DIPERNA—MAJDA MEASURES AND UNIFORM INTEGRABILITY

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The purpose of this note is to discuss the relationship among Rosenthal’s modulus of uniform integrability, Young measures and DiPerna-Majda measures. In particular, we give an explicit characterization of this modulus and state a criterion of the uniform integrability in terms of these measures. Further, we show applications to Fatou’s lemma. Finally, we get an assertion related to equivalence of the weak and strong convergence in $L^p(\Omega; \mathbb{R}^m)$ under certain conditions and make some remarks on “concentrations” of non-uniformly integrable sequences.

Key words. Bounded sequences, DiPerna-Majda measures, Fatou’s lemma, relative sequential weak compactness, uniform integrability, Young measures.

AMS subject classification: 28A05, 28A20, 28A33, 40A30

INTRODUCTION

We will consider a problem of uniform integrability of bounded sequences in $L^1(\Omega; \mathbb{R}^m)$ where $\Omega \subset \mathbb{R}^m$ is a bounded domain. The uniform integrability is equivalent to the relative (sequential) weak $L^1$-compactness of the sequence in question via the Dunford-Pettis compactness criterion; cf. e.g. [13, Sec. IV.8] or [31]. Briefly, any bounded and uniformly integrable sequence $\{u_k\}_{k \in \mathbb{N}}$ in $L^1(\Omega; \mathbb{R}^m)$ contains a subsequence (denoted by the same way) that \( w\text{-}\lim_{k \to \infty} u_k = u \) where $u \in L^1(\Omega; \mathbb{R}^m)$ and “w” denotes the weak limit. The opposite implication is also valid: weakly converging sequences in $L^1(\Omega; \mathbb{R}^m)$ are uniformly integrable. This additional requirement, namely the uniform integrability, on bounded sequences in $L^1(\Omega; \mathbb{R}^m)$ to be relatively weakly compact reflects the non-reflexiveness of $L^1(\Omega; \mathbb{R}^m)$. Let us recall that a bounded sequence $\{u_k\}_{k \in \mathbb{N}} \subset L^1(\Omega; \mathbb{R}^m)$ is said to be uniformly integrable if

$$\forall \varepsilon > 0 \ \exists K > 0 : \sup_{k \in \mathbb{N}} \int_{\{x \in \Omega; |u_k(x)| > K\}} |u_k(x)| \, dx \leq \varepsilon . \quad (1)$$

We refer e.g. to [9, 11, 13, 22] for other criteria insuring the relative weak compactness.
Saadoune and Valadier [25] introduced the so-called Rosenthal modulus of uniform integrability $\eta$; cf. also [7, 19]. Taking a bounded sequence $\{u_k\}_{k \in \mathbb{N}}$ in $L^1(\Omega; \mathbb{R}^m)$ then ("meas" stands for the Lebesgue measure on $\mathbb{R}^n$)

$$\eta(\{u_k\}_{k \in \mathbb{N}}) = \lim_{\varepsilon \to 0^+} \left[ \sup_{k \in \mathbb{N}} \left\{ \int_A |u_k(x)| \, dx; \ \text{meas}(A) \leq \varepsilon \right\} \right].$$

It is proved in [25] (see also [27]) that $\eta(\{u_k\}_{k \in \mathbb{N}})$ can be equivalently expressed as

$$\eta(\{u_k\}_{k \in \mathbb{N}}) = \lim_{K \to \infty} \left[ \sup_{k \in \mathbb{N}} \int_{\{x \in \Omega: |u_k(x)| \leq K\}} |u_k(x)| \, dx \right]. \quad (2)$$

We remark that the sequence $\{u_k\}_{k \in \mathbb{N}}$ is uniformly integrable if and only if $\eta(\{u_k\}_{k \in \mathbb{N}}) = 0$.

To understand better the meaning of Rosenthal's modulus let us suppose that $\{u_k\}_{k \in \mathbb{N}}$ in $L^1(\Omega; \mathbb{R}^m)$ is not uniformly integrable. This means that

$$\exists \varepsilon > 0 \ \forall K > 0 : \sup_{k \in \mathbb{N}} \int_{\{x \in \Omega: |u_k(x)| \geq K\}} |u_k(x)| \, dx > \varepsilon. \quad (3)$$

Then $\eta(\{u_k\}_{k \in \mathbb{N}})$ is the supremum of all of these $\varepsilon$'s. This means that for Rosenthal's modulus instead of $\varepsilon$ the sharp inequality (3) changes to $\geq$. In fact, this is the most convenient definition for our purposes and we are going to use it. Let us just mention that we will sometimes speak about the uniform integrability without saying explicitly that we mean that one in $L^1(\Omega; \mathbb{R}^m)$.

In what follows $L^p(\Omega; \mathbb{R}^m)$, $1 \leq p \leq +\infty$ is the usual Lebesgue space of measurable functions $\Omega \to \mathbb{R}^m$ that are integrable with their $p$-th power (for $1 \leq p < +\infty$) or essentially bounded on $\Omega$ (if $p = +\infty$). If $m = 1$, we write only $L^p(\Omega)$ instead of $L^p(\Omega; \mathbb{R}^1)$. For more information we refer e.g. to [13].

The aim of the note is twofold. Firstly, to show how modern mathematical apparatus of Young measures and their generalizations fits in the classical topic as the uniform integrability, secondly, to provide better understanding of these generalizations. The plan of this paper is as follows. First, we briefly introduce Young measures (see [4, 20, 26, 32, 33]) and their generalization called DiPerna-Majda measures; cf. [12, 21]. Afterwards, we study the relation between Rosenthal's modulus and measures of DiPerna and Majda. In particular, we give an explicit characterization of this modulus and show its intimate relationship to the support of these measures. We also touch properties of DiPerna-Majda measures which were analyzed in detail in [21]. This enables us to find a new characterization of uniformly integrable sequences. Further, we apply our results to the Fatou lemma getting thus simple and straightforward proofs of interesting inequalities involving Young measures. We finish the paper by a statement which says when the weak convergence in $L^p(\Omega; \mathbb{R}^m)$ implies the strong convergence and by a description of "concentrations" in terms of DiPerna-Majda measures.

**Young measures.** The Young measures [32] represent a modern mathematical tool to hold certain "limit" information about oscillations in nonlinear problems arising in optimal control theory, variational calculus, partial differential equations, game theory, etc; more
details about Young measures can be found, e.g., in [4, 6, 10, 15, 20, 21, 24, 23, 28, 29, 31]. The Young measures on a domain \( \Omega \subset \mathbb{R}^n \) are weakly measurable mappings \( x \mapsto \nu_x : \Omega \to rca(\mathbb{R}^m) \) with values in probability measures; “rca” denotes the set of regular countably additive set functions on the Borel \( \sigma \)-algebra on \( \mathbb{R}^m \) (cf. [13]) with a bounded total variation and the adjective “weakly measurable” means that, for any \( v \in C_0(\mathbb{R}^m) \), the mapping \( \Omega \to \mathbb{R} : x \mapsto \langle \nu_x, v \rangle = \int_{\mathbb{R}^m} v(\lambda) \nu_x(d\lambda) \) is measurable in the usual sense. Let us remind that, by the Riesz theorem, \( rca(\mathbb{R}^m) \), normed by the total variation, is a Banach space which is isometrically isomorphic with \( C_0(\mathbb{R}^m)^\ast \), where \( C_0(\mathbb{R}^m) \) stands for the space of all continuous functions \( \mathbb{R}^m \to \mathbb{R} \) vanishing at infinity. Let us denote the set of all Young measures by \( \mathcal{Y}(\Omega; \mathbb{R}^m) \). It is known that \( \mathcal{Y}(\Omega; \mathbb{R}^m) \) is a convex subset of \( L^\infty_w(\Omega; rca(\mathbb{R}^m)) \cong L^1(\Omega; C_0(\mathbb{R}^m))^\ast \), where the subscript “w” indicates the property “weakly measurable”. A classical result [10, 28, 32] is that, for every sequence \( \{ u_k \}_{k \in \mathbb{N}} \) bounded in \( L^\infty(\Omega; \mathbb{R}^m) \), there exists its subsequence (denoted by the same indices for notational simplicity) and a Young measure \( \nu = \{ \nu_x \}_{x \in \Omega} \in \mathcal{Y}(\Omega; \mathbb{R}^m) \) such that

\[
\forall v \in C_0(\mathbb{R}^m) : \quad \lim_{k \to \infty} v \circ u_k = v_{\nu} \quad \text{weakly}^* \text{ in } L^\infty(\Omega),
\]

where \( [v \circ u_k](x) = v(u_k(x)) \) and

\[
v_{\nu}(x) = \int_{\mathbb{R}^m} v(\lambda)\nu_x(d\lambda).
\]

Let us denote by \( \mathcal{Y}^\infty(\Omega; \mathbb{R}^m) \) the set of all Young measures which are created by this way, i.e. by taking all bounded sequences in \( L^\infty(\Omega; \mathbb{R}^m) \). Note that (4) actually holds for any \( v : \mathbb{R}^m \to \mathbb{R} \) continuous.

A generalization of this result was formulated by Maria Schonbek [26] (cf. also [4], for \( p = 1 \) especially [20] and [24] where further generalization in this direction has been performed) for the case \( 1 \leq p < +\infty \): for every sequence \( \{ u_k \}_{k \in \mathbb{N}} \) bounded in \( L^p(\Omega; \mathbb{R}^m) \) there exists its subsequence (denoted by the same indices) and a Young measure \( \nu = \{ \nu_x \}_{x \in \Omega} \in \mathcal{Y}(\Omega; \mathbb{R}^m) \) such that

\[
\forall v \in C_p(\mathbb{R}^m) : \quad \lim_{k \to \infty} v \circ u_k = v_{\nu} \quad \text{weakly in } L^1(\Omega),
\]

where

\[
C_p(\mathbb{R}^m) = \{ v \in C(\mathbb{R}^m); v(\lambda) = o(|\lambda|^p) \text{ for } |\lambda| \to \infty \}.
\]

We denote by \( \mathcal{Y}^p(\Omega; \mathbb{R}^m) \) the set of all Young measures which are created by this way, i.e. by taking all bounded sequences in \( L^p(\Omega; \mathbb{R}^m) \). The reader can find in [20] that \( \mathcal{Y}^p(\Omega; \mathbb{R}^m) = \{ v \in \mathcal{Y}(\Omega; \mathbb{R}^m); \int_{\Omega} \int_{\mathbb{R}^m} |\lambda|^p \nu_x(d\lambda) \, dx < +\infty \} \). We call a sequence \( \{ u_k \}_{k \in \mathbb{N}} \) satisfying (6) a generating sequence of \( \nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m) \).

**DiPerna-Majda measures.** Sometimes nonlinear problems may exhibit, beside the rapid-oscillation phenomena, also concentration effects which were previously neglected because the \( L^p \)-Young measures admits only test functions with the growth strictly lower than
DiPerna and Majda ([12]) developed a tool to handle both oscillation and concentration effects simultaneously. Let us take a complete (i.e. containing constants and separating points from closed subsets) separable (i.e. containing a countable subset which is dense with respect to the supremum norm) ring \( \mathcal{R} \) of continuous bounded functions \( \mathbb{IR}^m \rightarrow \mathbb{IR} \). It is known (cf. [14, §3.12.21]) that there is a one-to-one correspondence \( \mathcal{R} \leftrightarrow \beta_{\mathcal{R}} \mathbb{IR}^m \) between such rings and metrizable compactifications of \( \mathbb{IR}^m \); by a compactification we mean here a compact set, denoted by \( \beta_{\mathcal{R}} \mathbb{IR}^m \) into which \( \mathbb{IR}^m \) is embedded homeomorphically and densely. We will not distinguish between \( \mathbb{IR}^m \) and its image in \( \beta_{\mathcal{R}} \mathbb{IR}^m \). If we take, for example, the smallest complete ring containing only continuous functions possessing limits for \( |\lambda| \rightarrow \infty \), then the corresponding compactification is the one point (Alexandroff) compactification, i.e., \( \beta_{\mathcal{R}} \mathbb{IR}^m = \mathbb{IR}^m \cup \{ \infty \} \). See [12, 16, 24] for other examples.

DiPerna and Majda showed that, having a bounded sequence \( \{ u_k \}_{k \in \mathbb{N}} \) in \( L^p(\Omega; \mathbb{IR}^m) \) with \( 1 \leq p < +\infty \) and \( \Omega \) an open domain in \( \mathbb{IR}^n \), there exists its subsequence (denoted by the same indices), a positive Radon measure \( \sigma \in rca(\bar{\Omega}) \), and a Young measure \( \hat{\nu} \in \mathcal{Y}(\bar{\Omega}, \sigma; \beta_{\mathcal{R}} \mathbb{IR}^m) \) (i.e. we consider here the closure \( \bar{\Omega} \) of \( \Omega \) endowed with the Radon measure \( \sigma \) instead of the Lebesgue measure as previously) such that

\[
\forall g \in C(\bar{\Omega}) \forall v_0 \in \mathcal{R} : \lim_{k \to \infty} \int_{\Omega} g(x)v(u_k(x))dx = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}} \mathbb{IR}^m} g(x)v_0(\lambda)\hat{\nu}_x(d\lambda)\sigma(dx), \quad (7)
\]

where \( v(\lambda) = v_0(\lambda)(1 + |\lambda|^p) \). In particular, putting \( v_0 = 1 \in \mathcal{R} \) we can see that

\[
\lim_{k \to \infty} (1 + |u_k|^p) = \sigma \quad \text{weakly}^* \text{ in } rca(\bar{\Omega}).
\]

Let us again denote by \( D\mathcal{M}_R^p(\Omega; \mathbb{IR}^m) \) the set of all pairs \( (\sigma, \hat{\nu}) \in rca(\bar{\Omega}) \times \mathcal{Y}(\bar{\Omega}, \sigma; \beta_{\mathcal{R}} \mathbb{IR}^m) \) created by this way, i.e. \( D\mathcal{M}_R^p(\Omega; \mathbb{IR}^m) \) contains just such \( (\sigma, \hat{\nu}) \) for which there exists a sequence \( \{ u_k \}_{k \in \mathbb{N}} \) such that (7) holds; note that, taking \( v_0 = 1 \), we can see that such a sequence must be inevitably bounded in \( L^p(\Omega; \mathbb{IR}^m) \). Also here the sequence appearing in (7) is called the generating sequence of \( (\sigma, \hat{\nu}) \in D\mathcal{M}_R^p(\Omega; \mathbb{IR}^m) \). We refer to [21] for properties and the full explicit description of \( D\mathcal{M}_R^p(\Omega; \mathbb{IR}^m) \).

Recently T. Roubíček (see [20, 24]) proved the following result.

**Proposition 1.** Let \( \{ u_k \}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{IR}^m) \) be a generating sequence of \( (\sigma, \hat{\nu}) \in D\mathcal{M}_R^p(\Omega; \mathbb{IR}^m) \). Then \( \{ |u_k|^p \}_{k \in \mathbb{N}} \) is uniformly integrable if and only if

\[
\int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}} \mathbb{IR}^m \setminus \mathbb{IR}^m} \hat{\nu}_x(d\lambda)\sigma(dx) = 0. \quad (8)
\]

In particular, taking \( p = 1 \) we have that \( \{ u_k \}_{k \in \mathbb{N}} \) is uniformly integrable if and only if (8) is valid.
1. DIPERNA-MAJDA MEASURES AND THE ROSENTHAL MODULUS

Let us start with the following lemma.

**Lemma 1.** Let \( \{u_k\}_{k \in \mathbb{N}} \subset L^1(\Omega) \) be bounded. Then \( \eta(\{|u_k|\}_{k \in \mathbb{N}}) = \eta(\{1 + |u_k|\}_{k \in \mathbb{N}}) \).

**Proof.** This is easy. \( \square \)

Now we show that if \((\sigma, \dot{\nu}) \in \mathcal{DM}_R^p(\Omega; \mathbb{R}^m)\) then \(\int_{\Omega} \int_{\beta R \mathbb{R}^m \setminus \mathbb{R}^m} \dot{\nu}_x(d\lambda)\sigma(dx)\) does not depend on the particular compactification of \(\mathbb{R}^m\), i.e., on \(\beta R \mathbb{R}^m\). In other words, we show that it is only related to the generating sequence.

**Proposition 2.** Let \( R \) and \( R' \) be two separable complete and closed rings of continuous bounded functions \( \mathbb{R}^m \to \mathbb{R} \) and let \( \{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m) \), \( 1 \leq p < +\infty \) generate \((\sigma, \dot{\nu}) \in \mathcal{DM}_R^p(\Omega; \mathbb{R}^m)\) and also \((\sigma', \dot{\nu}') \in \mathcal{DM}_{R'}^p(\Omega; \mathbb{R}^m)\). Then

\[
\int_{\Omega} \int_{\beta R \mathbb{R}^m \setminus \mathbb{R}^m} \dot{\nu}_x(d\lambda)\sigma(dx) = \int_{\Omega} \int_{\beta R' \mathbb{R}^m \setminus \mathbb{R}^m} \dot{\nu}'_x(d\lambda)\sigma'(dx).
\]

**Proof.** Let us take \( g = 1 \) and \( v_0 = 1 \) in (7). Then we have

\[
\lim_{k \to \infty} \int_{\Omega} (1 + |u_k(x)|^p) \, dx = \int_{\Omega} \int_{\mathbb{R}^m} \dot{\nu}_x(d\lambda)\sigma(dx) + \int_{\Omega} \int_{\beta R \mathbb{R}^m \setminus \mathbb{R}^m} \dot{\nu}_x(d\lambda)\sigma(dx).
\]

On the other hand, we can write Lebesgue's decomposition of \(\sigma(dx) = d_\sigma(x)dx + \sigma^s(dx)\), where \(d_\sigma \in L^1(\Omega)\) is the density of the absolutely continuous part of \(\sigma\) with respect to the Lebesgue measure and \(\sigma^s\) is the singular part of \(\sigma\). It follows from [21, Th. 2] that the support of \(\sigma^s\), \(\text{supp} \, \sigma^s \subset \{x \in \Omega; \int_{\mathbb{R}^m} \dot{\nu}_x(d\lambda) = 0\}\). Therefore, we have

\[
\int_{\Omega} \int_{\mathbb{R}^m} \dot{\nu}(d\lambda)\sigma(dx) = \int_{\Omega} \int_{\mathbb{R}^m} \dot{\nu}(d\lambda)d_\sigma(x)dx + \int_{\Omega} \int_{\mathbb{R}^m} \dot{\nu}(d\lambda)d_\sigma(x)dx.
\]

Finally, it follows from [20, Formulae (13-15)] and [21, Th. 1] that \(\{u_k\}_{k \in \mathbb{N}}\) generates a Young measure \(\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)\) given for almost all \(x \in \Omega\) by

\[
\nu_x(d\lambda) = d_\sigma(x)\frac{\dot{\nu}_x(d\lambda)}{1 + |\lambda|^p}.
\]

Eventually, we can write (9) as

\[
\lim_{k \to \infty} \int_{\Omega} (1 + |u_k(x)|^p) \, dx = \int_{\Omega} \int_{\mathbb{R}^m} (1 + |\lambda|^p)\nu_x(d\lambda) \, dx + \int_{\Omega} \int_{\beta R \mathbb{R}^m \setminus \mathbb{R}^m} \dot{\nu}_x(d\lambda)\sigma(dx).
\]

The same procedure can be done also for \((\sigma', \dot{\nu}')\) and we obtain

\[
\lim_{k \to \infty} \int_{\Omega} (1 + |u_k(x)|^p) \, dx = \int_{\Omega} \int_{\mathbb{R}^m} (1 + |\lambda|^p)\nu_x(d\lambda) \, dx + \int_{\Omega} \int_{\beta R' \mathbb{R}^m \setminus \mathbb{R}^m} \dot{\nu}'_x(d\lambda)\sigma'(dx).
\]
The subtraction of the last equality from the last but one gives the assertion of the proposition.

\[\square\]

We will need the following auxiliary and easy lemma.

**Lemma 2.** Let \(\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)\) be bounded and generate \(\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)\). Then also any subsequence of \(\{u_k\}_{k \in \mathbb{N}}\) generates the same \(\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)\).

**Proof.** The proof is quite the same as Step 5 in the proof of [24, Prop. 3.2.9]. We include it just for reader’s convenience.

Let \(\{u_{k_i}\}_{i \in \mathbb{N}}\) be a subsequence of \(\{u_k\}_{k \in \mathbb{N}}\). This subsequence generates a Young measure \(\tilde{\nu} \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)\). We have that \(w\lim_{i \to \infty} v(u_{k_i}) = v_{\tilde{\nu}}\) in \(L^1(\Omega)\) and also \(w\lim_{i \to \infty} v(u_{k_i}) = v_\nu\) in \(L^1(\Omega)\) for any \(v \in C_0(\mathbb{R}^m)\) where \(v_{\tilde{\nu}}\) and \(v_\nu\) are given as in (5). Thus, \(v_\nu = v_{\tilde{\nu}}\) or saying otherwise

\[
\int_{\mathbb{R}^m} v(\lambda) \nu_x(d\lambda) = \int_{\mathbb{R}^m} v(\lambda) \tilde{\nu}_x(d\lambda) \text{ for a.a. } x \in \Omega.
\]

Let \(\{v^i\}_{i \in \mathbb{N}}\) denote a countable dense subset of the separable space \(C_0(\mathbb{R}^m)\). Then for any \(i \in \mathbb{N}\) there is \(A_i \subset \Omega\), \(\text{meas}(A_i) = 0\) and \(\int_{\mathbb{R}^m} v^i(x) \nu_x(d\lambda) = \int_{\mathbb{R}^m} v^i(x) \tilde{\nu}_x(d\lambda)\) for any \(x \in \Omega \setminus A_i\). Altogether we have for any \(i \in \mathbb{N}\) and any \(x \in \hat{\Omega}\)

\[
\int_{\mathbb{R}^m} v^i(\lambda) \nu_x(d\lambda) = \int_{\mathbb{R}^m} v^i(\lambda) \tilde{\nu}_x(d\lambda)
\]

where \(\hat{\Omega} = \Omega \setminus \bigcup_{i=1}^{\infty} A_i\). Note that \(\text{meas}(\hat{\Omega}) = \text{meas}(\Omega)\). Finally, we get by the continuity argument that (10) is fulfilled even for any \(v \in C_0(\mathbb{R}^m)\) and any \(x \in \hat{\Omega}\) which shows that \(\tilde{\nu}_x = \nu_x\) everywhere on \(\hat{\Omega}\) and therefore almost everywhere on \(\Omega\).

\[\square\]

**Remark 1.** (i) Proposition 2 and Lemma 2 lead us to the following conclusion. If \(\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)\) is bounded, generates a Young measure \(\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)\) and, moreover, there exists \(\lim_{k \to \infty} \int_{\Omega} |u_k(x)|^p \, dx\), then

\[
\lim_{k \to \infty} \int_{\Omega} |u_k(x)|^p \, dx = \int_{\Omega} \int_{\mathbb{R}^m} |\lambda|^p \nu_x(d\lambda) \, dx + \int_{\Omega} \int_{\beta \mathbb{R} \setminus \Omega} \tilde{\nu}_x(d\lambda) \sigma(dx) = \sigma(\hat{\Omega}) - \text{meas}(\Omega),
\]

where \((\sigma, \tilde{\nu}) \in \mathcal{DM}_R^p(\Omega; \mathbb{R}^m)\) is an arbitrary DiPerna-Majda measure generated by some subsequence of \(\{u_k\}_{k \in \mathbb{N}}\). Indeed, for such subsequence, say \(\{u_{k_i}\}_{i \in \mathbb{N}}\), the last equality obviously holds and also \(\lim_{k \to \infty} \int_{\Omega} |u_k(x)|^p \, dx = \lim_{i \to \infty} \int_{\Omega} |u_{k_i}(x)|^p \, dx\).

(ii) Following the same proof as that of the previous lemma we can show that once \(\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m), 1 \leq p < +\infty\) is bounded and generates a Young measure \(\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)\) than this sequence generates also \(\tilde{\nu} \in \mathcal{Y}^1(\Omega; \mathbb{R}^m)\) and \(\nu = \tilde{\nu}\).

The next proposition characterizes \(\eta\).

**Proposition 3.** Let \(\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m), 1 \leq p < +\infty\) generate \((\sigma, \tilde{\nu}) \in \mathcal{DM}_R^p(\Omega; \mathbb{R}^m)\). Then

\[
\eta(\{\|u_k\|^p\}_{k \in \mathbb{N}}) = \int_{\Omega} \int_{\beta \mathbb{R} \setminus \Omega} \tilde{\nu}_x(d\lambda) \sigma(dx).
\]
Proof. The main idea is basically taken from Tomáš Roubíček’s proof of Proposition 1 Utilizing Lemma 1 it is sufficient to look for Rosenthal’s modulus of \( \{1 + |u_k|^p\}_{k \in \mathbb{N}} \). If \( \{ |u_k|^p \}_{k \in \mathbb{N}} \) is uniformly integrable the assertion follows from Proposition 1. Let us suppose that \( \{ |u_k|^p \}_{k \in \mathbb{N}} \subset L^1(\Omega) \) is not uniformly integrable. First, let us abbreviate

\[
T = \int_{\Omega} \int_{\beta^{-1} \mathbb{R}^m \setminus \mathbb{R}^m} \hat{\nu}_x(d\lambda) \sigma(dx) .
\]

Further, we define for any \( \varrho \geq 0 \) the function \( v_0^\varrho : \mathbb{R}^m \to \mathbb{R} \)

\[
v_0^\varrho(\lambda) = \begin{cases} 
0 & \text{if } |\lambda| \leq \varrho \\
|\lambda| - \varrho & \text{if } \varrho \leq |\lambda| \leq \varrho + 1 \\
1 & \text{if } |\lambda| \geq \varrho + 1 .
\end{cases}
\]

Note that always \( v_0^\varrho \in \mathcal{R} \). We can estimate for any \( \varrho > 0 \)

\[
T = \int_{\Omega} \int_{\beta^{-1} \mathbb{R}^m \setminus \mathbb{R}^m} \hat{\nu}_x(d\lambda) \sigma(dx) 
\leq \int_{\Omega} \int_{\beta^{-1} \mathbb{R}^m} v_0^\varrho(\lambda) \hat{\nu}_x(d\lambda) \sigma(dx)
= \lim_{k \to \infty} \int_{\Omega} v_0^\varrho(u_k(x))(1 + |u_k(x)|^p)dx
\leq \sup_{k \in \mathbb{N}} \int_{\Omega \times \{ x \in \Omega \mid |u_k(x)| \geq \varrho \}} 1 + |u_k(x)|^p dx .
\]

This gives us that \( \eta(\{ u_k \}_{k \in \mathbb{N}}) \geq T \).

To finish the proof we have to show that \( T \) is the supremum of all of \( \hat{T} \)'s satisfying

\[
\forall K > 0 : \sup_{k \in \mathbb{N}} \int_{\{ x \in \Omega \mid |u_k(x)| \geq K \}} 1 + |u_k(x)|^p dx > \hat{T} .
\]

This will be done if for any \( \delta > 0 \) we find \( K(\delta) > 0 \) that \( \sup_{k \in \mathbb{N}} \int_{\{ x \in \Omega \mid |u_k(x)| \geq K(\delta) \}} 1 + |u_k(x)|^p dx < T + \delta \).

Let us define \( B_\varrho = \{ \lambda \in \mathbb{R}^m ; |\lambda| \leq \varrho \} \). We have from the Lebesgue dominated convergence theorem

\[
\lim_{\varrho \to -\infty} \int_{\Omega} \int_{\beta^{-1} \mathbb{R}^m \setminus B_\varrho} \hat{\nu}_x(d\lambda) \sigma(dx) = \int_{\Omega} \int_{\beta^{-1} \mathbb{R}^m \setminus \mathbb{R}^m} \hat{\nu}_x(d\lambda) \sigma(dx) = T .
\]

For any \( \delta > 0 \) we can find \( \varrho > 0 \) sufficiently large that

\[
\int_{\Omega} \int_{\beta^{-1} \mathbb{R}^m \setminus B_\varrho} \hat{\nu}_x(d\lambda) \sigma(dx) < T + \frac{\delta}{4} .
\]

On the other hand, there is \( k_\delta > 0 \) that for any \( k > k_\delta \)

\[
\left| \int_{\Omega} \int_{\beta^{-1} \mathbb{R}^m} v_0^\varrho(\lambda) \hat{\nu}_x(d\lambda) \sigma(dx) - \int_{\Omega} v_0^\varrho(u_k(x))(1 + |u_k(x)|^p)dx \right| < \frac{\delta}{4} .
\]

As \( v_0^\varrho = 0 \) on \( B_\varrho \) we have also

\[
\int_{\Omega} \int_{\beta^{-1} \mathbb{R}^m} v_0^\varrho(\lambda) \hat{\nu}_x(d\lambda) \sigma(dx) = \int_{\Omega} \int_{\beta^{-1} \mathbb{R}^m \setminus B_\varrho} v_0^\varrho(\lambda) \hat{\nu}_x(d\lambda) \sigma(dx) .
\]
Altogether we obtain that 
\[ \int \Omega v_0^0(u_k(x))(1 + |u_k(x)|^p)dx < T + \delta/2 \text{ for any } k > k_\delta. \]
Thus, for any \(k > k_\delta\) also
\[ \int_{\{ x \in \Omega; \ |u_k(x)| \geq \epsilon + 1 \}} 1 + |u_k(x)|^p dx \leq \int \Omega v_0^0(u_k(x))(1 + |u_k(x)|^p)dx < T + \frac{\delta}{2} \]
and finally,
\[ \sup_{k > k_\delta} \int_{\{ x \in \Omega; \ |u_k(x)| \geq \epsilon + 1 \}} 1 + |u_k(x)|^p dx < T + \frac{3\delta}{4}. \]
We end up the proof recalling that the finite set \(\{1 + |u_k|, k = 1, \ldots, k_\delta\}\) is obviously uniformly integrable, hence, we can take \(\delta > 0\) such that \(\sup_{k \in \{1, \ldots, k_\delta\}} \int_{\{ x \in \Omega; \ |u_k(x)| \geq \delta \}} 1 + |u_k(x)|^p dx \leq \delta/4.\)
Eventually, we get
\[
\sup_{k \in \mathbb{N}} \int_{\{ x \in \Omega; \ |u_k(x)| \geq \max(\epsilon + 1, \delta) \}} 1 + |u_k(x)|^p dx
\leq \sup_{k \in \{1, \ldots, k_\delta\}} \int_{\{ x \in \Omega; \ |u_k(x)| \geq \max(\epsilon + 1, \delta) \}} 1 + |u_k(x)|^p dx
+ \sup_{k > k_\delta} \int_{\{ x \in \Omega; \ |u_k(x)| \geq \max(\epsilon + 1, \delta) \}} 1 + |u_k(x)|^p dx
< T + \frac{\delta}{4} + \frac{3\delta}{4} = T + \delta.
\]
As \(\delta > 0\) has been arbitrary we see that \(T = \sup \{ \tilde{T}; \ \tilde{T} \text{ satisfies (11)} \}\) and thus \(T = \eta(\{|u_k|^p\}_{k \in \mathbb{N}})\). The proposition is proved.

**Remark 2.** (i) Let us mention that due to [25, Th 4.5] any bounded sequence \(\{u_k\}_{k \in \mathbb{N}} \subset L^1(\Omega; \mathbb{R}^m)\) contains a subsequence with the Rosenthal modulus equal to \(\eta(\{|u_k\}_{k \in \mathbb{N}})\).

(ii) Our previous result says that once \(\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m), 1 \leq p < +\infty\) generates \((\sigma, \hat{\nu}) \in \mathcal{DM}_R^p(\Omega; \mathbb{R}^m)\) then for any \(K > 0\)
\[
\sup_{k \in \mathbb{N}} \int_{\{ x \in \Omega; \ |u_k(x)| \geq K \}} |u_k(x)|^p dx \geq \int_{\Omega} \int_{\mathbb{R}^m \setminus \mathbb{R}^m} \hat{\nu}_x(\lambda)\sigma(d\lambda). \tag{12}
\]
Moreover, this lower bound is sharp in the sense that the number on the right-hand side is the largest possible. This clarifies the relationship between generating sequences of the DiPerna-Majda measure and the value of the integrals in (8). Especially, note that (12) is fulfilled for all of sequences that generate the particular DiPerna-Majda measure \((\sigma, \hat{\nu}), i.e.,\) whenever \(\{u_k\}_{k \in \mathbb{N}}\) and \(\{v_k\}_{k \in \mathbb{N}} \in L^p(\Omega; \mathbb{R}^m)\) generate \((\sigma, \hat{\nu})\) then \(\eta(\{|u_k|^p\}_{k \in \mathbb{N}}) = \eta(\{|v_k|^p\}_{k \in \mathbb{N}})\).

(iii) If \(\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)\) generates except a Young measure \(\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)\) also some DiPerna-Majda measure, we can write
\[
\lim_{k \to \infty} \int_{\Omega} |u_k(x)|^p dx = \int_{\Omega} \int_{\mathbb{R}^m} |\lambda|^p \nu_x(d\lambda) dx + \eta(\{|u_k|^p\}_{k \in \mathbb{N}}) .
\]

(iv) It follows from the proof of the previous proposition that if \(\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)\) generates \((\sigma, \hat{\nu}) \in \mathcal{DM}_R^p(\Omega; \mathbb{R}^m)\) then
\[
\eta(\{v_0(u_k)\}_{k \in \mathbb{N}}) = \int_{\Omega} \int_{\mathbb{R}^m \setminus \mathbb{R}^m} v_0(\lambda)\hat{\nu}_x(dx)\sigma(dx),
\]
where \(0 \leq v_0 \in \mathcal{R}\). (We can take any \(\mathcal{R}\) that contains \(v_0\).)
2. A CRITERION OF UNIFORM INTEGRABILITY AND OTHER APPLICATIONS

A uniform integrability criterion. First, we give a criterion of the uniform integrability.

**Proposition 4.** Let $\mathcal{R}$ be a separable complete and closed ring of continuous functions $\mathbb{R}^m \rightarrow \mathbb{R}$. Let $\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)$, $1 \leq p < +\infty$ be bounded. Then $\{|u_k|^p\}_{k \in \mathbb{N}}$ is uniformly integrable if and only if

$$S := \sup_{(\sigma, \hat{\nu}) \in \mathcal{U}} \int_{\Omega} \int_{\mathbb{R}^m \setminus \mathbb{R}^m} \hat{\nu}_x(d\lambda)\sigma(dx) = 0,$$

where $\mathcal{U}$ is a set of all $(\sigma, \hat{\nu}) \in \mathcal{D}\mathcal{M}_R^p(\Omega; \mathbb{R}^m)$ that are generated by some subsequence of $\{u_k\}_{k \in \mathbb{N}}$.

**Proof.** If $\{|u_k|^p\}_{k \in \mathbb{N}}$ is uniformly integrable, then its Rosenthal modulus equals to zero. Moreover, any subsequence of $\{|u_k|^p\}_{k \in \mathbb{N}}$ has the zero modulus. It follows that (13) is fulfilled.

When $\{|u_k|^p\}_{k \in \mathbb{N}}$ is not uniformly integrable then there is $\varepsilon > 0$ that

$$\forall K > 0 : \sup_{k \in \mathbb{N}} \int_{\{x \in \Omega; |u_k(x)| \geq K\}} |u_k(x)|^p \, dx > \varepsilon.$$

Let us define a subsequence of $\{u_k\}$ as follows: for $K \in \mathbb{N}$ let $v_K := u_k$ for such $u_k$ that

$$\int_{\{x \in \Omega; |u_k(x)| \geq K\}} |u_k(x)|^p \, dx > \varepsilon,$$

$v_l \neq u_k$ for any $l < K$ and the index $k$ is the smallest possible. It is easy to see that $\{|v_k|^p\}_{k \in \mathbb{N}}$ and any of its subsequences are not uniformly integrable and thus (13) cannot be satisfied.

In particular, if $\beta \mathbb{R}^m$ is just the one point Alexandroff compactification of $\mathbb{R}^m$, i.e., if $\beta \mathbb{R}^m \setminus \mathbb{R}^m = \{\infty\}$, then (13) reduces to

$$\forall (\sigma, \hat{\nu}) \in \mathcal{U} \quad \hat{\nu}_x(\infty) = 0 \quad \text{for } \sigma\text{-almost all } x \in \bar{\Omega}.$$

**Remark 3.** One can see that $S = \eta(|u_k|^p)_{k \in \mathbb{N}}$. Indeed, we have $\eta(|u_k|^p)_{k \in \mathbb{N}} \geq S$. If $\eta(|u_k|^p)_{k \in \mathbb{N}} > S$, then we would be able to extract a subsequence from $\{u_k\}_{k \in \mathbb{N}}$ which generates $(\sigma, \hat{\nu}) \in \mathcal{D}\mathcal{M}_R^p(\Omega; \mathbb{R}^m)$ and for which $\int_{\Omega} \int_{\mathbb{R}^m \setminus \mathbb{R}^m} \hat{\nu}_x(d\lambda)\sigma(dx) > S$ contrary to the definition of $S$ in Proposition 4.

**Applications to Fatou's lemma.** Let us apply our results to the generalized Fatou lemma.

**Proposition 5.** (see [19, 25]) Let $\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)$, $1 \leq p < +\infty$ be bounded and such that there exists $\lim_{k \to \infty} \int_{\Omega} |u_k(x)|^p \, dx$. Then

$$\int_{\Omega} \liminf_{k \to \infty} |u_k(x)|^p \, dx \leq \lim_{k \to \infty} \int_{\Omega} |u_k(x)|^p \, dx - \eta(|u_k|^p)_{k \in \mathbb{N}}.$$  
(14)
Taking our characterization of the Rosenthal modulus into the play we can come up with the following assertions.

**Corollary 1.** Under the assumptions of Proposition 5 (14) is equivalent to

\[
\int_{\Omega} \liminf_{k \to \infty} |u_k(x)|^p \, dx \leq \lim_{k \to \infty} \int_{\Omega} |u_k(x)|^p \, dx - \sup_{(\sigma, \nu) \in \mathcal{U}} \int_{\beta_{R} \mathbb{R}^m \setminus \mathbb{R}^m} \dot{\nu}_x(d\lambda)\sigma(dx),
\]

where \(\mathcal{U} \subset \mathcal{DM}^p_R(\Omega; \mathbb{R}^m)\) contains all of DiPerna-Majda measures generated by some subsequence of \(\{u_k\}_{k \in \mathbb{N}}\).

**Proof.** It follows immediately from Proposition 5 and Remark 3. \(\square\)

**Proposition 6.** Let \(\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m), 1 \leq p < +\infty\) be bounded and generate a Young measure \(\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)\). Then

\[
\int_{\Omega} \liminf_{k \to \infty} |u_k(x)|^p \, dx \leq \int_{\mathbb{R}^m} |\lambda|^p \nu_x(d\lambda) \, dx \leq \liminf_{k \to \infty} \int_{\Omega} |u_k(x)|^p \, dx.
\]

**Proof.** The second inequality is standard and can be found e.g. in [20, 29]. Let us prove the first one. Let \(\{u_{k_i}\}_{i \in \mathbb{N}}\) be such subsequence of \(\{u_k\}_{k \in \mathbb{N}}\) that generates except the Young measure \(\nu\) also some DiPerna-Majda measure. This means due to Remark 2 that

\[
\lim_{i \to \infty} \int_{\Omega} |u_{k_i}(x)|^p \, dx = \int_{\mathbb{R}^m} |\lambda|^p \nu_x(d\lambda) \, dx + \eta(\{u_{k_i}^p\}_{i \in \mathbb{N}}),
\]

where \(\eta(\{u_{k_i}^p\}_{i \in \mathbb{N}})\) is given by Proposition 3. Applying now Proposition 5 to this subsequence we have

\[
\int_{\Omega} \liminf_{i \to \infty} |u_{k_i}(x)|^p \, dx \leq \int_{\mathbb{R}^m} |\lambda|^p \nu_x(d\lambda) \, dx
\]

and because \(\int_{\Omega} \liminf_{k \to \infty} |u_k(x)|^p \, dx \leq \int_{\Omega} \liminf_{i \to \infty} |u_{k_i}(x)|^p \, dx\) we obtain the assertion. \(\square\)

**Corollary 2.** Let \(\{u_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m), 1 \leq p < +\infty\) be bounded. Then

\[
\int_{\Omega} \liminf_{k \to \infty} |u_k(x)|^p \, dx \leq \inf_{\nu \in \mathcal{U}} \int_{\mathbb{R}^m} |\lambda|^p \nu_x(d\lambda) \, dx \leq \liminf_{k \to \infty} \int_{\Omega} |u_k(x)|^p \, dx,
\]

where \(\mathcal{U} \subset \mathcal{Y}^p(\Omega; \mathbb{R}^m)\) contains all of Young measures generated by some subsequence of \(\{u_k\}_{k \in \mathbb{N}}\).

**Proof.** Let \(\{v_k\}_{k \in \mathbb{N}}\) be a subsequence of \(\{u_k\}_{k \in \mathbb{N}}\) that generates \(\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m)\). According to the Proposition 6

\[
\int_{\Omega} \liminf_{k \to \infty} |v_k(x)|^p \, dx \leq \int_{\mathbb{R}^m} |\lambda|^p \nu_x(d\lambda) \, dx,
\]

which gives

\[
\int_{\Omega} \liminf_{k \to \infty} |u_k(x)|^p \, dx \leq \int_{\mathbb{R}^m} |\lambda|^p \nu_x(d\lambda) \, dx.
\]
The first inequality follows straightforwardly.

Now let \( \{w_k\}_{k \in \mathbb{N}} \) be such subsequence of \( \{u_k\}_{k \in \mathbb{N}} \) that \( \lim_{k \to \infty} \int_\Omega |w_k(x)|^p \, dx = \liminf_{k \to \infty} \int_\Omega |u_k(x)|^p \, dx \) and that it generates \( \tilde{\nu} \in \mathcal{Y}^p(\Omega; \mathbb{R}^m) \). Then due to the previous proposition

\[
\int_\Omega \int_{\mathbb{R}^m} |\lambda|^p \tilde{\nu}_x(d\lambda) \, dx \leq \liminf_{k \to \infty} \int_\Omega |u_k(x)|^p \, dx
\]

from which we have the second inequality.

\[\square\]

**Remark 4.** Propositions 5 and 6 remain valid for any function \( 0 \leq v \in C_p(\mathbb{R}^m) \) instead of \( |\cdot|^p \).

**Generalized Visintin’s theorem.** In this part of the paper we generalize a result of A. Visintin ([30, Th. 1], see also [1, 2, 3]) getting thus conditions under which the weak convergence in \( L^p(\Omega; \mathbb{R}^m) \) implies the strong one. In spite that the generalization is fairly elementary, it demonstrates the usefulness of Young measures.

We recall that a point \( a \in \mathcal{A} \) is called extreme in \( \mathcal{A} \) if the following implication holds for any \( a_0, a_1 \in \mathcal{A} \) and any \( 0 < \lambda < 1 \): \( a = \lambda a_0 + (1 - \lambda) a_1 \Rightarrow a_0 = a_1 \). We denote by ”co” the convex hull.

**Proposition 7.** ([30, Th. 1]) Let \( \{u_k\}_{k \in \mathbb{N}} \subseteq L^1(\Omega; \mathbb{R}^m) \) converge weakly to \( u_0 \in L^1(\Omega; \mathbb{R}^m) \) for \( k \to \infty \). If \( u_0(x) \) is an extreme point of \( \text{co}\{u_k(x)\}; \ k \geq 0 \) for almost all \( x \in \Omega \), then \( \lim_{k \to \infty} \|u_k - u_0\|_{L^1(\Omega; \mathbb{R}^m)} = 0 \).

A. Visintin pointed out that the above theorem is not valid for \( L^p(\Omega; \mathbb{R}^m) \), \( p > 1 \); cf. [30, Counterexample 1]. Nevertheless, we can modify his result.

**Proposition 8.** Let \( \{u_k\}_{k \in \mathbb{N} \cup \{0\}} \subseteq L^p(\Omega; \mathbb{R}^m) \), \( 1 \leq p < +\infty \) and let \( \{u_k\}_{k \in \mathbb{N}} \) converge weakly to \( u_0 \) for \( k \to \infty \). If \( u_0(x) \) is an extreme point of \( \text{co}\{u_k(x)\}; \ k \geq 0 \) for almost all \( x \in \Omega \) then \( \lim_{k \to \infty} \|u_k - u_0\|_{L^q(\Omega; \mathbb{R}^m)} = 0 \) for any \( 1 \leq q < p \). Moreover, if \( \{|u_k|^p\}_{k \in \mathbb{N}} \) is uniformly integrable, then also \( \lim_{k \to \infty} \|u_k - u_0\|_{L^p(\Omega; \mathbb{R}^m)} = 0 \).

**Proof.** First, the case \( p = 1 \) is solved by Proposition 7. Suppose that \( 1 < p < +\infty \). We divide the proof into three steps.

**Step 1.** As \( \{u_k\}_{k \in \mathbb{N}} \) tends to \( u_0 \) weakly in \( L^p(\Omega; \mathbb{R}^m) \) it converges weakly in \( L^1(\Omega; \mathbb{R}^m) \) too. Moreover, \( u_0 \in \text{co}\{u_k(x)\}; \ k \geq 0 \) is extreme for almost all \( x \in \Omega \), which means due to Proposition 7 that \( \lim_{k \to \infty} u_k = u_0 \) strongly in \( L^1(\Omega; \mathbb{R}^m) \) and \( \{u_k\}_{k \in \mathbb{N}} \) generates \( \tilde{\nu} \in \mathcal{Y}^1(\Omega; \mathbb{R}^m) \) where \( \tilde{\nu}_x = \delta_{u_0(x)} \) for almost all \( x \in \Omega \); cf. also [29, Proof of Th. 22].

Let \( \{w_k\}_{k \in \mathbb{N}} \subseteq \{u_k\}_{k \in \mathbb{N}} \) be a subsequence generating \( \nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^m) \). Then it also generates \( \tilde{\nu} \in \mathcal{Y}^1(\Omega; \mathbb{R}^m) \), \( \tilde{\nu}_x = \delta_{u_0(x)} \). According to Remark 1 (ii) \( \nu_x = \delta_{u_0(x)} \) for almost all \( x \in \Omega \). Further, keep in mind that \( \{w_k\}_{k \in \mathbb{N}} \) converges weakly to \( u_0 \) in \( L^p(\Omega; \mathbb{R}^m) \).

**Step 2.** Since \( 1 < p < +\infty \), we can take \( 1 < q < p \) and apply (6) with \( v(\lambda) = |\lambda|^q \). Thus we get (remember \( \nu_x = \delta_{u_0(x)} \))

\[
\lim_{k \to \infty} \int_\Omega |w_k(x)|^q \varphi(x) \, dx = \int_\Omega |u_0(x)|^q \varphi(x) \, dx \text{ for any } \varphi \in L^\infty(\Omega) .
\]  

\[\text{(15)}\]
In particular, taking $\varphi = 1$ we obtain $\lim_{k \to \infty} \|w_k\|_{L^q(\Omega; \mathbb{R}^m)} = \|u_0\|_{L^q(\Omega; \mathbb{R}^m)}$. This together with the weak convergence of $\{w_k\}$ to $u_0$ in $L^q(\Omega; \mathbb{R}^m)$ gives the strong convergence of $\{w_k\}$ in $L^q(\Omega; \mathbb{R}^m)$ and therefore also in $L^1(\Omega; \mathbb{R}^m)$.

If, moreover, $\{|w_k|^p\}_{k \in \mathbb{N}}$ is uniformly integrable, so is $\{|w_k|^p\}_{k \in \mathbb{N}}$ and we may even use the test function $v(\lambda) = |\lambda|^p$ in (15), which ends up in the strong convergence also in $L^p(\Omega; \mathbb{R}^m)$.

**Step 3.** It follows from Steps 1 and 2 that from any weakly convergent subsequence of $\{u_k\}_{k \in \mathbb{N}}$ we can extract a subsequence, i.e., $\{w_k\}_{k \in \mathbb{N}}$ converging strongly to $u_0$ in $L^q(\Omega; \mathbb{R}^m)$, which gives that the whole sequence $\{u_k\}$ converges strongly to $u_0$. This argument has been already used by Visintin; cf. [30].

We immediately have the following consequence.

**Corollary 3.** Let $\{u_k\}_{k \in \mathbb{N} \cup \{0\}} \subset L^\infty(\Omega; \mathbb{R}^m)$ and $u_k \to u_0$ for $k \to \infty$ weakly * in $L^\infty(\Omega; \mathbb{R}^m)$. If $u_0(x)$ is an extreme point of $\overline{co}\{u_k(x); k \geq 0\}$ for almost all $x \in \Omega$ then $\lim_{k \to \infty} \|u_k - u_0\|_{L^p(\Omega; \mathbb{R}^m)} = 0$ for any $1 \leq p < +\infty$.

**Remarks on concentrations.** Let $p = 1$, $n = m = 1$, $\Omega = (0, 1)$ and $\beta_\mathbb{R} \mathbb{R} = \mathbb{R} \cup \{\infty\}$ for a moment. There are two basic types of non-uniformly integrable bounded sequences in $L^1(\Omega)$ generating a DiPerna-Majda measure.

**Example 1.**

$$u_k(x) = \begin{cases} k & \text{if } x \in \left(\frac{1}{2} - \frac{1}{k}, \frac{1}{2} + \frac{1}{k}\right) \\ 0 & \text{otherwise} \end{cases}.$$

**Example 2.**

$$v_k(x) = \begin{cases} 0 & \text{if } x \in \left(\frac{1}{k} - \frac{1}{2k}, \frac{1}{k} + \frac{1}{2k}\right) \cap (0, 1), l \in \mathbb{N} \\ k & \text{otherwise.} \end{cases}$$

Similar examples to these have been already used in [24]; cf. also [5]. It can be shown (see [24]) that $\{u_k\}_{k \in \mathbb{N}}$ generates a DiPerna-Majda measure $(\sigma^1, \nu^1) \in \mathcal{DM}_\mathbb{R}^1(\Omega; \mathbb{R})$ such that $\sigma^1(dx) = dx + 2\delta_{0.5}$ and $\nu^1_x = \delta_0$, $x \neq 0.5$ and $\nu^1_{1/2} = \delta_\infty$.

The sequence $\{v_k\}_{k \in \mathbb{N}}$ generates $(\sigma^2, \nu^2) \in \mathcal{DM}_\mathbb{R}^1(\Omega; \mathbb{R})$ where $\sigma^2(dx) = 2dx$ and $\nu^2_x = 0.5\delta_0 + 0.5\delta_\infty$ for $x \in (0, 1)$; cf. [24].

The sequence $\{u_k\}$ concentrates around the point $x = 0.5$ meanwhile $\{v_k\}$ exhibits a "continuous" concentration smeared out uniformly throughout the whole $\Omega$. In [21] it is shown that these two different situations are reflected in the properties of $\sigma^1$ and $\sigma^2$. The measure $\sigma^1$ is not absolutely continuous with respect to the Lebesgue measure but $\sigma^2$ is.

Taking now $(\sigma, \nu) \in \mathcal{DM}_\mathbb{R}^p(\Omega; \mathbb{R}^m)$ (for simplicity $\beta_\mathbb{R} \mathbb{R}^m = \mathbb{R}^m \cup \{\infty\}$) generated by some $\{u_k\}_{k \in \mathbb{N}}$ and having $E \subset \Omega$, $\sigma$-measurable, with the characteristic function $\chi_E$ then it follows from [21, Th 2.] that the following three basic situations can take place:
(i) \( \{ \chi_{Ew_k} \}_{k \in \mathbb{N}} \) is uniformly integrable if and only if \( \hat{\nu}_x(\infty) = 0 \) for \( \sigma \)-almost all \( x \in E \),

(ii) \( \{ \chi_{Ew_k} \}_{k \in \mathbb{N}} \) exhibits a “point” concentration at \( x \in E \) if and only if \( \hat{\nu}_x(\infty) = 1 \) and \( \sigma(\{x\}) > 0 \), (see Example 1),

(iii) \( \{ \chi_{Ew_k} \}_{k \in \mathbb{N}} \) shows a “continuous” concentration on \( E \) if and only if \( 0 < \hat{\nu}_x(\infty) < 1 \) for \( \sigma \)-almost all \( x \in E \), (see Example 2).

If (i), or (iii) is valid then the restriction of \( \sigma \) on \( E \) is absolutely continuous with respect to the Lebesgue measure.

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