices are index sets.) Let G_k, $k = 1, \ldots, m$, be the subgraph of G whose vertex set is V_k and whose edge set $\mathcal{E}(G_k)$ consists of all edges of G both of whose vertices lie in V_k. Since $\{i, j\} \in \mathcal{E}(G)$, the edge set of G, implies by hypothesis that $\{i, j\} \in V_k$ for some $k = 1, \ldots, m$, we have $\bigcup_{k=1}^{m} (G_k) = \mathcal{E}(G)$. Therefore, $\bigcup_{k=1}^{m} G_k$ is a "treelike decomposition of G" as defined in [2] and equality holds in (4.3) by the theorem of [2]. (Note that the positive definiteness of A was not used in this direction.)

For necessity, assume that equality holds in (4.3). The condition on the 0 entries of A^{-1} may then be demonstrated using a result of [10] by considering the partial Hermitian matrix with specified entries coinciding with those of A exactly in the positions involved in the principal minors which compose the right hand side of (4.3). However, we give a self-contained proof here.

For this, suppose that A is a positive definite matrix for which

$$\det A = \frac{\prod_{k=1}^{m} A_{V_k}}{\prod_{\{V_i, V_j\} \in \xi(F)} A_{V_i \cap V_j}} .$$

We wish to show that $(A^{-1})_{ij} = 0$ whenever $\{i,j\}$ is contained in none of the vertex sets V_k, $k = 1,\ldots,m$. Suppose that A is a real matrix. Consider the class $\mathcal{a}$ of all positive definite symmetric matrices B whose entries agree with those of A on the diagonal and in the positions i, j for which $\{i,j\}$ is contained in at least one of the vertex sets V_k, $k = 1,\ldots,m$, and consider the remaining entries as "free" real variables. By Theorem 1

$$\det B \leqslant \frac{\prod_{k=1}^{m} B_{V_k}}{\prod_{\{V_i, V_j\} \in \xi(F)} B_{V_i \cap V_j}}$$

for all $B \in \mathcal{a}$. But the right hand side (involving exactly the entries which agree with those of A) is constant for all $B \in \mathcal{a}$ and equal to det A, per assumption. This means that det(·) attains a maximum at A over the class $\mathcal{a}$. Since $\mathcal{a}$ is an open set, when embedded in the natural space associated with the "free" variables, A must be a critical point with respect to the "free" variables. Thus,

$$\frac{\partial}{\partial b_{ij}} \det B \bigg|_{B=A} = 0$$

whenever $\{i,j\}$ is contained in none of the vertex sets V_k, $k = 1,\ldots,m$. Since

$$(A^{-1})_{ij} = \frac{\operatorname{cof}(a_{ij})}{\det A} = \frac{1}{2\det A} \cdot \frac{\partial}{\partial b_{ij}} \det B \bigg|_{B=A} ,$$

the i, j entry of A^{-1} is zero wnenever $\{i,j\}$ is contained in none of the vertex sets V_k, $k = 1,\ldots,m$. This completes the proof for real A.

If A is complex Hermitian, a similar argument applies with the real and imaginary parts of the appropriate b_{ij} taken as the "free" variables. ∎

Given index sets $V_1,\ldots,V_m \subseteq N$ satisfying (4.1), we call that portion of an n-by-n matrix A which lies in the principal submatrices $A(V_i)$, $i = 1,\ldots,m$, The $V_1,\ldots,V_m$ profile of A. Note that it is just principal minors from the appropriate profile of (a positive definite matrix) A which enter into the right hand side of (4.3). We finally indicate that not only is equality possible in each of the inequalities guaranteed by Theorem 1, but equality is possible for the corresponding profile of any positive definite Hermitian matrix. This means that Theorem 1 is quite strong in that equality is possible in a very strong sense.

Theorem 4. Let $V_1,\ldots,V_m \subseteq N$ be index sets for which (4.1) holds. Let F be a spanning forest of the intersection graph of $V_1,\ldots,V_m$ for which (4.2) holds, so that (4.3) holds with $G = F$ for all n-by-n positive definite Hermitian matrices. For each n-by-n positive definite Hermitian matrix B, there exists a unique n-by-n positive definite Hermitian matrix A whose $V_1,\ldots,V_m$ profile agrees with that of B and such that equality holds in (4.3) with $G = F$.

Proof: Here, we apply the results of [10]. Let $\mathcal{B}$ be the partial Hermitian matrix with specified entries, agreeing with those of B, exactly in the $V_1,\ldots,V_m$ profile. We know that there exist positive definite completions of $\mathcal{B}$, as B is an example. According to [10] there is a unique determinant maximizing positive definite completion of $\mathcal{B}$, and the inverse of this matrix necessarily has all entries equal to 0 outside the $V_1, \ldots, V_m$ profile. Call this matrix A. By the theorem of [2], the determinant of A is given by the right hand side of (4.3). Thus A is the (unique) matrix whose existence the theorem asserts. ∎

6. Relationship to Earlier Inequalities

In the late 1960's a series of inequalities generalizing Hadamard-Fischer appeared, including, for example, [3, 4, 6, 7, 11]. Our results here show that we have indicated all possible extensions of the type of (4.3). However, it is of interest to compare the inequalities here to some prior inequalities. Those of [4] are among the strongest for their simplicity and breadth. There it is shown that if $V_1, \ldots, V_m \subseteq N$ are arbitrary index sets satisfying (4.1), then

$$\det A \leqslant \frac{\prod_{k=1}^{m} A_{V_k}}{\prod_{k=2}^{m} A_{U_k}} , \tag{6.1}$$

in which $U_k \subseteq N$ is the set of indices appearing at least k times among the V's, $k = 2, \ldots, m$. Our Theorem 1, of course, does not directly apply to an arbitrary collection of index sets $V_1, \ldots, V_m$, as the intersection graph G_I may not have a spanning forest satisfying the intersection property (4.2). However, for a collection of index sets to which Theorem 1 does apply, it generally gives a stronger inequality than (6.1) based upon simpler information. In addition, Theorem 1 can be applied to an arbitrary collection of index sets by using it on various subcollections (and possibly principal submatrices). We illustrate comparison of (4.3) to the sample collection of inequalities (6.1) with a few examples and then indicate why (4.3) is a generally stronger inequality.

If $n = 3$ and $V_1 = \{1,2\}$, $V_2 = \{2,3\}$ and $V_3 = \{1,3\}$, then there is no spanning forest of the intersection graph satisfying (4.2). Thus, Theorem 1 does not directly apply, while (6.1) yields the inequality

$$(\det A)^2 \leqslant A_{\{1,2\}} A_{\{2,3\}} A_{\{1,3\}}$$

However, two applications of Theorem 1, to the collection $\{V_1, V_2\}$ and to $\{V_1, V_3\}$, yield

$$\det A \leqslant \frac{A_{\{1,2\}} A_{\{2,3\}}}{a_{22}} ,$$

$$\det A \leqslant \frac{A_{\{1,2\}} A_{\{1,3\}}}{a_{11}} ,$$

and, therefore,

$$(\det A)^2 \leqslant A_{\{1,2\}} A_{\{2,3\}} A_{\{1,3\}} \frac{A_{\{1,2\}}}{a_{11} a_{22}} .$$

The latter is stronger than that derived from (6.1) as the factor $\frac{A_{\{1,2\}}}{a_{11} a_{22}} \leqslant 1$ by Hadamard's inequality. Alternatively, consider the collection of index sets $V_1 = \{1,2\}$, $V_2 = \{2,3\}$, $V_3 = \{3,4\}$, $V_4 = \{4,5\}$ for $n = 5$. In this case, Theorem 1 and (6.1) both apply directly. The inequality (6.1) gives

$$\det A \leqslant \frac{A_{\{1,2\}} A_{\{2,3\}} A_{\{3,4\}} A_{\{4,5\}}}{A_{\{2,3,4\}}} ,$$

while Theorem 1 yields

$$\det A \leqslant \frac{A_{\{1,2\}} A_{\{2,3\}} A_{\{3,4\}} A_{\{4,5\}}}{a_{22} a_{33} a_{44}} .$$

The latter is a stronger (and simpler) bound, as $a_{22} a_{33} a_{44} \geqslant A_{\{2,3,4\}}$, again by Hadamard's inequality. This illustrates the fact that the indices tend to occur in a more decoupled manner in the denominator of (4.3) than in (6.1), making (4.3) the stronger and simpler.

Observation. Under the assumptions of Theorem 1 both inequalities (4.3) and (6.1) hold and the right hand side of each has the same numerator. However, the denominator of the right hand side of (4.3) is at least as great as that of (6.1), and the index set of every minor in the denominator of the right hand side of (4.3) is contained in one of those for (6.1). Thus, (4.3) is generally a tighter and simpler bound than (6.1).

Proof: Consider the collection of index sets $V_i \cap V_j$ appearing in the denominator of the right hand side of (4.3); call them $W_1, \ldots, W_{m-1}$. Notice that U_k is just the collection of indices appearing at least $k-1$ times among the W's, $k = 2, \ldots, m$. Therefore, applying (6.1) to the collection $W_1, \ldots, W_{m-1}$ and the principal submatrix $A(U_2)$, we have

$$\frac{A_{U_2} A_{U_3} \cdots A_{U_m}}{A_{W_1} A_{W_2} \cdots A_{W_{m-1}}} \leqslant 1 \quad .$$

This verifies the statement about the relative sizes of the two denominators. Moreover, the intersection property implies that each $W_i \subset U_2$, verifying the statement about index set inclusion. Note also that the relation of the U's to the W's implies that the first ℓ U's always contain as many indices, counting multiplicity, as any ℓ W's.

7. Application to Positive Definite Completions.

We have already made use of the ideas of [10], which was partly motivated by the relationship with determinantal inequalities. There it was shown that if a partial Hermitian matrix has a positive definite completion, then there is a unique one with maximum determinant and it is the same as the unique one whose inverse has zeroes in the positions of the unspecified entries of the partial Hermitian matrix. This means that knowledge of the maximum determinant implies a sharp bound in terms of the specified entries for any positive definite matrix with these specified entries. Actual values for these bounds were not explored in [10]. In case $V_1, \ldots, V_m$ are index sets satisfying the conditions of Theorem 1 and A is a partial Hermitian matrix whose specified entries occur exactly in the $V_1, \ldots, V_m$ profile, then Theorem 1 explicitly gives the best bound in terms of the specified minors. In case the specified entries do not coincide with the profile, a subset of the specified entries might be used to give a bound using Theorem 1, but in general, this will not be the best possible. The question of the best bound seems to be open when Theorem 1 does not fully apply. It appears that the answer cannot have such a simple form. For example, positive definite matrices of the form

$$\begin{bmatrix} 2 & -1 & a & -1 \\ -1 & 2 & -1 & b \\ \bar{a} & -1 & 2 & -1 \\ -1 & \bar{b} & -1 & 2 \end{bmatrix}$$

must have determinants bounded by the sharp bound of $16\sqrt{3} - 25$, but it is difficult to imagine a "simple" way (with Theorem 1 as the standard) this might be calculated from the specified entries.

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