SOME APPLICATIONS OF ELEMENTARY LINEAR ALGEBRA IN COMBINATORICS

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Some Applications of Elementary Linear Algebra in Combinatorics*

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Linear algebra has been used with great effectiveness in combinatorics and graph theory. It is sometimes surprising how elementary ideas of linear algebra have far reaching consequences. What are these elementary ideas? Linear independence, rank, determinant, eigenvalues, dimension. Ideas that one learns about in a first course in linear algebra. Many of these applications are accessible to students in a first course. In this note we discuss three such applications. Of these, two can be obtained by purely combinatorial arguments, but all the known arguments for the third use ideas from linear algebra.

Some of the simplest square (symmetric) matrices are \(O_n\) (the zero matrix of order \(n\)), \(I_n\) (the identity matrix of order \(n\)), and \(J_n\) (the all 1’s matrix of order \(n\)). The fundamental linear algebraic invariants of these matrices are easy to determine and are given in the table below (\(e_i\) denotes the \(i\)th standard basis vector of the real \(n\)-dimensional vector space \(\mathbb{R}^n\)).

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<tr>
<td>rank</td>
<td>0</td>
<td>$n$</td>
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<td>1</td>
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<td>eigenvalues</td>
<td>$0$ ($n$ times)</td>
<td>$1$ ($n$ times)</td>
</tr>
<tr>
<td>eigenvectors</td>
<td>$e_i$ ($1 \leq i \leq n$)</td>
<td>$e_i$ ($1 \leq i \leq n$)</td>
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</table>

In the table we have given $n$ linearly independent eigenvectors for each matrix. Since $O_n$ and $I_n$ each have only one distinct eigenvalue, it follows that every nonzero vector is an eigenvector of both matrices. The vectors $e_i - e_{i+1}$, $(i = 1, 2, \ldots, n - 1)$ span the subspace $W$ of $\mathbb{R}^n$ consisting of those vectors $x = (x_1, x_2, \ldots, x_n)$ with $x_1 + x_2 + \cdots + x_n = 0$ and hence every nonzero vector of $W$ is an eigenvector of $J_n$ with eigenvalue 0.

What happens when we take linear combinations of the matrices? The zero matrix doesn’t contribute anything, so all linear combinations are of the form

$$aI_n + bJ_n \quad (a \text{ and } b \text{ are real numbers}).$$

The vectors $e_i - e_{i+1}$ are eigenvectors of $aI_n + bJ_n$ with eigenvalue $a(1) + b(0) = a$, and $\sum_{i=1}^{n} e_i$ is an eigenvector with eigenvalue $a(1) + b(n) = a + bn$. Hence

$$a \ (n - 1 \text{ times}) \text{ and } a + bn \text{ are the eigenvalues of } aI_n + bJ_n, \quad (1)$$

and because the determinant of a matrix equals the product of its eigenvalues,

$$\det(aI_n + bJ_n) = a^{n-1}(a + bn).$$

Since $aI_n + bJ_n$ is a symmetric matrix, its rank equals the number of nonzero eigenvalues and can be easily determined as a function of $a$ and $b$ using (1).

These calculations involve only elementary linear algebraic notions but nonetheless have a combinatorial consequence which is difficult to obtain without linear algebra. Let $A_1, A_2, \ldots, A_n$ be a family of $n$ distinct subsets of an $m$-element set $X = \{x_1, x_2, \ldots, x_m\}$. Suppose that

(a) each set $A_i$ contains exactly $k$ elements, and

(b) every pair of sets $A_i$ and $A_j$ with $i \neq j$ has exactly $\lambda$ elements in common.

Then it turns out that the number of sets $A_i$ is at most the number of elements of $X$, that is, $n \leq m$. The connection between this result and linear algebra is provided by the incidence matrix of the given sets. The incidence matrix of this family of subsets of $X$ is the $n$ by $m$ matrix $B = [b_{ij}]$ where

$$b_{ij} = \begin{cases} 1 & \text{if } x_j \in A_i \\ 0 & \text{if } x_j \not\in A_i \end{cases}.$$
For example, let $m = 7$ and

$$A_1 = \{x_1, x_2, x_4\}, A_2 = \{x_2, x_3, x_5\}, A_3 = \{x_3, x_4, x_6\}, A_4 = \{x_4, x_5, x_7\},$$

$$A_5 = \{x_5, x_6, x_1\}, A_6 = \{x_6, x_7, x_2\}, A_7 = \{x_7, x_1, x_3\}.$$  

(2)

The incidence matrix is

$$B = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

Since each set contains three elements ($k = 3$), each row of $B$ contains 3 ones. Since each pair of sets has one common element ($\lambda = 1$), the inner product of any two distinct rows of $B$ equals 1. This means that each diagonal element of $BB^t$ equals 3 and each off-diagonal element equals 1. Equivalently,

$$BB^t = 2I_7 + J_7.$$

In general, the matrix $BB^t$ is the intersection matrix of the family of sets $A_1, A_2, \ldots, A_n$ since the entry in row $i$ and column $j$ equals the number of elements of $A_i \cap A_j$. If the sets $A_1, A_2, \ldots, A_n$ have properties (a) and (b), then

$$BB^t = (k - \lambda)I_n + \lambda J_n.$$

Thus (1) implies that the eigenvalues of $BB^t$ are

$$k - \lambda \text{ (n - 1 times) and } k + (n - 1)\lambda.$$ 

Since the sets are distinct, we have $k > \lambda$. Hence none of the eigenvalues of $BB^t$ equals zero and the rank of the matrix $BB^t$ of order $n$ equals $n$. Since the rank of a product of matrices does not exceed the rank of either factor, we now conclude that the rank of $B$ is at least $n$. But the rank cannot exceed the number $n$ of its rows and this implies that the rank of $B$ equals $n$. The rank of $B$ also cannot exceed the number $m$ of its columns and thus we obtain $n \leq m$. Therefore:

*If we have $n$ distinct subsets of cardinality $k$ of a set of $m$ elements such that each pair of sets have $\lambda$ elements in common, then the number of sets is at most the number of elements.*
More generally we see that if \( B \) is any \( n \) by \( m \) real matrix satisfying the equation \( BB^t = aI_n + bJ_n \) where \( a \neq 0 \) and \( a + bn \neq 0 \) (equivalently, \( \det(aI_n + bJ_n) \neq 0 \)), then \( n \leq m. \)\(^1\)

The above result is one form of a famous inequality in the theory of combinatorial designs known as Fisher’s inequality (see e.g. [8]). Although it is possible to prove Fisher’s inequality in other ways, none is as elegant and illuminating as the proof given above based on elementary, but powerful, linear algebra.

An equally famous result in combinatorics is the so-called marriage theorem of P. Hall (see e.g. [1], [6], [8]). Again let \( A_1, A_2, \ldots, A_n \) be a family of \( n \) subsets of a set \( X = \{x_1, x_2, \ldots, x_m\} \) of \( m \) elements. A system of distinct representatives, abbreviated SDR, of \( A_1, A_2, \ldots, A_n \) is a family \( a_1, a_2, \ldots, a_n \) of distinct elements such that \( a_i \in A_i \) for each \( i = 1, 2, \ldots, n \). For example, \( x_1, x_2, x_3, x_4, x_5, x_6, x_7 \) and \( x_2, x_5, x_3, x_4, x_6, x_7, x_1 \) are both SDR’s of the family (2). If the family \( A_1, A_2, \ldots, A_n \) has an SDR, then clearly for each \( k = 1, 2, \ldots, n \) the union of any \( k \) sets contains at least \( k \) elements, that is,

\[
| \bigcup_{i \in J} A_i | \geq |J| \quad (J \subseteq \{1, 2, \ldots, n\} )\]

To show that condition (3) is sufficient for there to be an SDR, we replace the 1’s in the incidence matrix \( B \) by numbers which are chosen to be far from identical. Let \( R = [r_{ij}] \) be the \( n \) by \( m \) matrix where \( r_{ij} = 0 \) if and only if \( x_j \not\in A_i \). We choose the nonzero elements of \( R \) to be real numbers which are algebraically independent over the field of rational numbers \( \mathbb{Q} \).\(^2\)

Suppose that the rank of \( R \) equals \( n \). Then some submatrix \( S \) of \( R \) of order \( n \) has rank \( n \) and hence \( \det S \neq 0 \). To simplify the notation, assume that the columns of \( S \) are the first \( n \) columns of \( R ).\(^3\) Then

\[
\det S = \sum_{(i_1, i_2, \ldots, i_n)} \pm r_{1i_1} r_{2i_2} \cdots r_{ni_n} \neq 0,
\]

where the summation extends over all permutations \( (i_1, i_2, \ldots, i_n) \) of \( \{1, 2, \ldots, n\} \) and the choice of + or − depends on whether the permutation is even or odd. Since the determinant of \( S \) is not zero, there is a permutation \( (k_1, k_2, \ldots, k_n) \) such that

\[
r_{1k_1} r_{2k_2} \cdots r_{nk_n} \neq 0.
\]

\(^1\)Of course, we can generalize even further: if \( B \) is an \( n \) by \( m \) matrix satisfying \( BB^t = C \) where \( C \) is a nonsingular matrix of order \( n \), then the rank of \( B \) equals \( n \) and hence \( n \leq m \). Thus if the intersection matrix of a family of \( n \) subsets of an \( m \) element set is nonsingular, then the number of sets is at most the number of elements.

\(^2\)That is, they do not satisfy any nonzero polynomial equation with rational coefficients.

\(^3\)This can be achieved by changing the order of the columns of \( R \), that is, by relabeling the elements of \( X \).
Thus

\[ r_{1k_1} \neq 0 \implies x_{k_1} \in A_1 \]
\[ r_{2k_2} \neq 0 \implies x_{k_2} \in A_2 \]
\[ \vdots \]
\[ r_{nk_n} \neq 0 \implies x_{k_n} \in A_n \]

Since \((k_1, k_2, \ldots, k_n)\) is a permutation, \(x_{k_1}, x_{k_2}, \ldots, x_{k_n}\) is a family of distinct elements and thus an SDR of \(A_1, A_2, \ldots, A_n\).

Now suppose that the rank of \(R\) is less than \(n\). Thus the rows of \(R\) are linearly dependent. We choose a minimal set of linearly dependent rows of \(R\) and without loss of generality we assume that these are the first \(k\) rows.\(^4\) This means that the \(k\) by \(n\) submatrix \(R'\) determined by the first \(k\) rows of \(R\) has rank equal to \(k - 1\) and every proper subset of the rows of \(R'\) is a linearly independent set. Since the rank of \(R'\) is \(k - 1\), there are \(k - 1\) linearly independent columns of \(R'\) such that every other column of \(R'\) is a linear combination of them. Without loss of generality we assume that the first \(k - 1\) columns of \(R'\) are linearly independent.\(^5\) Thus \(R\) has the form

\[
\begin{bmatrix}
R_1 & R_2 \\
R_3 & R_4
\end{bmatrix}
\]

where \(R_1\) is a \(k\) by \(k - 1\) matrix and each column of \(R_2\) is a linear combination of the columns of \(R_1\). There is a nontrivial linear combination of the rows of \(R_1\) which equals zero, that is, there is a nonzero vector \(u = (u_1, u_2, \ldots, u_k)\) such that

\[ uR_1 = 0. \]

The entries \(u_i\) of \(u\) are real numbers which can be expressed as polynomials in the nonzero elements of \(R_1\).\(^6\) Since each column of \(R_2\) is a linear combination of the columns of \(R_1\) we also have \(uR_2 = 0\).\(^7\) Hence

\[ u \begin{bmatrix} R_1 & R_2 \end{bmatrix} = 0. \]

Since the first \(k\) rows form a minimal linearly dependent set of rows, no entry \(u_i\) of \(u\) equals zero. Let \((r_{1j}, r_{2j}, \ldots, r_{kj})\) be any column of \(R_2\). Then

\[ u_1 r_{1j} + u_2 r_{2j} + \cdots + u_k r_{kj} = 0. \quad (4) \]

\(^4\)This can be achieved by changing the order of the rows of \(R\), that is, by relabeling the sets \(A_1, A_2, \ldots, A_n\). Note that \(k > 1\) since (3) implies that no row of \(A\) contains only 0's.

\(^5\)See footnote 3.

\(^6\)This follows by solving the system \(uR_1 = 0\) of \(k - 1\) equations in the \(k\) unknowns \(u_i\) by Gaussian elimination.

\(^7\)There is a subtle point to be appreciated here. Since the first \(k\) rows of \(R\) are linearly dependent we could have chosen \(u\) so that \(u[R_1\ R_2] = 0\). But then we would have to say that the entries \(u_i\) are polynomials in the nonzero entries of \(R_1\) and \(R_2\). This would have made the rest of the proof invalid.
Since no $u_i$ equals zero, if any of the numbers $r_{ij}, (i = 1, 2, \ldots, k; j = k, k + 1, \ldots, n)$ is different from zero, then (4) is a non-zero polynomial equation with rational coefficients contradicting the algebraic independence of the nonzero elements of $R$. Therefore $R_2 = O$ implying that the union of the first $k$ sets contains only $k - 1$ elements.\footnote{The fact that it turns out to be the first $k$ sets is a consequence of the relabeling of the sets that was done within the proof.} Thus the supposition that the rank of $R$ is less than $n$ leads to a contradiction of (3). We conclude:

The family of sets $A_1, A_2, \ldots, A_n$ has an SDR if and only if $\left| \bigcup_{i \in J} A_i \right| \geq |J|$ for all $J \subseteq \{1, 2, \ldots, n\}$.

The idea of the above proof is due to Edmonds [3]. Other proofs of the marriage theorem of a combinatorial nature are available (see e.g. [6]). However for the next result every known proof uses some basic notions of linear algebra.

Consider the complete graph $K_n$ of order $n$. It has $n$ vertices $\{1, 2, \ldots, n\}$ and an edge $\{i,j\}$ joining each pair of distinct vertices $i$ and $j$. In Figure 1 we have drawn the graph $K_4$. A biclique (short for complete bipartite subgraph) of $K_n$ is obtained by choosing two disjoint subsets $X$ and $Y$ of the vertices of $K_n$ and all the edges $(X,Y)$ joining them. Since each edge of $K_n$ is a biclique, it is possible to partition the edges of $K_n$ into bicliques. Hence it is natural to ask for the minimum number of bicliques into which the edges of $K_n$ can be partitioned. Figure 2 exhibits three biclique partitions of $K_4$. The first of these can be generalized to give a biclique partition of $K_n$ into the $n-1$ bicliques

\[
\langle\{1\}, \{2,3,\ldots,n\}\rangle, \langle\{2\}, \{3,4,\ldots,n\}\rangle, \ldots, \langle\{n-1\}, \{n\}\rangle.
\]

It is a surprising result of Graham and Pollak [4],[5] that, trivial as this biclique partition seems, it is impossible to partition the edges of $K_n$ into fewer than $n-1$ bicliques.

de Caen and Hoffman’s proof [2] of the Graham Pollak theorem uses a connection between biclique partitions of $K_n$ and tournaments (or tournament matrices). By orienting each edge of $K_n$ we obtain a tournament of order $n$. A tournament of order $n$ can be regarded as the result of a round-robin tournament with $n$ teams (the $n$ vertices) in which every team plays every other team exactly once. Assuming there are no ties, if we orient an edge from team $i$ to team $j$ whenever $i$ beats $j$ we obtain a tournament of order $n$. The tournament matrix associated with a tournament of order $n$ is the matrix $T = [t_{ij}]$ of order $n$ where

\[
t_{ij} = \begin{cases} 
1 & \text{if team } i \text{ beats team } j \\
0 & \text{if } i = j \text{ or team } i \text{ loses to team } j.
\end{cases}
\]

Thus for a tournament matrix we have $T + T^t = -I_n + J_n$.

Let

\[
\langle X_1, Y_1\rangle, \ldots, \langle X_k, Y_k\rangle
\]

(5)
Figure 1: $K_4$ – the complete graph on 4 vertices.

Figure 2: Three biclique partitions of $K_4$.

Figure 3: Tournaments associated with the biclique partitions of $K_4$ in Figure 2.
be a partition of $K_n$ into $k$ bicliques. We obtain a tournament by orienting the edges of $(X_i, Y_i)$ from $X_i$ to $Y_i$ for each $i = 1, 2, \ldots, k$. Tournaments corresponding to the biclique partitions of $K_4$ in Figure 2 are drawn in Figure 3. If $Z$ is a subset of $\{1, 2, \ldots, n\}$, then we let $Z^T$ denote the characteristic vector of $Z$ whose $i$th coordinate is 1 if $i$ is in $Z$ and is 0 otherwise. Since (5) is a biclique partition of $K_n$, the corresponding tournament matrix satisfies

$$T = X_1^T Y_1 + X_2^T Y_2 + \cdots + X_k^T Y_k$$

(6)

where each $X_i^T Y_i$ is a matrix of rank 1. For the tournament matrices corresponding to the tournaments in Figure 3 we have, respectively,

$$\begin{align*}
\begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} &= \begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix},
\end{align*}$$

and

$$\begin{align*}
\begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} &= \begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\end{align*}$$

and

$$\begin{align*}
\begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} &= \begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\end{align*}$$

It follows from (6) that the tournament matrix corresponding to a partition of $K_n$ into $k$ bicliques has rank at most $k$. We now show that the rank of each tournament matrix of order $n$ is $n - 1$ or $n$.

Suppose that the rank of a tournament matrix $T$ of order $n$ is strictly less than $n - 1$. Then the $n + 1$ by $n$ matrix obtained from $T$ by appending a row of all 1's has rank at most $n - 1$. Hence there exists a nonzero vector $x = (x_1, x_2, \ldots, x_n)$ such that

$$Tx^t = 0 \text{ and } \sum_{i=1}^{n} x_i = 0.$$ 

On the one hand we have

$$x(T + T^t)x^t = x(Tx^t) + (xT^t)x^t = 0 + 0 = 0.$$
On the other hand we have

\[ x(T + T')x^t = x(-I_n + J_n)x^t = -xx^t + xJ_nx^t = -xx^t. \]

Hence \( xx^t = 0 \) contradicting the fact that \( x \) is a nonzero vector. We conclude that the rank of \( T \) is at least \( n - 1 \) and therefore:

The complete graph \( K_n \) of order \( n \) cannot be partitioned into fewer than \( n - 1 \) bicliques.

Other proofs of the theorem are given in [7] and [9]. The argument above works under more general circumstances: if \( T \) is any real matrix of order \( n \) satisfying the equation \( T + T^t = aI_n + bJ_n \) where \( a \neq 0 \), then the rank of \( T \) is at least \( n - 1 \).

We have illustrated how elementary linear algebra can be a powerful tool for proving combinatorial theorems. Each of the examples given can be appreciated by someone with an understanding of basic ideas of linear algebra. It is also true that combinatorial ideas have been used in the study of linear algebra. The book [1] is devoted to the interplay between linear algebra and combinatorics.

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