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IN CRYSTALS

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NEKHOROSHEV’S THEOREM, ERGODICITY, AND THE MOTION OF ENERGETIC CHARGED PARTICLES IN CRYSTALS

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Abstract. We present a mathematical theory of energetic charged particle motion in perfect crystals. This theory is entirely classical and deterministic and applies for the most part only to highly energetic, positively charged particles moving in an idealized crystal. Nevertheless, it is the first time the existence of “channeling motions” has been rigorously deduced for a full three degree-of-freedom model, and it gives a satisfying mathematical unity to seemingly disparate phenomena, embracing “axial” and “planar” channeling, as well as certain nonchanneling motions in crystals. The theory makes use of a new elementary result in ergodic theory, as well as results in Hamiltonian perturbation theory due to Nekhoroshev, and more recently to Benettin, Gallavotti, Galgani, and Giorgilli.

After brief discussions of the physics of energetic charged particle motions in crystals and of the particular motions known as channeling, we outline the relevant mathematical results in some detail without proving them completely. The presentation at this level should permit the reader to grasp the essentials and to refer to more complete proofs as desired.

Key Words. Nekhoroshev’s theorem, particle channeling, Hamiltonian perturbation theory

** This article summarizes the first author’s Ph.D. thesis and will appear shortly in Foundations of Classical and Quantum Dynamics, a Festschrift in honor of William Sáenz’ sixty-fifth birthday.

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Introduction

The following is a summary of the first author’s Ph.D. thesis,\textsuperscript{12} carried out under the supervision of the second author. Because of the role William Sáenz played in its development, each of us takes special pleasure in submitting this contribution to the Festschrift honoring his sixty-fifth birthday. Much of what is reported here was accomplished in Professor Sáenz’ presence, and it is certain that this work benefited from his criticism, and especially from his infectious enthusiasm and sense of humor.

The subject of this article is a mathematical theory of energetic particle motion in perfect crystals. This theory is rudimentary in that it is entirely classical and deterministic and applies for the most part only to positively charged particles moving in a highly idealized crystal. Nevertheless, it is the first time the existence of “channeling motions” has been rigorously deduced for a full three degree-of-freedom model, and it gives a satisfying mathematical unity to seemingly disparate phenomena, embracing “axial” and “planar” channeling, as well as certain nonchanneling motions to be described below.

The theory makes use of an elementary result in ergodic theory, as well as results in Hamiltonian perturbation theory due to Nekhoroshev,\textsuperscript{23,24} and more recently to Benettin, Gallavotti, Galgani, and Giorgilli.\textsuperscript{6,7} Our aim here is to outline these results in some detail without proving them completely; we hope the presentation at this level of detail will permit the reader to grasp the essentials and to refer to more complete proofs\textsuperscript{13} as desired. Before beginning however, for the benefit of the uninitiated it seems appropriate to give the barest of introductions to the subfield of particle-solid interactions to which these results apply.

1. Energetic Particle Motions in Crystals; Particle Channeling

Consider an energetic beam of charged particles incident on a crystalline target in a random direction (more will be said about the meaning of a “random” direction below). A variety of processes typically ensue: matter is ejected from the crystal (sputtering), particles veer violently from their initial paths upon encountering crystal nuclei (Rutherford scattering), sometimes also reacting with the nuclei (through absorption, fission, or other means). Particles lose energy and are slightly deflected through collisions with electrons (electron multiple scattering); energy loss also results from radiation emission as light particles are accelerated through encounters with the crystal lattice (bremsstrahlung). The proportion of energy loss due to radiation increases with particle energy, and for ultrarelativistic particles the radiative energy loss exceeds the collisional loss. When energy loss processes slow particles sufficiently, they may come to rest inside the crystal (implantation).

If the crystal is now reoriented so that the beam is incident in a “non-random” direction, i.e., a direction nearly aligned with a low-order axis or plane, the results are dramatically different. Experimentally, the average depth of penetration into a thick crystal is greatly increased, as is the incidence of transmission through a thin crystal, while the rate of particle backscattering may decrease by as much as a factor of 100. These changes result from the way particles interact with the rows or planes of atoms with which their trajectories are initially parallel. In fact, sequential collisions of particles with the lattice are strongly correlated, and soft collisions gently steer the beam along crystal rows or planes. This steering process is called channeling.
Channeling has led to a better understanding of the properties of solids, and has had numerous technological applications. For example, it has been used as a material analysis tool to study crystal defects, surfaces and interfaces, and to determine the location of crystal impurities. It has been used to measure nuclear lifetimes, to study the strain in “strained-layer superlattices,” and to deflect high energy particle beams. In fact, a bent crystal can deflect beams of much higher energy than a traditional magnetic septum, though of course transmission is dramatically reduced. Channeled electrons and positrons also emit electromagnetic radiation, an effect that has received increasing attention since its discovery in the seventies. Channeling has possible applications at low energies as a solid state probe, and at high energies as a monoenergetic gamma ray source and (more speculatively) as a cosmic ray telescope; there are also interesting problems related to the radiation reaction in the classical field equations and in strong field QED. The reader unfamiliar with the physics of channeling or its numerous applications is urged to consult the literature. Lindhard’s seminal paper$^{22}$ has been expanded in a set of Aarhus lecture notes which are available.$^1$ A review article by Gemmel$^{18}$ contains general discussions of particle channeling and applications and an extensive bibliography of the literature up to 1974. Five books have appeared more recently. A basic Soviet book on channeling, dechanneling, and channeling radiation by Kumakhov and Shirmer, published in Russian in 1980, has recently appeared in English.$^{20}$ Feldman, Mayer, and Picraux$^{17}$ review channeling as a material science tool, and their book contains an extensive bibliography of low-energy channeling through 1982. Ohtsuki$^{25}$ discusses several theoretical aspects of channeling and includes a detailed analysis of dechanneling. Sáenz and Überall$^{26}$ edited a collection of articles and Kumakhov and Komarov$^{19}$ have written a book which together give a detailed review of the theoretical and experimental status of channeling radiation and coherent bremsstrahlung. Results of a recent workshop on relativistic channeling are presented in Ref.$^9$. Proceedings from International Conferences on Atomic Collisions in Solids (1979, ‘81, ‘83, ‘85, ‘87, ‘89) contain many articles related to particle channeling (cf. corresponding issues of Nucl. Instr. Methods B), and more recently, articles for a general audience have appeared.$^{27}$

2. Channeling Theories

The reader can no doubt appreciate that channeling is a very complex process, and that a complete understanding of it and its applications, if achieved, would involve concepts from a great many branches of physics, including condensed matter physics, classical and quantum mechanics, classical and quantum electrodynamics, as well as mathematical concepts from dynamical systems, probability, and stochastic processes. Indeed, as can be seen by consulting Refs.$^1$, 9, or 18, the theoretical channeling literature is quite extensive, overlapping the theory in each of the fields just mentioned, and more. However, our concern here is not with efforts to push the limits of theoretical detail and agreement with experiment. We concentrate instead on making the basic qualitative explanations of axial and planar channeling given above into a more rigorous theory, and we ask the question: Is the simple geometry of an ideal crystal and its classical interaction with energetic, positive particles sufficient to deduce the major features of observed channeling processes—especially the distinctions between random, planar, and axial motions and their associated continuum models?
The individual who first seriously addressed this question was J. Lindhard, and he did so only a short time after the discovery of channeling in the early sixties. His papers\textsuperscript{21,22} include detailed physical arguments; by translating the simplest classical cases considered in these articles into the language of Hamiltonian systems, we may paraphrase that part of the theory which ignores multiple scattering as follows. We begin by considering the perfect crystal model, in which particle motion is governed by the three degree-of-freedom Hamiltonian

\[ \mathcal{H} = \frac{1}{2m} p^2 + V(q), \text{ where } p, q \in \mathbb{R}^3, \text{ and } p^2 = p \cdot p. \]  

(1)

Here \( V \) is the periodic crystal potential, usually expressed as a sum of thermally averaged screened Coulomb atomic potentials at the lattice sites. The relevance of this model in channeling physics and some of the effects it ignores are discussed in Ref. 12. We simply point out that the most important of these effects are lattice vibrations and electron multiple scattering, usually described as perturbations to the motion generated by \( \mathcal{H} \). Another natural starting point might be to take \( V \) as a sum of screened Coulomb potentials without thermal averaging. However, in a systematic treatment of the lattice vibrations as a perturbation, one is led naturally from screened Coulomb atomic potentials to their thermal average (cf. Ref. 28 or Andersen's article in Ref. 9).

Lindhard argued that for incident particle directions nearly parallel to low-index axes, the motions of \( \mathcal{H} \) are well described by the so-called impulse-momentum approximation, wherein, for grazing angles, the influence of a string of atoms on a particle may be approximated by a sequence of soft collisions with individual atoms in the string. The impulse-momentum approximation is thus a physical version of the oft-cited heuristic explanation of channeling given above. Lindhard goes on to show that, because particles experience a large number of soft collisions with atoms in a given string, another approximation is justified in which each discrete string of atoms is replaced by a "continuum string." In terms of the Hamiltonian (1), this amounts to replacing the crystal potential \( V \) by the axial continuum potential \( V_A \) obtained from \( V \) by averaging along the direction of the strings considered. This greatly simplifies (1); the position \( q_1' \) along the axial direction is now a cyclic coordinate, so the momenta \( p_2' \) in this direction is constant, and the motion in the plane transverse to the axial direction is governed by the two degree-of-freedom axial continuum Hamiltonian

\[ \mathcal{H}_A = \frac{1}{2m} (p_1'^2 + p_2'^2) + V_A(q_1', q_2'). \]  

(2)

Although Lindhard did not express his result in this way originally, and though he did not transform system (1) into system (2) nor give rigorous estimates of how solutions of (2) approximate those of (1), it is clear that he did establish a strong link between (1) and (2), and that this link is one of the cornerstone achievements in channeling theory.

It is not clear however, that a similarly convincing connection has ever been established between (1) and the one degree-of-freedom planar continuum Hamiltonian

\[ \mathcal{H}_P = \frac{1}{2m} p_1''^2 + V_P(q_1''). \]  

(3)
in which the conjugate coordinates $p_1''$ and $q_1''$ are momentum and displacement in the
direction transverse to the planes in question, and the planar continuum potential $V_P$
is the average of $V$ parallel to the planes. Although this planar continuum model
shares the intuitive appeal of the axial continuum model and has gained the same basic
currency among channeling theorists, the arguments used to derive it from the perfect
crystal model have not been wholly convincing. The problem is that the impulse-
momentum approximation breaks down, because, while it is clear how particles can
undergo successive grazing collisions with a segment of collinear atoms, it is not clear
how they can do so with a region of coplanar atoms. This criticism does not detract
from the utility of the planar continuum model, the usefulness of which has been borne
out through repeated experiment, but it does point out the basic theoretical difficulty
in the way it is descended from the perfect crystal model.

In the nineteen seventies, it was noticed that the equations of motion for the per-
fected crystal model could be scaled so as to appear in standard form for the method
of averaging, a very general perturbation technique for ordinary differential equations
with a small parameter (see Ref. 16 for a detailed discussion of averaging for periodic
systems). In Ref. 8 this technique was used to show that, under a rather restrictive
assumption, solutions to the continuum string model (the axial continuum model re-
stricted to one string in the plane) were in fact the first term in an asymptotic expansion
of solutions to the perfect string model. It was stressed that the averaging technique
actually transforms the perfect string into the continuum string to a certain order, and
so gives approximations to the exact perfect string solutions which (i) may be rigorously
compared to the exact solutions, and (ii) display effects of the perfect string periodicity
which are (by definition) absent from the continuum solutions alone. This early aver-
aging result has since been expanded to more general derivations of axial continuum
models from perfect crystal models and, inasmuch as Nekhoroshev-type perturba-
tion theory may be viewed as a specialized averaging method for Hamiltonian systems,
the results given here are the theoretical culmination of deterministic averaging ideas
applied to the channeling problem.

In summary, the particle motion theory presented below began with ideas from
the averaging theory for axial channeling and developed into a model embracing es-
sentially all energetic particle motions in cubic perfect crystals. Consideration of the
full three degree-of-freedom system (1) led to small divisor and ergodicity problems not
present in the axial averaging theory. In the process, it was also discovered that the
scaling could be carried out so as to preserve the Hamiltonian structure of (1); this
allows Nekhoroshev-type perturbation techniques to be applied in place of the more
general averaging technique. The results obtained in this way give a very satisfying
mathematical interpretation of axial and planar channeling as motion at resonance in
the underlying three degree-of-freedom Hamiltonian. Moreover, stability results are ob-
tained for channeling motions on very long time intervals, along with a description of
trajectories and a “maximum time until close encounter” for nonchanneling motions.

3. Formulation of the Problem

Hamiltonian perturbation theory can be brought to bear on the particle-crystal
problem by first reducing it to the “cubic perfect crystal model”; that is, to the classical
Hamiltonian system (1) above where $V$ has period $d$ in each of its coordinates. In order
to simplify later estimates, we require the potential $V$ in (1) to be real analytic. This requirement is not as severe as it might seem, for although a screened Coulomb potential is not analytic, its thermal average typically is, since it is obtained by convolving the screened atomic potentials with the spatial probability distribution of the nuclei around the lattice sites. It should be pointed out that while this leads to the periodic potential which best models the thermal lattice vibrations of the crystal, it has no “hard core” and so cannot faithfully represent the process of close encounter (e.g. Rutherford scattering) between charged particles and nuclei. This objection is removed, at least mathematically, by introducing what we call the channeling criterion. We assume that the potential $V$ has been adjusted by means of an additive constant so that its minimum value is zero; we then let $\mathcal{E}_M$ denote its maximum value over $\mathbb{R}^3$ and define the subsets of configuration space

$$\mathcal{B}(\mathcal{E}_\perp) = \{q \in \mathbb{R}^3 \mid V(q) \geq \mathcal{E}_\perp\},$$

for potential energies $\mathcal{E}_\perp (0 \leq \mathcal{E}_\perp \leq \mathcal{E}_M)$. If, as is assumed here, the potential governs the motion of positively charged particles, then clearly for sufficiently large $\mathcal{E}_\perp < \mathcal{E}_M$, the set $\mathcal{B}(\mathcal{E}_\perp)$ is the disjoint union of (slightly deformed) balls centered on the lattice sites. By choosing a physically suitable value for $\mathcal{E}_\perp$, we may distinguish particle trajectories which come too close to nuclei to be governed by the thermally averaged potential as those which enter the region $\mathcal{B}(\mathcal{E}_\perp)$. More precisely, fix such a $\mathcal{E}_\perp$ and consider a solution $(p(\tau), q(\tau))$ of the equations of motion corresponding to (1). Such a solution is a channeling solution on the time interval $\mathcal{I}$ provided

$$q(\tau) \notin \mathcal{B}(\mathcal{E}_\perp), \quad \text{or equivalently} \quad V(q(\tau)) < \mathcal{E}_\perp \quad \forall \tau \in \mathcal{I}.$$  

(5)

This is the channeling criterion, and it is assumed that the perfect crystal model is a good approximation for particle trajectories that satisfy it. A trajectory which fails to satisfy this criterion for the first time at $t_1$ is assumed to suffer a “close encounter” with a nucleus, and is not viewed as a good approximation to an actual particle trajectory for subsequent times $t > t_1$. While the channeling criterion is independent of the incident energy $\mathcal{E}$, channeling motions are such that $\mathcal{E} \gg \mathcal{E}_\perp$, and this defines the fundamental perturbation parameter in our analysis.

The Hamiltonian (1) may be transformed to nondimensional, nearly integrable form as follows. Restricting attention to particles of fixed energy $\mathcal{E}$, i.e. to trajectories $(p(\tau), q(\tau))$ satisfying $\mathcal{H}(p(\tau), q(\tau)) = \mathcal{E}$, we define the scaled momentum (actions) $I \in \mathbb{R}^3$, the scaled position (angles) $\theta \in T^3$, the scaled potential $W$ and scaled time $t$ by

$$I = (m\mathcal{E})^{-1/2}p, \quad \theta = \frac{1}{d}q, \quad W(\theta) = \frac{1}{\mathcal{E}_\perp}V(\theta d), \quad t = \frac{1}{\tau_0}\tau.$$  

(6)

Here $\tau_0 = d\sqrt{m/\mathcal{E}}$ is the time required for a particle to travel a distance $\sqrt{2}d$ in the potential-free case. The choice of scaling is motivated by the assumption $\mathcal{E} \gg \mathcal{E}_\perp$, so that trajectories satisfying the channeling criterion maintain a kinetic energy of approximately $\mathcal{E}$. The transformed Hamiltonian $\hat{H}$ now reads $\hat{H}(I, \theta) = \frac{1}{2}I^2 + (\mathcal{E}_\perp/\mathcal{E})W(\theta)$, or, writing $\epsilon = \mathcal{E}_\perp/\mathcal{E} \ll 1$, this becomes

$$H(I, \theta) = \frac{1}{2}I^2 + \epsilon W(\theta), \quad \text{where} \quad W \in C^\omega(T^3),$$

(7a)
and \( 0 \leq W(\theta) \leq \mathcal{E}_0 / \mathcal{E}_\perp \), \( (7b) \)

and we are interested in the behavior of solutions on the surface \( H = 1 \) for small \( \epsilon \) and long times. Here “long” is relative to \( \tau_0 \), and \( O(\epsilon^{-1/2}) \) is a typical planar channeling period.

It is no surprise that the scaled Hamiltonian appears in action-angle form, since this scaling views the potential as a perturbation of rectilinear motion in the lattice \( T^3 \). System (7) is called the “nearly integrable form of the perfect crystal model” or the “scaled perfect crystal model.”

The region of close encounter \( \mathcal{B}(\mathcal{E}_\perp) \) described in equation (4) is immediately reformulated in terms of the scaled variables as

\[
\mathcal{C}(1) = \{ \theta \in T^3 \mid W(\theta) \geq 1 \},
\]

and so the criterion (5) for a solution \((I(t), \theta(t))\) of Hamilton’s equations corresponding to (7) to be a channeling trajectory on the time interval \( \mathcal{I} \) becomes

\[
\theta(t) \notin \mathcal{C}(1), \quad \text{or equivalently} \quad W(\theta(t)) < 1 \quad \forall \ t \in \mathcal{I}.
\]

4. Outline of Methods and Results

A. The Physical Results

As formulated above, the problem of particle motion in a perfect crystal leads naturally to the following question: Among all possible orientations with which energetic charged particles may impinge upon a perfect crystal, which orientations lead to trajectories that avoid close encounters with nuclei on reasonable time intervals (or equivalently for reasonable crystal depths)? The response may be divided into three cases:

1) For orientations away from crystal axes or planes, close encounters occur rapidly, within times that may be estimated as a function of particle energy and the way orientations “away from axes or planes” are defined. In addition, until close encounters occur, particle trajectories are approximated by rectilinear trajectories.

2) For positive particles oriented sufficiently near planar directions and away from axes, close encounters are avoided on exponentially long time intervals \([0, T_0]\), where \( T_0 \) is \( O(\epsilon^\tau \exp(\epsilon^{-\tau/4})) \), \( \tau > 0 \). On these intervals, particle trajectories are given in terms of solutions of a transformed Hamiltonian which agrees to leading order with the associated planar continuum model (3). It follows that the transverse energy of (3) and the longitudinal momentum (both of which are conserved in the planar continuum model) are approximately conserved on \([0, T_0]\) and that particles remain confined between planes of nuclei.

3) For positive particles oriented sufficiently near axial directions, close encounters are avoided on exponentially long time intervals \([0, T_0]\), with \( T_0 \) as above. On these intervals, particle trajectories are given in terms of solutions of a transformed Hamiltonian which agrees to leading order with the associated axial continuum
model (2). It follows that the transverse energy of (2) and the longitudinal momentum (both of which are conserved in the axial continuum model) are approximately conserved on $[0, T_0]$ and that particles remain away from the strings of nuclei.

Result 1) may have potential use in the physics of ion implantation, while results 2) and 3) appear to embody the first rigorous derivations of the planar continuum model and of the conservation of transverse energy and longitudinal momentum for planar and axial channeling motions on long time intervals. It is not yet clear that these are the only possibilities for small $\epsilon$. For example, there may be an interesting class of motions in the planar/axial transition region not included in results 2) or 3). Furthermore, the present results are for positive particles and the corresponding negative particle case would be interesting but more difficult, as particles are steered toward the close encounter region rather than away from it.

We now turn to a brief outline of the mathematical tools used to prove the physical results just described, beginning with a partial motivation for the use of Nekhoroshev’s theorem.

**B. How Nekhoroshev’s Theorem Arises**

As remarked in the previous section, the scaled perfect crystal Hamiltonian (7) is a perturbation of the free-particle Hamiltonian $H(I, \theta) = \frac{1}{2} I^2$, the solutions of which are simply the rectilinear trajectories $I(t) = I_0$, $\theta(t) = \theta_0 + I_0 t$. It is natural to ask in what sense these simple solutions are good approximations to the exact solutions of (7), and an immediate response is found by applying the KAM theorem (a good outline of KAM theory may be found in Ref. 4). In this case, the KAM theorem says roughly that, for $\epsilon$ sufficiently small, given any solution $(I(t), \theta(t))$ of (7) with “highly nonresonant” initial action $I_0$ (which physically is the initial “direction” here), the drift in direction $||I(t) - I_0||$ remains uniformly small ( $O(\sqrt{\epsilon})$ ) for all time. The highly nonresonant initial directions $I_0$ lie on smooth “KAM tori,” which are small ( $O(\sqrt{\epsilon})$ ) perturbations of the uniformly circular tori defined in phase space by the constant actions $J_0$ satisfying infinitely many incommensurability relations of the form

$$|k \cdot J_0| \geq C \epsilon^\alpha |k|^{-p} \quad \forall k \in \mathbb{Z}^3, k \neq 0,$$  \hspace{1cm} (10)

where $C$, $\alpha$, and $p$ are appropriate positive constants and $|k| = |k_1| + |k_2| + |k_3|$. The $J_0$ satisfying these *Diophantine conditions* form a Cantor set of large relative measure in action space; more precisely, the complement of this set has relative measure $O(\epsilon^\alpha)$ as $\epsilon \to 0$.

There are two reasons that the conclusions of the KAM theorem have limited usefulness here. First, the Cantor set of initial conditions is a set without interior; it is interpenetrated throughout with initial conditions whose trajectories may fail to satisfy the KAM theorem’s conclusions. Second, as will be discussed below, this same set of initial conditions is a subset of the *nonchanneling directions*, in the sense that trajectories with these initial conditions quickly enter the restricted neighborhoods $\mathcal{C}(1)$ surrounding the lattice sites. The conclusions of the KAM theorem may then be interpreted physically by saying that particles impinging on the crystal in highly nonresonant directions continue through the crystal in almost unperturbed straight lines, passing through the
softened nuclei like high speed projectiles. This is the most spectacular example of how the thermally averaged potential fails to model close encounter processes adequately.

Both of these difficulties combine to point the way to Nekhoroshev's theorem. This theorem—the proof of which was executed by the student of Arnol'd whose name it bears—is closely related to the KAM theorem, but has definite advantages over it in most physical applications. To put it roughly, Nekhoroshev's theorem trades the near conservation of action on infinite time intervals for near conservation on finite but very long time intervals; in return it gains a set of initial conditions nicer than (10) for which the description is valid. This set closely approximates the KAM initial conditions, and is "nice" in that it consists of finitely many connected components, each with nonempty interior. In fact, in typical versions of Nekhoroshev's theorem, one pieces together estimates on this set with estimates on its complement to obtain near conservation of action on all of phase space; in the present application precise estimates need to be tailored to individual parts of action space with particular resonance properties.

In view of the difficulties with KAM cited above, Nekhoroshev's theorem seems more promising, since only nice initial conditions have physical relevance, and since the infinite time intervals of the KAM theorem are superfluous here. This promise is further borne out by the tools (resonant normal forms) which the proof of Nekhoroshev's theorem provides for describing motion near the resonances of (1) which are sufficiently far from (10); these motions turn out to be precisely the channeling motions of (1).

C. Outline of the Mathematical Results

Having argued on physical grounds that Nekhoroshev's theorem is the "natural" approach to the problem formulated in Section 3, the task remains to formulate and prove in a mathematical way the assertions stated physically in Section 4A. Here we give an overview of the relevant mathematical techniques and of the way they fit together before entering into more detail below.

As in the KAM theorem, the idea behind Nekhoroshev's theorem consists in seeking a sequence of canonical transformations of phase space \((I, \theta)\) which bring the Hamiltonian (7) into simpler form. To get an idea of how this works, let us follow the Italian authors of Refs. 6, 7 in seeking a near-identity canonical change of variables \((I, \theta) \mapsto (J, \phi)\) which will bring (7) into "normal form" by (almost) replacing the perturbing potential with a suitable average. The Lie method is the conceptually simplest way of introducing canonical transformations in this context; for this purpose we consider a function \(\chi : D \to \mathbb{R}\) (the Lie generating function) analytic on a certain domain in phase space \(D = K \times T^3\), where \(K\) is a "nice" subset of action space. Formally, the \(\epsilon\)-near-identity transformation \(T\) generated by such a function acts on smooth functions of phase space by way of the operator \(e^{\mathcal{L}_{\chi}}\), where \(\mathcal{L}_\chi = \{ , \chi\}\) is the Poisson bracket of \(\chi\) with operands. To first order, the transformed Hamiltonian \(H'\) obtained in this way from \(H\) in (7) is

\[
H'(J, \phi) = (e^{\mathcal{L}_{\chi}}H)(J, \phi) = H(J, \phi) + \epsilon\{H, \chi\} + O(\epsilon^2)
\]

\[
= \frac{1}{2} J^2 + \epsilon(W(\phi) - J \cdot \frac{\partial \chi}{\partial \phi}) + O(\epsilon^2).
\]
Attempting to eliminate the $O(\epsilon)$ term leads to what Arnol’d calls the homological equation. Expressed in terms of the complex Fourier coefficients of $\chi$ and $W$, the homological equation reads

$$2\pi i (k \cdot J) \chi_k(J) = W_k \quad \forall k \in \mathbb{Z}^3.$$  \hspace{1cm} (12)

Since (12) has no solution for $k = 0$, we see immediately that the best that can be done is to reduce the $O(\epsilon)$ term to $W_0$, the space average of $W$ (whence the physical name “spatial continuum model” for the associated normal form). Even so, because of the small denominators $k \cdot J$, the convergence of the Fourier series for $\chi$ defined by solving (12) for the remaining $k$ is problematic, and the nicest domain of definition is obtained by moving Fourier coefficients with indices of order higher than some “ultraviolet cutoff” $N(\epsilon)$ into a remainder term. The resulting series comprising the low-order harmonics then has well-behaved small denominators on the subset of action space defined by

$$\{ J \in \mathbb{R}^3 \mid |k \cdot J| \geq C\epsilon^\alpha |k|^{-p} \land 0 < |k| \leq N \},$$  \hspace{1cm} (13)

where $|k| = |k_1| + |k_2| + |k_3|$, and $C, \alpha, p > 0$ are appropriate constants. This set eliminates planar slabs centered on the low-order resonant planes $k \cdot J = 0$ in action space, and it is this domain which converges with increasing $N$ to a highly nonresonant subset of action space as occurs typically in the KAM theorem.

Although the domain (13) has nonempty interior and is “large” (i.e., its complement vanishes with $\epsilon$), it does not include the low-order channeling directions; that is, directions $J$ near to where the small denominators $k \cdot J$ vanish for nonzero $k \in \mathbb{Z}^3$ with small norm. It turns out that these directions are contained in the domains of transformations which reduce $W$ not to its spatial average, but instead to its average over angles parallel to these directions. To see how this comes about, note that if a denominator $k \cdot J$ vanishes, then so does any denominator $j \cdot J$, where $j$ is codirectional with $k$; i.e., $j \in \mathcal{M}_k$ where $\mathcal{M}$ is the set of all $rk \in \mathbb{Z}^3$, $r \in \mathbb{R}$. If one deletes from the Fourier series of the generating function $\chi$ all Fourier coefficients with small denominators generated by a submodule $\mathcal{M}_k$, a modified transformation will be defined in and around the $J$-plane $k \cdot J = 0$, and it will reduce $W$ to the “resonant subsseries” $\sum_{j \in \mathcal{M}_k} W_j e^{2\pi ij \cdot \phi}$, which is precisely the average of $W$ over angles parallel to the $J$-plane, i.e., a planar continuum potential. In this case the transformed Hamiltonian is said to be in resonant normal form to first order. What Nekhoroshev calls an “analytic lemma” is then obtained by carrying the transformation to optimal order in $\epsilon$ for any suitable submodule of $\mathbb{Z}^3$; this results, roughly speaking, in a near-identity canonical transformation

$$T : K \times \mathbb{T}^3 \rightarrow T(K \times \mathbb{T}^3),$$  \hspace{1cm} (14)

where $K$ is a compact subset of action space which is nonresonant with respect to $\mathcal{M}$ to order $N$:

$$I \in K \Rightarrow |k \cdot I| \geq \frac{3}{2} A\epsilon^\alpha |k|^{-p} \quad \text{for all } k \notin \mathcal{M} \quad \text{with } |k| \leq N$$  \hspace{1cm} (15)
for suitable positive constants $A$, $\alpha$, and $p$. On $K \times T^3$, $T$ transforms the Hamiltonian 
(7) to the resonant normal form $H' = H \circ T$, which may be written

$$H'(J, \phi) = \frac{1}{2} J^2 + \epsilon G(J, \phi) + R(J, \phi), \quad \text{where}$$

$$G(J, \phi) = \sum_{k \in \mathcal{M}} G_k(J) e^{2\pi i k \cdot \phi}, \quad \text{and}$$

$$R(J, \phi) = \sum_{k \notin \mathcal{M}} R_k(J) e^{2\pi i k \cdot \phi}. \quad (16a, 16b, 16c)$$

Furthermore, the remainder $R$ is exponentially small in the sup norm over the relevant domain:

$$\|R\|_{K \times T^3} \leq 2E^2 e^{-\epsilon - \tau/4}, \quad (17)$$

where $\tau$, $E$ are again suitable prespecified positive numbers. Here $\mathcal{M}$ was introduced as a one-dimensional maximal submodule of $\mathbb{Z}^3$, but the procedure outlined works for $\mathcal{M} = \{0\}$ as discussed in conjunction with Eq. (12), or for $\mathcal{M}$ a two-dimensional maximal submodule constructed from two independent vectors in $\mathbb{Z}^3$, which corresponds to an axial direction in action space. More details are given in Section 6.

Once the transformation to normal form is accomplished, conclusions may be drawn concerning the behavior of trajectories of (1). Although the transformation to normal form corresponding to an arbitrary maximal submodule $\mathcal{M}$ is carried out by way of a single theorem, in investigating the trajectories we must first single out the case corresponding to the trivial submodule ($\mathcal{M} = \{0\}$). This is because trajectories with initial conditions governed by this nonresonant normal form quickly fail to satisfy the channeling criterion of Eqs. (4), (5), (8) and (9). This is a very intuitive result; to appreciate the geometry, the following arboreal metaphor is useful. Imagine standing in a large orchard of trees planted in regular rows and columns: looking around, one sees not only the corridors formed by the spaces between the main rows and columns of trees; one sees myriad corridors of varying width in many directions. In the limit of slenderer and slenderer trees, one sees corridors in every direction rationally related to the main rows and columns; in practice, the tree trunks obstruct all but finitely many “low-order” corridors. Outside these low-order corridors, one’s line of sight penetrates only a certain depth into the orchard, and this depth depends on the width of the tree trunks. Thinking now of the tree trunks as the close encounter region $C(1)$, relation (13) defines simultaneously what is meant by directions “outside the low-order corridors,” as well as the initial directions which give rise to nearly rectilinear trajectories. If we can determine the “maximum line of sight” allowed by the truncated Diophantine conditions (13) and $C(1)$, we may then couple this with an appropriate rectilinear approximation derived from the nonresonant normal form to arrive at an estimate of the maximum time to close encounter for “nonchanneling trajectories.” This estimate of the line of sight is expressed mathematically as an “estimate of the rate of ergodization” for linear flows of the torus, and is discussed below in Section 5.

For normal forms based on low-order nontrivial maximum submodules, it is possible to prove that trajectories beginning well inside the corresponding set of initial conditions will remain trapped there by reason of the convexity of the unperturbed energy surface.
This trapping mechanism was first pointed out by Nekhoroshev, and was discussed in detail in Ref. 7. In the present context, this culminates in a theorem which rigorously demonstrates the physical characteristics of channeling listed in Section 4A; this theorem is discussed below in Section 7.

5. Ergodization Rates for Linear Flow on the Torus

In this section we present and sketch the proofs of results that will appear with complete proofs in Ref. 14. We are concerned here with the rate at which linear flow fills the 3-torus when subject to truncated nonresonance conditions of the form (13). Although the result stated here is needed only for the 3-torus, its generalization to the $n$-torus presents no added difficulties. In contrast, similar results can be proved for the 2-torus using more elementary techniques.

We will work on the flat $n$-torus ($n \geq 2$) with the usual Euclidean metric. This space may be viewed as the fundamental cube in $\mathbb{R}^n$

$$\bar{\chi} = \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \cdots \times \left[ -\frac{1}{2}, \frac{1}{2} \right]$$

(18)

with opposite faces identified. The unique coordinates $\mathbf{\theta} = (\theta_1, \ldots, \theta_n)$ of a point in $T^n$ will belong to the half open cube

$$\chi = \left[ -\frac{1}{2}, \frac{1}{2} \right) \times \cdots \times \left[ -\frac{1}{2}, \frac{1}{2} \right)$$

(19)

Although strictly speaking the coordinates $\mathbf{\theta}$ belong to $\mathbb{R}^n$, we will abuse the notation slightly and identify points in $T^n$ with their coordinates by writing $\mathbf{\theta} \in T^n$. In these coordinates, the standard universal covering projection $p: \mathbb{R}^n \to T^n$ takes the form

$$p(x) = \left( \{x_1 + 1/2\} - 1/2, \ldots, \{x_n + 1/2\} - 1/2 \right),$$

(20)

where $\{x\}$ denotes the fractional part of $x$ ($\{x\} = x - [x]$, where $[x]$ is the greatest integer less than or equal to $x$). The covering map is now used to define operations on $T^n$: each operation (multiplication by scalars, addition, additive inversion, and translation by points in $\mathbb{R}^n$) is simply the projection of the usual operation with coordinates in $\mathbb{R}^n$; e.g., the coordinates of the sum of two points $\mathbf{\theta}, \mathbf{\phi} \in T^n$ are $p(\mathbf{\theta} + \mathbf{\phi})$.

The flat metric $d$ may now be defined for any pair of points $\mathbf{\theta}, \mathbf{\phi} \in T^n$ by

$$d(\mathbf{\theta}, \mathbf{\phi}) = \left\| \mathbf{\theta} - \mathbf{\phi} \right\|,$$

(21)

where the norm is the usual Euclidean norm on $\mathbb{R}^n$. The simple expression for the metric results from the “centered” coordinates on the torus, and we see that the distance between points on the torus is at most $\sqrt{n}/2$.

For $0 \leq R \leq 1/2$, the metric allows us to define the open ball of radius $R$ centered on $\mathbf{\theta}$ in $T^n$ by $B_R(\mathbf{\theta}) = \{ \mathbf{\phi} \in T^n \mid d(\mathbf{\theta}, \mathbf{\phi}) < R \}$. For our purposes it will not matter that the boundary of the closed ball $B_{1/2}(\mathbf{\theta})$ intersects itself in $n$ points; we will still make use of it.

Finally, if $\mathbf{\alpha}$ is a vector in $\mathbb{R}^n$ with unit length (denoted $\mathbf{\alpha} \in S^{n-1}$), a “rectilinear orbit” (called a “straight winding” by V.I. Arnol’d) of $T^n$ with direction vector $\mathbf{\alpha}$
will be defined as follows. For each $t \in \mathbb{R}$, we first consider the one-parameter family of translation maps $\alpha_t : T^n \to T^n$ defined by $\alpha_t(\theta) = \theta + ta$. A rectilinear orbit of $T^n$ with direction vector $\alpha$ and initial condition $\theta$ is then the image of $\theta$ under the flow defined by $\alpha_t$ over some closed interval in $\mathbb{R}$:

$$
\bigcup_{a \leq t \leq b} \alpha_t(\theta).
$$

One final comment about the notation: both the Euclidean norm $\|k\| = (\sum_i k_i^2)^{1/2}$ and the vector index norm $|k| = \sum_i |k_i|$ will be used for integer vectors $k \in \mathbb{Z}^n$. This is because the Euclidean norm is most natural here, while the application to the channeling problem requires the index norm. This means that factors of $\sqrt{n}$ will occasionally arise through use of the relation $\|k\| \leq |k| \leq \sqrt{n}\|k\|$.

It is well known (see, e.g. Refs. 2, 3) that orbits $\bigcup_{0 \leq t < \infty} \alpha_t(\theta)$ are either degenerate (i.e., they fill tori of dimension strictly less than $n$), or are dense in $T^n$, according as to whether or not $\alpha$ satisfies a resonance condition $\alpha \cdot k = 0$ for some $k \in \mathbb{Z}^n \setminus \{0\}$, and that this property of orbits is independent of the initial condition. For nonresonant $\alpha$, the denseness of $\bigcup_{0 \leq t < \infty} \alpha_t(\theta)$ implies an “ergodization” result; i.e., for nonresonant $\alpha$, given $R > 0$, there exists a $T > 0$ such that no point in $T^n$ is more than $R$-distant from any rectilinear orbit $\bigcup_{0 \leq t \leq T} \alpha_t(\theta)$. It is easy to see that this condition is equivalent to the following

**Definition.** Given $R > 0$, the direction vector $\alpha \in S^{n-1}$ is said to **ergodize** $T^n$ to within $R$ after time $T$ if

$$
\bigcup_{0 \leq t \leq T} \alpha_t(\mathcal{B}_R(\theta)) = T^n
$$

for all $\theta \in T^n$.

In words, this is expressed by saying that given $R > 0$, the image of any ball of radius $R$ under the flow generated by $\alpha$ between 0 and $T$ is all of $T^n$. The property of ergodizing the torus to within $R$ after time $T$ is independent of initial condition in the following sense: for any pair $\theta, \phi \in T^n$, $\bigcup_{0 \leq t \leq T} \alpha_t(\mathcal{B}_R(\theta)) = \bigcup_{0 \leq t \leq T} \alpha_t(\mathcal{B}_R(\phi)) + \{\theta - \phi\}$, so that if the ergodization condition $\bigcup_{0 \leq t \leq T} \alpha_t(\mathcal{B}_R(\theta)) = T^n$ is satisfied for one $\theta \in T^n$, then it is satisfied for all $\theta \in T^n$.

If all that can be said about the direction vector $\alpha$ is that it is nonresonant, then for fixed $R$, the time $T$ required to ergodize $T^n$ is of course highly dependent on $\alpha$, and may in general be arbitrarily long (think of a sequence of nonresonant $\alpha$'s converging to a low order resonant $\alpha$). Furthermore, with the above definition of ergodization, even orbits given by resonant $\alpha$ can ergodize $T^n$ for fixed $R > 0$, provided they are resonant at sufficiently high order (i.e., if $\alpha \cdot k = 0$ only for integer vectors $k$ with large norm).

We now state and sketch the proof of the ergodization result in the case of **untruncated** nonresonance conditions. Specifically, for $2p > n$ we will consider $\alpha$ belonging to the set

$$
\mathcal{D}(p, C) = \{ \alpha \in S^{n-1} \mid |\alpha \cdot k| > \frac{C}{|k|^p} \forall k \in \mathbb{Z}^n \setminus \{0\} \}.
$$

Direction vectors in $\mathcal{D}(p, C)$ are called highly nonresonant and are said to satisfy a Diophantine condition of order $p$. 

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The preliminary version of the main result of this section may be stated as follows:

**Theorem 1.** Let $0 < R \leq 1$. Given $\alpha \in \mathcal{D}(p,C)$, rectilinear orbits of $T^n$ with direction vector $\alpha$ will ergodize $T^n$ to within $R$ after time $T$, where

$$T = \frac{2n^{p/2} \|V\|_\Delta}{C \pi R^{p+n/2}}$$

is independent of $\alpha$.

The factor $\|V\|_\Delta$ is a Sobolev norm of a particular minimizing function $V_\bullet$ about which more will be said below (see comments following Eq. (36)). The idea for the proof is to trace the time averages of smooth functions with compact support as they evolve in opposite time directions under the flow defined by an arbitrary direction vector in $\mathcal{D}(p,C)$. If the supports of these functions are concentrated inside balls of radius $R/2$ centered on two arbitrary points in $T^n$, the supports of their evolving time averages do not necessarily intersect until the flows has ergodized $T^n$ to within $R$. As long as the supports of the time averages are disjoint, their inner product is zero, and this fact may be used to derive an expression for the ergodization time. The small divisors in this expression are controlled by the Diophantine condition on $\alpha$, and this control gives an estimate of the ergodization time for all Diophantine $\alpha$.

Before giving the main ideas of the proof, a few more definitions and notational conventions must be spelled out.

We define $\mathcal{A}^{R/2}_{(p)}$ to be a set of functions on $T^n$, each element $u$ of which is $p$ times continuously differentiable, is nonnegative on its support contained in $\mathcal{B}_{R/2}(0)$, and satisfies $\int_{T^n} u = 1$.

Given $u \in \mathcal{A}^{R/2}_{(p)}$ and $z \in T^n$, $u_z(x) = u(x - z)$ will denote the translation of $u$ by $z$; the support of $u_z$ has simply been shifted from the ball $\mathcal{B}_{R/2}(0)$ centered at 0 to the ball $\mathcal{B}_{R/2}(z)$ centered at $z$. The set of all translates of functions in $\mathcal{A}^{R/2}_{(p)}$ will be denoted by $\mathcal{T} \mathcal{A}^{R/2}_{(p)}$.

For $u_z \in \mathcal{T} \mathcal{A}^{R/2}_{(p)}$, we define $\langle u_z \rangle_t^\alpha : T^n \to \mathbb{R}$, the $t$-average of $u_z$ in the direction $\alpha$, by

$$\langle u_z \rangle_t^\alpha(x) = \frac{1}{t} \int_0^t (u_z \circ \alpha_r)(x) \, dr.$$  \hspace{1cm} (25)

It is important here to see that

$$\text{supp} \left( \langle u_z \rangle_t^\alpha \right) \subset \bigcup_{0 \leq r \leq t} \alpha_{-r}(\mathcal{B}_{R/2}(z)),$$  \hspace{1cm} (26)

which the reader is invited to check.

If $u_y \in \mathcal{T} \mathcal{A}^{R/2}_{(p)}$, and $u$ is expressed in terms of its Fourier series, then $u_y$ may be written

$$u_y(x) = \sum_{k \in \mathbb{Z}^n} u_k e^{2\pi i k \cdot (x-y)}.$$  \hspace{1cm} (27)
We thus compute \( \langle u_y \rangle_t^\alpha(x) \)
\[
= \frac{1}{t} \int_0^t (u_y \circ \alpha_r)(x) \, dr = \frac{1}{t} \int_0^t \sum_{k \in \mathbb{Z}^n} u_k e^{2\pi i k \cdot (x-y-r \alpha)} \, dr
\]
\[
= \sum_{k \cdot \alpha \neq 0} u_k e^{2\pi i k \cdot (x-y)} + \frac{1}{t} \sum_{k \cdot \alpha \neq 0} u_k e^{2\pi i k \cdot (x-y)} \left( e^{-2\pi i k \cdot \alpha} - \frac{1}{2\pi i k \cdot \alpha} \right). \tag{28}
\]

The proof may now be sketched. Let \( 0 < R \leq 1, \alpha \in \mathcal{D}(p, C) \) and \( u \in \mathcal{A}^{R^2}(p) \) be arbitrary, and let \( y, z \in \mathbb{T}^n, T > 0 \) be such that the images of the balls \( B_{R/2}(y) \) and \( B_{R/2}(z) \) do not intersect under forward and reverse orbits in the direction of \( \alpha \) between 0 and \( T/2 \):
\[
\left( \bigcup_{0 \leq t \leq T/2} \alpha_{-t}(B_{R/2}(y)) \right) \cap \left( \bigcup_{0 \leq s \leq T/2} \alpha_{s}(B_{R/2}(z)) \right) = \emptyset. \tag{29}
\]
Since the supports of \( \langle u_y \rangle_t^\alpha \) and \( \langle u_z \rangle_t^\alpha \) are contained in
\[
\bigcup_{0 \leq t \leq T/2} \alpha_{-t}(B_{R/2}(y)) \quad \text{and} \quad \bigcup_{0 \leq t \leq T/2} \alpha_{t}(B_{R/2}(z)), \tag{30}
\]
respectively, the inner product
\[
\left( \langle u_y \rangle_t^\alpha, \langle u_z \rangle_t^\alpha \right) = \int_{\mathbb{T}^n} \langle u_y \rangle_t^\alpha(x) \overline{\langle u_z \rangle_t^\alpha(x)} \, dx \tag{31}
\]
vanishes for \( 0 \leq t \leq T/2 \). Using the series expression (28) obtained above for the time averages of \( u_y \) and \( u_z \), we directly compute the integral for \( 0 \leq t \leq T/2 \):

\[
0 = \left( \langle u_y \rangle_t^\alpha, \langle u_z \rangle_t^\alpha \right) = \int_{\mathbb{T}^n} \sum_{k \cdot \alpha = 0 \atop l \cdot \alpha = 0} u_k \bar{u}_l e^{2\pi i [(k-l) \cdot x - (k \cdot \alpha \cdot y - l \cdot \alpha)]} \, dx
\]
\[
+ \frac{1}{t} \int_{\mathbb{T}^n} \left[ \sum_{k \cdot \alpha = 0 \atop l \cdot \alpha \neq 0} u_k \bar{u}_l e^{2\pi i [(k-l) \cdot x - (k \cdot \alpha \cdot y - l \cdot \alpha)]} \left( e^{2\pi i (k \cdot \alpha \cdot y - l \cdot \alpha)} - \frac{1}{2\pi i (k \cdot \alpha)} \right) \right] \, dx
\]
\[
+ \frac{1}{t^2} \int_{\mathbb{T}^n} \sum_{k \cdot \alpha \neq 0 \atop l \cdot \alpha \neq 0} u_k \bar{u}_l e^{2\pi i [(k-l) \cdot x - (k \cdot \alpha \cdot y - l \cdot \alpha)]} \left( e^{2\pi i (k \cdot \alpha \cdot y - l \cdot \alpha)} - \frac{1}{2\pi i (k \cdot \alpha)} \right) \left( e^{2\pi i (k \cdot \alpha \cdot y - l \cdot \alpha)} - \frac{1}{2\pi i (l \cdot \alpha)} \right) \, dx. \tag{32}
\]

Examination of the above integrals reveals that in each term, nonzero contributions can occur only when \( k = l \). Since \( k = l \) never occurs in the middle terms, they vanish, and
after integration the sum of the first and last terms reduces to

$$\sum_{k,\alpha=0} |u_k|^2 e^{2\pi ik \cdot (y-z)} - \frac{1}{t^2} \sum_{k,\alpha \neq 0} |u_k|^2 e^{2\pi ik \cdot (y-z)} \left( \frac{e^{2\pi ik \cdot \alpha} - 1}{4\pi^2 (k \cdot \alpha)^2} \right)^2. \quad (33)$$

Because \( \alpha \in \mathcal{D}(p, C) \), the only \( k \) for which \( k \cdot \alpha = 0 \) is \( k = 0 \). The first sum therefore reduces to \( |u_0|^2 = \left| \int_{T^n} u \right|^2 = 1 \), and after multiplying by \( t^2 \) in Eq. (32), we have

$$t^2 = \sum_{k \neq 0} |u_k|^2 e^{2\pi ik \cdot (y-z)} \left( \frac{e^{2\pi ik \cdot \alpha} - 1}{4\pi^2 (k \cdot \alpha)^2} \right)^2 \leq \frac{1}{\pi^2} \sum_{k \in \mathbb{Z}^n} |u_k|^2 \frac{1}{(k \cdot \alpha)^2}. \quad (34)$$

The Diophantine condition \( |\alpha \cdot k| > C|k|^{-p} \) then applies to the small divisors to ensure that this sum is strictly less than

$$\frac{1}{C^2 \pi^2} \sum_{k \in \mathbb{Z}^n} |k|^{2p} |u_k|^2. \quad (35)$$

Now \( u \in \mathcal{A}_{(p)}^{R/2} \) was arbitrary, which implies that

$$t^2 < \frac{1}{C^2 \pi^2} \inf_{u \in \mathcal{A}_{(p)}^{R/2}} \sum_{k \in \mathbb{Z}^n} |k|^{2p} |u_k|^2 \leq \frac{n^p}{C^2 \pi^2} \inf_{u \in \mathcal{A}_{(p)}^{1/2}} \sum_{k \in \mathbb{Z}^n} ||k||^{2p} |u_k|^2$$

$$= \frac{n^p \|U_*\|_{\Delta}^2}{C^2 \pi^2} = \frac{n^p \|V_*\|_{\Delta}^2}{C^2 \pi^2 R^{2p+n}}. \quad (36)$$

In the first equality, \( U_* \) represents the (unique) element of \( \mathcal{A}_{(p)}^{R/2} \) for which the infimum is attained, while the symbol \( \| \|_{\Delta} \) reflects the fact that the preceding sum is the square of a norm on \( \mathcal{A}_{(p)}^{1/2} \) (in fact a Sobolev norm). In the second equality, \( V_* \) is the element of \( \mathcal{A}_{(p)}^{1/2} \) which minimizes this Sobolev norm. Not surprisingly, it turns out that \( V_* \) is a scaling of \( U_* \) (in sloppy notation, \( U_*(x) = R^{-n} V_*(x/R) \)), and this allows the ergodization time to be expressed in reference to the “natural” (and smallest) norm \( \|V_*\|_{\Delta}^2 \), which occurs for those functions with the largest possible support. Further details may be found in Ref. 14.

The sketch of the proof is nearly complete. We have shown that if \( T \) is such that the nonintersection condition (29) holds, then for all \( 0 < t \leq T/2 \), the inequality (36) is true. In particular, for \( t = T/2 \), it follows that \( T^2 < 4n^p \|V_*\|_{\Delta}^2/(C^2 \pi^2 R^{2p+n}) \), and so if \( T = 2n^p/|V_*|_{\Delta}/(C^2 \pi R^{2p+n/2}) \), then condition (29) must be false. In other words, it must be true that

$$\left( \bigcup_{0 \leq t \leq T/2} \alpha_{-t}(\mathcal{B}_{R/2}(y)) \right) \cap \left( \bigcup_{0 \leq s \leq T/2} \alpha_s(\mathcal{B}_{R/2}(z)) \right) \neq \emptyset, \quad (37)$$

in which case there exist \( s, t \in [0, T/2] \) such that \( \alpha_{-t}(\mathcal{B}_{R/2}(y)) \cap \alpha_s(\mathcal{B}_{R/2}(z)) \neq \emptyset. \)

It is quite easy to show (see Corollary 4.1 of Ref. 14) that if this intersection condition
holds for arbitrary \( y \) and \( z \) in \( T^n \), then

\[
\bigcup_{0 \leq t \leq T} \alpha_t(B_{R/2}(z)) = T^n, \tag{38}
\]

and it follows that any \( \alpha \in \mathcal{D}(p, C) \) ergodizes the torus to within \( R \) after time

\[
T = \frac{2n^{p/2}\|V_*\|_{\Delta}}{C\pi R^{p + n/2}}. \tag{39}
\]

In the application to the perfect crystal model, we need to estimate the ergodization time not on the set \( \mathcal{D}(p, C) \), but rather on a superset of it of the form

\[
\mathcal{D}(p, C, N) = \{ \alpha \in S^{n-1} \mid |\alpha \cdot k| > \frac{C}{|k|^p} \forall k \in \mathbb{Z}^n \setminus \{0\} \exists |k| \leq N \} \tag{40}
\]

where the \( \alpha \) satisfy nonresonance conditions only up to a certain order \( N \).

It is intuitively clear that, for fixed \( R > 0 \), rectilinear orbits arising from resonant direction vectors will ergodize the torus, provided they are resonant at sufficiently high order. Such orbits are indeed degenerate, but at high enough order, they build a “web” fine enough to trap the \( n \)-ball of radius \( R \). In fact, given \( R \) and given a Diophantine condition, it is possible to show that for \( N > N^* \), where

\[
N^* = \kappa R^{-\left(\frac{p + n/2}{p - n/2}\right)} + n/2, \tag{41}
\]

when the Diophantine condition is truncated at order \( N \), the ergodization time found in the untruncated case is retained up to a factor depending on \( N \). (Here \( \kappa \) is a constant depending on \( n, p \), and \( V_* \); it is given explicitly in Ref. 14.) This result is stated as

**Theorem 2.** Let \( 0 < R \leq 1 \), and choose \( N > N^* \). Then given \( \alpha \in \mathcal{D}(p, C, N) \), \( T^n \) will be ergodized to within \( R \) by rectilinear orbits with direction vector \( \alpha \) after time

\[
T = \frac{2n^{p/2}\|V_*\|_{\Delta}}{C\pi R^{p + n/2}} \left[ 1 - \left( \frac{N^* - n/2}{N - n/2} \right)^{p - 2} \right]^{-1/2}. \tag{42}
\]

The proof of this theorem closely parallels but is more complicated than the proof of Theorem 1. Essentially, the test functions \( u(x) = \sum_k u_k e^{2\pi ik \cdot x} \) with compact support are replaced by approximating trigonometric polynomials \( u^\leq(x) = \sum_{|k| \leq N} u_k e^{2\pi ik \cdot x} \), and the orthogonality condition (32) is replaced by approximate orthogonality

\[
\left| \langle \frac{1}{N} \sum_{\alpha} u_{\alpha}^\wedge, \frac{1}{N} \sum_{\beta} u_{\beta}^\wedge \rangle \right| \leq \delta \text{ for an appropriately small } \delta. \tag{43}
\]

Because of the rapid decrease of the Fourier coefficients \( u_k \) with increasing \( |k| \), the size of \( \delta \) may be controlled and the previous estimates go through in an approximate way, leading to Theorem 2 as well as to the estimate (41) of the “critical cutoff.” Details may be found in Sections 4.3 and 4.4 of Ref. 14.

### 6. The Transformation to Normal Form

The transformation of (7) to the normal form (16) is the most complicated step required by our techniques, and it is the step which we outline in barest detail. Full
details may be found in Chapter 3 of Ref. 12; similar transformations are described in Refs. 6, 7, 23, and 24. Nekhoroshev proves the existence of the transformation in what he calls the “Analytic Lemma,” because the transformation simply changes the form of (7) on certain regions of phase space; the separate issue of the behavior of trajectories on those regions is the subject of the “geometric” part of the proof (see Section 7).

In order to state an abbreviated form of the Analytic Lemma, a few definitions and notational conventions must be spelled out. First, in order to discuss resonances, we recall that a finite subset \( \{ k^{(i)} \}_{i=1}^m \) of \( \mathbb{Z}^3 \) generates a submodule of \( \mathbb{Z}^3 \) defined as \( \{ z \in \mathbb{Z}^3 | z = \sum n_i k^{(i)} , n_i \in \mathbb{Z} \} \). Each submodule of \( \mathbb{Z}^3 \) has dimension 0, 1, 2, or 3, in the sense that the smallest vector subspace of \( \mathbb{R}^3 \) containing the submodule has that dimension. If \( \{ k^{(i)} \}_{i=1}^m \) generates a submodule of dimension \( n \), then the maximal submodule generated by \( \{ k^{(i)} \}_{i=1}^m \) is the largest submodule of dimension \( n \) containing \( \{ k^{(i)} \}_{i=1}^m \). In the remainder of this article, the symbol \( \mathcal{M} \) will refer to a 0-, 1- or 2-dimensional maximal submodule of \( \mathbb{Z}^3 \). Not surprisingly, every \( \mathcal{M} \) is generated respectively by integer combinations of a 1- or 2-element basis in \( \mathbb{Z}^3 \); the order \( |\mathcal{M}| \) of a maximal submodule is the smallest nonnegative integer \( r \) such that \( \mathcal{M} \) admits a basis of vectors with norm less than or equal to \( r \) (the norm here is the one defined following Eq. (13)).

Before stating the lemma we introduce the notation

\[
\Pi_{\mathcal{M}} f = \sum_{k \in \mathcal{M}} f_k(I) e^{2\pi i j \cdot \phi} \tag{43}
\]

for the resonant subseries of \( f \) corresponding to the maximal submodule \( \mathcal{M} \). (\( \Pi_{\mathcal{M}} \) is a projection operator, as the notation suggests.) We also use \( \| f \|_D \) to denote \( \sup_{x \in D} |f(x)| \) (or \( \max_i \sup_{x \in D} |f_i(x)| \) if \( f \) has components \( f_i \)). We may now state the

**Analytic Lemma.** Let \( \mathcal{M} \neq \mathbb{Z}^3 \) be a maximal submodule of \( \mathbb{Z}^3 \) of dimension 0, corresponding to “random” directions in action space, or of dimension 1 or 2, corresponding respectively to planar or axial directions in action space which are orthogonal to \( k \in \mathcal{M} \). Let

\[
H(I, \theta) = h(I) + f(I, \theta), \tag{44a}
\]

\[
h(I) = \frac{1}{2} I^2 \tag{44b}
\]

be analytic on a certain complex extension \( D \) of the real domain \( K \times T^3 \), where \( K \subset \mathbb{R}^3 \) is a compact subset of action space from which \( \epsilon \)-dependent neighborhoods of low order resonant planes \( I \cdot k = 0, k \notin \mathcal{M} \) have been removed in order to control small divisors. Suppose that

\[
\| H \|_D \leq E \quad \text{and} \quad \| f \|_D \leq \epsilon E, \tag{45}
\]

where \( \epsilon \) is sufficiently small. Then there exists a bijective near-identity transformation \( T : D_\infty \rightarrow T(D_\infty) \) with both \( T \) and \( T^{-1} \) canonical and analytic and with \( D_\infty \) a subset of \( D \) which closely approximates it. On \( D_\infty \), \( T \) transforms \( H \) to \( H' = H \circ T \), which may be written

\[
H'(J, \phi) = h(J) + G(J, \phi) + R(J, \phi), \quad \text{where} \tag{46}
\]
\[ G(J, \phi) = \sum_{k \in \mathcal{M}} G_k(J) e^{2\pi i k \cdot \phi}, \quad \text{and} \]
\[ R(J, \phi) = \sum_{k \notin \mathcal{M}} R_k(J) e^{2\pi i k \cdot \phi}. \]  

Furthermore,
\[ \|R\|_{D_{\infty}} \leq 2Eee\exp(-\epsilon^{-\tau/4}), \quad \text{and} \]
\[ \frac{1}{\epsilon} \|G - \Pi_{\mathcal{M}} f\|_{D_{\infty}} \leq \frac{1}{2} \epsilon^\tau. \]

A sequence of remarks will help clarify the content of the lemma.

1) There are various implicit and explicit constants in the lemma: \( \alpha, \tau, d, c > 1/2, p > 4 \) and \( a = \alpha + (p + 1)\tau \) are positive numbers satisfying certain consistency relations (an example of such a set of numbers is \( \alpha = 1/8, \tau = 1/72, d = 1/2, c = 5/8, p = 5, a = 5/24 \)).

2) The planar slabs removed from action space are such that \( I \in K \Rightarrow |k \cdot I| \geq \frac{3}{2}Ae^\alpha|k|^{-p} \forall k \notin \mathcal{M} \) and \( |k| \leq N \). The vectors \( k \notin \mathcal{M} \) with \( |k| \leq N \) define a finite number of directions in \( \mathbb{Z}^3 \) and for each of these directions there is a \( k \) with minimal norm, say \( k' \). It is easy to see that the region of action space removed by the \( k \)'s in the direction of \( k' \) is a planar slab centered on the resonant plane \( k' \cdot I = 0 \) with thickness \( \frac{3}{2}Ae^\alpha|k'|^{-p} \). This makes precise the description of the compact set \( K \) in the lemma.

3) Essentially, \( D \) contains \( K \times T^3 \) but protrudes a sup-norm distance \( Ae^\alpha \) into the complex domain surrounding \( K \) and a distance \( \sigma \) into the complex domain surrounding \( T^3 \).

4) The transformation \((I, \theta) = T(J, \phi)\) is "near-identity" in the sense that
\[ \|I - J\|_{K_{\infty} \times T^3} \leq \frac{A}{4} \epsilon^c, \quad \text{and} \]
\[ \|\theta - \phi\|_{K_{\infty} \times T^3} \leq \frac{\sigma}{4} \epsilon^d. \]

5) A discussion of the transformed Hamiltonian (46) as a generalized continuum model when \( \mathcal{M} \neq \{0\} \) is given in section 7C. Here it suffices to say that for the perfect crystal model \( f(I, \theta) = W(\theta) \) and Eq. (50) says that \( G \) is close to the associated continuum potential.

6) In Ref. 12, a complete statement and proof of the lemma is given including crude estimates of the size of \( \epsilon \) and a more detailed characterization of the domain \( K_{\infty} \).

7) The Analytic Lemma may be proved by applying finitely many successive transformations of the type described in Section 4C. It is remarkable that the resulting remainder \( R \) is exponentially small; this is achieved by fixing \( \epsilon \) sufficiently small in advance, then taking the number of successive transformations to be a function of \( \epsilon \) (for the present result, \( [\epsilon^{-\tau/4}] \) transformations will suffice). We will not give the proof of this result, which depends on numerous technical estimates. However, we can give the
reader an idea of the proof by indicating how each transformation sets the stage for its successor. Beginning as in the Analytic Lemma with $H = h + f$, we split $f$ into its resonant and nonresonant parts $G = \Pi_M f$ and $R = (1 - \Pi_M)f$ so that $H = h + G + R$. Employing now the transformation $H' = e^{Lx}H$ described in (11) (the $\epsilon$-dependence in (11) is incorporated into $\chi$), we write the identity $H' = H' + H - H + \{H, \chi\} - \{H, \chi\}$ as

$$H' = H' + h + G + R - H + \{h, \chi\} + \{G + R, \chi\} - \{H, \chi\}$$

$$= h + G + R^\leq + \{h, \chi\} + S,$$

(53)

where we have introduced the cut off part of $R$ (recall $N = \epsilon^{-\tau}$):

$$R^\leq(J, \phi) = \sum_{k \in M, |k| \leq N} R_k(J) e^{2\pi i k \cdot \phi},$$

(54)

and the $N$-tail of $R$:

$$R^\geq = R - R^\leq,$$

(55)

and where

$$S = R^\geq + \{G + R, \chi\} + H' - H - \{H, \chi\}.$$  

(56)

This suggests choosing $\chi$ so as to eliminate $R^\leq$ by way of the homological equation $R^\leq + \{h, \chi\} = 0$. The various parts of $S$ are then controlled as follows. The term $R^\geq$ is small because of the rapid decay of the Fourier coefficients of $R$ and the choice of $N$; $\{G + R, \chi\} = \{f, \chi\}$ is small because $f$ and $\chi$ are; and finally, $H' - H - \{H, \chi\}$ is small because it is the remainder of the first-order Taylor's expansion of $H'$ along the "flow" of $\chi$, and $\chi$ is small. Once the size of $S$ is controlled, the new $G$ and $R$ for the succeeding transformation are defined by $G' = G + \Pi_M S$ and $R' = (1 - \Pi_M)S$. It can be verified that succeeding $G$'s remain $0(\epsilon)$ and that the $R$'s shrink as needed. The estimation process is aided by the analyticity in $D_\infty$ of the relevant functions; in most cases this permits Cauchy estimates to be used in place of more tedious estimates of individual series.

7. The Generalized Continuum Models

A. Geometric Considerations

With the analytical tools of Sections 5 and 6 in hand, we are in position to give a mathematical description of particle motions in the perfect crystal model. We begin by proving a result analogous to Nekhoroshev's "geometric lemma," which says that as long as trajectories remain inside the domain of a particular normal form, the transformed actions are very nearly required to move in an affine subspace in action space which is parallel to the submodule defining the normal form.

First, we give more detailed terminology concerning the geometry of the resonances. To each maximal submodule $M$ corresponds a resonance in action space, namely the subspace of all $J$ orthogonal to each $k \in M$. Actions in the neighborhood of a resonance (and sufficiently far from other low-order resonances) are channeling directions; this is formalized by first defining the resonant zone corresponding to $M$ by

$$Z_M(C, \alpha) = \{I \in \mathbb{R}^3 \mid |k \cdot I| \leq C e^\alpha |k|^{-p} \forall k \in M, 0 < |k| \leq |M|\}$$

(57)
for appropriate positive values of $C$, $\alpha$ and $p$. In the case where $\mathcal{M}$ is one dimensional, the zone $\mathcal{Z}_\mathcal{M}(C, \alpha)$ will also be denoted $\mathcal{Z}_k(C, \alpha)$, where $k$ is the unique generator (up to inversion) of $\mathcal{M}$. The case $\mathcal{M} = \{0\}$ also deserves comment; since $|\{0\}| = 0$, no $k \in \mathbb{Z}^3$ satisfy $0 < |k| \leq |\{0\}|$, and the "zone" $\mathcal{Z}_{\{0\}}(C, \alpha)$ degenerates to $\mathbb{R}^3$ for any $C$, $\alpha$.

Next, given $\mathcal{M}$, its associated resonant block of order $N \geq |\mathcal{M}|$ is defined as the subset of action space

$$\tilde{\mathcal{Z}}^{C, \alpha}_\mathcal{M}(N, C_1, \alpha_1) = \mathcal{Z}_\mathcal{M}(C_1, \alpha_1) \setminus \bigcup_{k \in \mathcal{M}, |k| \leq N} \text{Int} \mathcal{Z}_k(C, \alpha),$$

(58)

where $\text{Int}$ denotes interior. These actions correspond to channeling directions; physically, it is interesting to note that when $\text{dim} \mathcal{M} = 1$ (corresponding to a planar direction), low-order "axial" zones are removed, as they should be. In the degenerate case $\mathcal{M} = \{0\}$, the corresponding block is called a "nonresonant block," and since the underlying zone is all of $\mathbb{R}^3$, it is denoted $\tilde{\mathcal{Z}}^{C, \alpha}_0(N)$ without reference to $C_1$ or $\alpha_1$. By carefully applying the definition, we see that the nonresonant block is the set

$$\tilde{\mathcal{Z}}^{C, \alpha}_0(N) = \mathbb{R}^3 \setminus \bigcup_{0 < |k| \leq N} \text{Int} \mathcal{Z}_k(C, \alpha).$$

(59)

Finally, we give some terminology concerning the transformed Hamiltonian (46). By removing the remainder $R$ we obtain the so-called effective Hamiltonian

$$\tilde{H}(J, \phi) = h(J) + G(J, \phi) = \frac{1}{2}J^2 + \sum_{k \in \mathcal{M}} G_k(J)e^{2\pi ik \cdot \phi}.$$  

(60)

The idea to bear in mind is that properties of (60) are very nearly enjoyed by (46). The first thing to notice about (60) is that is that it is cyclic in certain linear combinations of $\phi$; that is, $\tilde{H}$ is independent of any $\phi$ in the orthogonal complement of $\mathcal{M}$. In particular, if $\mathcal{M} = \{0\}$ is the trivial submodule, its orthogonal complement is all of $\mathbb{R}^3$ and (60) is completely cyclic, or independent of $\phi$. If $\mathcal{M} = \{(n, 0, 0) | n \in \mathbb{Z}\}$ then $G$ is cyclic in $\phi_2$ and $\phi_3$ and if $\mathcal{M} = \{(n, m, 0) | n, m \in \mathbb{Z}\}$ then $G$ is cyclic in $\phi_3$. If $f = W$ in (44) then $G$ in (46) is close to the planar and axial continuum models discussed in conjunction with Eqs. (3) and (2) respectively. In general, to emphasize the dependence of $G$ on only those angle variables lying in span$\mathcal{M}$, we will often write $G(J, \phi) = G(J, \phi^*)$, where $\phi^*$ denotes the projection of $\phi$ onto span$\mathcal{M}$. This cyclic or partly cyclic property is nearly true of (46), and we may prove the following

**Geometric Proposition.** Assume the hypotheses of the Analytic Lemma. Let $(J_0, \phi_0) \in K_\infty \times \mathbb{T}^3$ be any initial condition in the real part $K_\infty \times \mathbb{T}^3$ of the domain $D_\infty$ of the transformed Hamiltonian (46) and let $(J, \phi)$ be the solution of Hamilton's equations corresponding to (46). Denote by $T_0$ the solution's (possibly infinite) first time of escape from $K_\infty \times \mathbb{T}^3$. If $\mathcal{M}$ is the maximal submodule to which (46) is adapted,
denote by $J^*$ the projection of $J$ onto span$\mathcal{M}$ and set $\hat{J} = J - J^*$ so that we have the orthogonal splitting $J = J^* + \hat{J}$. Then for $0 \leq t \leq T_0$,

$$\|\hat{J}(t) - \hat{J}_0\|_\infty \leq \frac{8E}{\sigma} t ee^{-\epsilon^{-\epsilon / 4}}.$$  \hfill (61)

**Proof.** By the Analytic Lemma, for $0 \leq t \leq T_0$ we have

$$J(t) = \hat{J}(t) - \int_0^t \frac{\partial R}{\partial \phi}(J(t'), \phi(t')) dt' ,$$  \hfill (62)

where

$$\hat{J}(t) = J_0 - \int_0^t \frac{\partial G}{\partial \phi}(J(t'), \phi(t')) dt' =$$

$$J_0 - 2\pi i \sum_{k \in \mathcal{M}} k G_k(J(t')) e^{2\pi i k \cdot \phi(t')} dt' =$$

$$\hat{J}_0 + J^*_0 - 2\pi i \sum_{k \in \mathcal{M}} k \int_0^t G_k(J(t')) e^{2\pi i k \cdot \phi(t')} dt'.$$  \hfill (63)

The last term belongs to span$\mathcal{M}$, and we see that $\hat{J}$ remains entirely in the affine subspace "parallel" to $\mathcal{M}$ passing through $J_0$:

$$\hat{J}(t) = \hat{J}_0 + (\hat{J}(t))^*.$$  \hfill (64)

From the equation

$$J(t) = \hat{J}(t) + J^*(t) = \hat{J}_0 + (\hat{J}(t))^* - \left(\int_0^t \frac{\partial R}{\partial \phi}(t') dt'\right)^\wedge - \left(\int_0^t \frac{\partial R}{\partial \phi}(t') dt'\right)^* ,$$  \hfill (65)

we identify

$$\hat{J}(t) = \hat{J}_0 - \left(\int_0^t \frac{\partial R}{\partial \phi}(t') dt'\right)^\wedge \quad \text{and}$$

$$J^*(t) = (\hat{J}(t))^* - \left(\int_0^t \frac{\partial R}{\partial \phi}(t') dt'\right)^*.$$  \hfill (66)

(67)

Now because the initial condition $(J_0, \phi_0)$ is real, the solution $(J, \phi)$ is real for all $t$. Thus for $0 \leq t \leq T_0$,

$$\|\hat{J}(t) - \hat{J}_0\|_\infty \leq \|\left(\int_0^t \frac{\partial R}{\partial \phi}(t') dt'\right)^\wedge\|_{K_\infty \times T^3} \leq \|\int_0^t \frac{\partial R}{\partial \phi}(t') dt'\|_{K_\infty \times T^3} \leq$$

$$t \|\frac{\partial R}{\partial \phi}\|_{K_\infty \times T^3} \leq \frac{4t}{\sigma} \|R\|_{D_\infty} \leq \frac{8tE}{\sigma} e^{-\epsilon^{-\epsilon / 4}},$$  \hfill (68)

where the second to last inequality follows from a Cauchy's estimate, and the last inequality uses (49). $\Box$
Using this or a similar proposition, it is possible to prove simultaneously the confinement of motions of (46) within blocks corresponding to any submodule \( \mathcal{M} \) (cf. Ref. 7). Our purpose here however is to examine these motions separately, and for the sake of clarity of notation we treat the nonresonant case \( \mathcal{M} = \{0\} \) separately in the next subsection.

**B. The Spatial Continuum Model**

In the case \( \mathcal{M} = \{0\} \), every vector \( J \in \mathbb{R}^3 \) is orthogonal to \( \mathcal{M} \) and the splitting \( J = J^* + \hat{J} \) described in the Geometric Proposition degenerates to \( J = \hat{J} \). Therefore for any initial condition \((J_0, \phi_0)\) lying in the real part \( K_\infty \times T^3 \) of the domain \( D_\infty \) of (46), by the Geometric Proposition the real solution \((J, \phi)\) of (46) satisfies

\[
\|J(t) - J_0\|_\infty \leq \frac{8tE}{\sigma} e^{-\epsilon^{-t/4}}
\]

for as long as it remains in \( K_\infty \times T^3 \).

This near constancy of the transformed actions is exploited by way of a “bootstrap argument” to show that solutions of (46) beginning with actions well inside a nonresonant block \( \hat{Z}_0 \) remain inside the block for exponentially long times. It is first necessary to show that points beginning inside a block remain nonresonant when allowed to wander slightly outside the block in \( \mathbb{R}^3 \). In Section 5.2 of Ref. 12, it is shown that if \( I_0 \in \hat{Z}_0^{\frac{9}{8} A_0^a}(N) \) and if \( \|I_0 - J_0\|_\infty \leq A e^a \), then for every \( k \neq 0 \) with \(|k| \leq N\), \(|k \cdot J_0| \geq \frac{3}{2} A e^a \).

We may now state a theorem concerning particles impinging upon the perfect crystal in nonresonant or nearly nonresonant directions. The symbol \( S'_{\gamma^2} \) will denote the set of actions energetically accessible to real-valued motions of (7); it is the spherical shell of thickness \( O(\epsilon) \) given by \( S'_{\gamma^2} = \{ I \in \mathbb{R}^3 | 2 - 2eE_M/e_\perp \leq I_1^2 + I_2^2 + I_3^2 \leq 2 \} \).

**Theorem 3.** (Rectilinear Trajectories) Assume the hypotheses of the Analytic Lemma with \( H \) given by (7), \( \mathcal{M} = \{0\} \), and with \( K \) a certain compact approximating superset of \( \hat{Z}_0^{\frac{9}{8} A_0^a}(N) \cap S'_{\gamma^2} \). Let

\[
(I_0, \theta_0) \in \left( \hat{Z}_0^{\frac{9}{8} A_0^a}(N) \cap S'_{\gamma^2} \right) \times T^3
\]

be an initial condition for (7) which is nonresonant to order \( N = \epsilon^{-r} \). Then for \( 0 \leq t \leq \frac{A \sigma}{32E} e^{c-1} e^{-r/4} \), the solution \((I, \theta)\) of (7) with initial condition \((I_0, \theta_0)\) satisfies

\[
\|I(t) - I_0\|_\infty \leq \frac{3}{4} A e^c,
\]

\[
\|\theta(t) - \theta_0 - tI_0\|_\infty \leq \frac{3}{4} A t e^c.
\]

The proof of this theorem is a straightforward application of the Analytic Lemma and the Geometric Proposition. By the Analytic Lemma, real trajectories arising from initial conditions (40) are governed by a nonresonant normal form until their actions exit the compact set \( K \). But as long as the nonresonant normal form is valid, the
Geometric Proposition ensures that the actions are nearly frozen; in fact they require an exponentially long time to move to the boundary of $K$. The nonresonant normal form is in effect throughout this long interval, and so the inequalities (51) and (52) of the Analytic Lemma may be applied to arrive at the estimates (71).

Theorem 3 may be interpreted as saying that for initial conditions which are nonresonant to a certain order $N$, motions of (7) have nearly constant action $I \simeq I_0$ on an exponentially long time interval, while the configuration or angle variable $\theta$ is approximated by the rectilinear motion

$$\dot{\theta}(t) = \theta_0 + t I_0$$

(72)
on any time interval $[0, \text{Const.} e^{-b}]$ with $\alpha < b < c$. (In the language of Section 5, (72) is a “straight winding of $T^3$”, once $I_0$ and $t$ are normalized.) This is a purely mathematical statement about solutions of the Hamiltonian (7), and we have postulated that solutions of (7) model particle motions in crystals only so far as they satisfy the channeling criterion (9). The following theorem uses the results of Section 5 to show that each trajectory satisfying (71) quickly fails the channeling criterion, after which it is presumed deflected and no longer well-modeled by (7).

**Theorem 4. Close Encounters** Assume the hypotheses of Theorem 3; i.e., assume the hypotheses of the Analytic Lemma with $H$ given by (7), $\mathcal{M} = \{0\}$, and $K$ defined appropriately. Let $2R > 0$ be the radius of the largest ball $B_{2R}$ contained in the region of close encounter $C(1) \subset T^3$ described in (8). Then for $\epsilon$ sufficiently small, the initial actions (or initial directions)

$$I_0 \in \hat{Z}_0^{\alpha} \cap S_{\epsilon/2}$$

(73)
are nonchanneling directions in the sense that given any initial condition

$$(I_0, \theta_0) \in \left(\hat{Z}_0^{\alpha} \cap S_{\epsilon/2}\right) \times T^3,$$

(74)
the corresponding solution $(I, \theta)$ of (7) fails to satisfy the channeling criterion (9) at some $t'$ in the (brief) time interval

$$[0, C_3 \epsilon^{-\alpha}],$$

(75a)
where

$$C_3 = \frac{3^{p/2} 5^{p/4} \Delta \|V_*\|}{5 \pi^{p/2} R^{p+3/2}},$$

(75b)
and the constant $\|V_*\|_\Delta$ appears in Theorem 2.

**Remark.** Recall that $\alpha < 1/2$ controls the $\epsilon$-dependence of the resonant zones removed from action space.

**Proof.** In order to apply Theorem 2, we first normalize the initial action vector $I_0$ by writing $\beta_0 = I_0/\|I_0\|$ (Euclidean norm). Then $\|\beta_0\| = 1$ and for all $k \in \mathbb{Z}^3 \setminus \{0\}$ with $|k| \leq N = \epsilon^{-\alpha}, \ |\beta_0 \cdot k| = \|I_0\|^{-1} |I_0 \cdot k| \geq (5/2 \sqrt{2}) \alpha \epsilon^\alpha |k|^{-p}$, so that in the notation of Theorem 2, $\beta_0 \in \mathcal{D}(p, (5/2 \sqrt{2}) \alpha \epsilon^\alpha, N)$. Defining the normalized time $\xi = t \|I_0\|$ and
restricting $\epsilon$ so that $N = \epsilon^{-r}$ is greater than the critical cutoff $N^*$ given in (41), we apply Theorem 2 to find that for any $\theta_0 \in T^3$, the winding $\beta(\xi) = \theta_0 + \beta_0 \xi$ ($= \theta_0 + tI_0$) ergodizes $T^3$ to within $R$ after (normalized) time

$$T = \frac{3p/2 \|V_*\|_\Delta}{5A e^\alpha \pi R^{p+3/2}} \left[ 1 - \left( \frac{N^* - 3/2}{N - 3/2} \right)^{p-\delta} \right]^{-1/2}. \quad (76)$$

Now since $\frac{3}{2} - p < 0$ and since $N = \epsilon^{-r}$, for small enough $\epsilon$ this last expression is less than or equal to

$$T_0 = \frac{3p/2 \|V_*\|_\Delta}{5A \pi R^{p+3/2}} \epsilon^{-\alpha}. \quad (77)$$

By definition, the ergodization result means that every point in $T^3$ is a distance less than or equal to $R$ from the winding

$$\bigcup_{0 \leq \xi \leq T^*} \beta(\xi). \quad (78)$$

Now let $\vartheta$ be the center of the ball $B_{2R}$, and let $\xi' \in [0, T_0]$ be such that $\|\beta(\xi') - \vartheta\| \leq R$. Finally, take $t' = \xi'/\|I_0\|$ and note that $t'$ satisfies (75a), since $0 \leq t' = \xi'/\|I_0\| \leq \xi' \leq T_0 = C_3 \epsilon^{-\alpha}$; this is because $\|I_0\| \geq 1$ for $\epsilon$ small enough and $I_0 \in \mathcal{S}_{\xi, \vartheta}$. Therefore

$$\|\theta(t') - \vartheta\| \leq \|\theta(t') - \beta(\xi')\| + \|\beta(\xi') - \vartheta\| =$$

$$\|\theta(t') - \theta_0 - \beta_0 \xi'\| + \|\beta(\xi') - \vartheta\| \leq$$

$$\|\theta(t') - \theta_0 - I_0 \xi'\| + R. \quad (79)$$

Another restriction on $\epsilon$ gives $t' \leq C_3 \epsilon^{-\alpha} \Rightarrow t' \leq \frac{A_1}{2e} \epsilon^{-1} e^{-r/4}$, so that (71b) applies to show that the Euclidean norm above is less than or equal to $\frac{3}{4} \sqrt{3} A t' e^c \leq \frac{3}{4} \sqrt{3} A C_3 \epsilon^{-\alpha}$, which is in turn less than or equal to $R$ for small enough $\epsilon$. Thus

$$\|\theta(t') - \vartheta\| \leq 2R, \quad (80)$$

and we have produced a $t' \in [0, C_3 \epsilon^{-\alpha}]$ at which the angle $\theta$ belongs to $B_{2R} \subset C(1)$. The solution $(I, \theta)$ therefore fails the channeling criterion at some time $t'' \leq t'$, and the theorem is proved. $\square$

Theorems 3 and 4 taken together provide the picture of the “spatial continuum model” which was outlined in Section 4. Namely, sufficiently energetic particles injected into a classical crystal in the nonresonant (or nonchanneling) directions $I_0 \in \tilde{\mathcal{S}}_0^{\frac{3}{2} A, \alpha}(N) \cap \mathcal{S}_{\xi, \vartheta}$ will follow a roughly straight path to their first collision with a region of close encounter $C(1)$, and this collision happens quickly, at a time no later than $C_3 \epsilon^{-\alpha}$ from the time of injection. Moreover it may be shown that, with increasing particle energy, the volume in action space of the admissible nonresonant actions $\tilde{\mathcal{S}}_0^{\frac{3}{2} A, \alpha}(N) \cap \mathcal{S}_{\xi, \vartheta}$ approaches the volume of the set of all admissible actions $\mathcal{S}_{\xi, \vartheta}$ for the perfect crystal Hamiltonian (7). Finally, it should be mentioned that although the emphasis here is on positively charged particles, Theorems 3 and 4 make no use of the nonnegativity of $W$; they hold irrespective of particle charge.
C. Channeling

With the addition of a physically motivated assumption about the potential \( W \), we can show the existence of solutions of the scaled perfect crystal Hamiltonian (7) which satisfy the channeling criterion (9) on exponentially long time intervals. These solutions have characteristics which physicists recognize as the traits of axial or planar channeling; namely, they stay away from close encounter regions and so have a low probability of encountering nuclei, their “transverse energy” and “longitudinal momenta” are nearly conserved, and they are approximated in some sense by solutions of an appropriate generalized (axial or planar) continuum Hamiltonian, which in turn is the ordinary (axial or planar) continuum Hamiltonian to leading order. To make what follows more concrete, while reading this section the reader is invited to consider two specific submodules of \( \mathbb{Z}^3 \), namely the submodule generated by \((1, 0, 0) \in \mathbb{Z}^3\) (corresponding to planar channeling) and the submodule generated by \((1, 0, 0)\) and \((0, 1, 0)\) (corresponding to axial channeling). It is convenient to take the lattice atoms to be at the corners of the unit cube and the close encounter region as the union of small balls surrounding the nuclei of the lattice atoms.

We turn now to the relevant assumptions and definitions concerning the scaled potential \( W \). Given \( \mathcal{M} \) and given \( \theta \in T^3 \sim \mathbb{R}^3 / \mathbb{Z}^3 \), write \( \theta = \theta^* + \tilde{\theta} \), where \( \theta^* \) and \( \tilde{\theta} \) are the projections onto \( \text{span}\mathcal{M} \) and \( (\text{span}\mathcal{M})^\perp \) in \( \mathbb{R}^3 \). The resonant subseries \( \Pi \mathcal{M} W(\theta^*) \) corresponding to \( \mathcal{M} \) is then just the average over the variables \( \tilde{\theta} \) determined by \( \mathcal{M} \); that is, \( \Pi \mathcal{M} W(\theta^*) \) is a continuum potential. We next consider regions in configuration space bounded by equipotential surfaces of the continuum potentials. Given \( \mathcal{M} \) and any number \( Q \geq 0 \), define the closed subset \( \mathcal{A}_\mathcal{M}(Q) \) as

\[
\mathcal{A}_\mathcal{M}(Q) = \{ \theta \in T^3 | \Pi \mathcal{M} W(\theta^*) \leq Q \}.
\] (81)

Our assumption about \( W \) is then the following:

**Assumption A.** There exists a critical order \( M^* \geq 1 \) such that for any \( \mathcal{M} \) with order \( |\mathcal{M}| \leq M^* \), there are numbers \( Q' \) and \( Q'' \), \( 0 \leq Q' < Q'' < 1 \) with the property that given any \( Q \in [Q', Q''] \), \( \mathcal{A}_\mathcal{M}(Q) \neq \emptyset \) and \( \mathcal{A}_\mathcal{M}(Q) \cap \mathcal{C}(1) = \emptyset \), where \( \mathcal{C}(1) = \{ \theta \in T^3 | W(\theta) \geq 1 \} \).

This physical assumption says that at sufficiently low order, there are equipotential surfaces of the continuum potential \( \Pi \mathcal{M} W \) which do not intersect the restricted region \( \mathcal{C}(1) \). If \( \dim \mathcal{M} = 1 \), these surfaces are planes; if \( \dim \mathcal{M} = 2 \), the surfaces are cylindrical sheets or tubes. The assumption therefore says that at sufficiently low order, there are clear planar or axial pathways through the crystal which do not meet the close encounter region \( \mathcal{C}(1) \). We will not examine this assumption further, since a calculation of \( M^* \) requires specific knowledge about the location of lattice sites in the crystal which we do not assume here. We simply remark that the assumption is satisfied by any physical cubic crystal for reasonable values of \( \mathcal{E}_{\perp} \), which ultimately defines the size of \( \mathcal{C}(1) \).

Finally, we define the transverse energy of a trajectory with respect to a particular continuum potential by

\[
E_{\perp}(I, \theta) = \frac{1}{2} (I^*)^2 + \epsilon \Pi \mathcal{M} W(\theta^*),
\] (82)
where again $I^*$ and $\theta^*$ denote the projections of $I$ and $\theta$ onto $\text{span}\mathcal{M}$. We now state an abbreviated form of the

**Channeling Theorem.** Let $\mathcal{M}$ be a 1- or 2-dimensional submodule of $\mathbb{Z}^3$ with order $|\mathcal{M}| \leq M^*$. Fix suitable positive values for $E$ and $A$ (which determines the width of resonant zones), and assume the hypotheses of the Analytic Lemma which allows for a transformation of the Hamiltonian (7) to the resonant normal form (46). Fix the maximum initial (scaled) transverse energy $Q \geq Q'$ and the maximum change in (scaled) transverse energy $\delta > 0$ such that $Q + \delta \leq Q''$, where $Q' < Q'' < 1$ are defined in Assumption A; set $C = |\mathcal{M}|^{p+1}(2Q/3)^{1/2}$. Then there exists a sufficiently small $\epsilon > 0$ such that any initial condition $(I_0, \theta_0)$ for (7) with initial transverse energy

$$E_\perp(I_0, \theta_0) \leq \epsilon Q$$

(83)

and with suitable initial direction

$$I_0 \in \mathcal{Z}^A_{\mathcal{M}}(N, C, 1/2)$$

(84)

gives rise to a solution $(I, \theta)$ of (7) which satisfies the channeling criterion (7) on the exponentially long time interval $[0, T_0]$, where

$$T_0 = \frac{\sigma}{48E} e^{\tau} e^{-\tau/4} - 1.$$  

(85)

This solution is approximated by a “generalized continuum model” solution in the sense that $(I, \theta) = T(J, \phi)$, where $T$ is the near-identity transformation bringing (7) into the normal form (46) and $(J, \phi)$ is the solution to (46) with initial condition $(J_0, \phi_0) = T^{-1}(I_0, \theta_0)$. Furthermore, on the interval $[0, T_0]$, the longitudinal momentum $\widehat{I}(t)$ is nearly constant:

$$||\widehat{I}(t) - \widehat{I}_0||_\infty \leq \frac{3}{4} A \epsilon^{\epsilon},$$

(86)

as is the transverse energy:

$$\frac{1}{\epsilon} |E_\perp(I(t), \theta(t)) - E_\perp(I_0, \theta_0)| \leq \delta.$$  

(87)

A complete statement and a proof of this theorem can be found in Ref. 12. It should be stressed that the normal form (46) cannot be immediately applied to obtain the result; the theorem relies on the near-conservation of transverse energy and longitudinal momentum to ensure that trajectories beginning well inside the domain of a particular resonant normal form remain inside that domain. This phenomenon of “trapping into resonance” is a general characteristic of quasi-convex Hamiltonians (Hamiltonians whose unperturbed energy surfaces are convex) as was originally pointed out by Nekhoroshev and described in detail in Ref. 7.

A few comments should help clarify the meaning of the channeling theorem. First, the statement that the solutions $(I, \theta)$ discussed in the theorem “are approximated by solutions of a generalized continuum model” simply indicates that the difference between
$(I, \theta)$ and $(J, \phi)$ is uniformly small on $[0, T_0]$, while $(J, \phi)$ is a solution of (46), which on the same interval may be written

$$H'(J, \phi) = \frac{1}{2}J^2 + \epsilon \Pi_M W(\phi^*) + O(\epsilon^{1+\gamma})$$  \hspace{1cm} (88)$$

by virtue of (49) and (50). Dropping the $O(\epsilon^{1+\gamma})$ term from (88) and renaming its phase variables $(p, q)$, we recover the ordinary continuum model

$$H_c(p, q) = \frac{1}{2}p^2 + \epsilon \Pi_M W(q^*)$$  \hspace{1cm} (89)$$

which splits into the independent systems

$$H_\perp(p^*, q^*) = \frac{1}{2}(p^*)^2 + \epsilon \Pi_M W(q^*) \hspace{1cm} (90a)$$

$$H_\parallel(\dot{p}, \dot{q}) = \frac{1}{2}\dot{p}^2. \hspace{1cm} (90b)$$

If $\dim \mathcal{M} = 1$, $H_\perp$ is a 1 degree-of-freedom planar continuum Hamiltonian, the scaled version of (3); if $\dim \mathcal{M} = 2$ it is the scaled version of a 2 degree-of-freedom axial continuum Hamiltonian (2). This is the sense in which (46) is a generalized continuum model, and it is no surprise that the $O(\epsilon^{1+\gamma})$ remainder (comprising $[\epsilon^{-\gamma/4}]$ terms) is needed to obtain the uniform approximation on exponentially long time intervals. In general, the continuum solution $(p, q)$ can uniformly approximate the exact solution $(I, \theta)$ to (7) on time intervals of length at most $O(\epsilon^{-1/2})$ (cf. Ref. 15).

A second point of interest is the set of admissible initial conditions

$$\left( \mathcal{Z}_\mathcal{M}^{A, \alpha}(N, C, 1/2) \times T^3 \right) \cap \mathcal{Q}, \hspace{1cm} (91a)$$

where

$$\mathcal{Q} = \{(I_0, \theta_0) \mid E_\perp(I_0, \theta_0) \leq Q\}. \hspace{1cm} (91b)$$

Without further knowledge about $W$, the set $\mathcal{Q}$ cannot be described, but it is not hard to see that the set of initial directions $I_0$ in $\mathcal{Q}$ is contained in the zone $\mathcal{Z}_\mathcal{M}(C, 1/2)$ which in turn contains the block $\mathcal{Z}_\mathcal{M}^{A, \alpha}(N, C, 1/2)$. This block consists of the zone $\mathcal{Z}_\mathcal{M}(C, 1/2)$ with other low-order zones removed; physically these removed zones correspond to low order axes. The geometry of these zones and blocks is discussed in detail in Ref. 12; there it is shown that the volume of a block $\mathcal{Z}_\mathcal{M}^{A, \alpha}(N, C, 1/2)$ approaches the volume of the zone $\mathcal{Z}_\mathcal{M}(C, 1/2)$ on which it is based with decreasing $\epsilon$. In fact, in the axial case, $\epsilon$ may be made small enough to ensure that the block coincides with its zone, in other words the axes may be prevented from overlapping by taking $\epsilon$ sufficiently small.

Finally, there is the related question of how the blocks fit together. Once the various constants defining the blocks are fixed in accordance with the requirements of the theorems, and $\epsilon$ is allowed to vary only within the restrictions permitted in those theorems, it is not hard to see that for sufficiently small $\epsilon$, axial blocks (which reduce to axial zones with decreasing $\epsilon$) are contained in the complement of planar blocks, which in turn are contained in the complement of the nonresonant block. We thus obtain three well-defined regions of phase space corresponding to distinct behavior of solutions.
of (7). However, because of the requirement \( c > \alpha \) in the Analytic Lemma (one of the “consistency relations” not stated above), for sufficiently small \( \varepsilon \) there is a subset of action space with relative volume \( O(\varepsilon^\alpha - \varepsilon^5) \) which is not contained in any of the blocks. Since the requirement \( c > \alpha \) appears unavoidable, this leaves open the question of what the motion is like for initial actions “between blocks,” where the interesting axial-planar transition takes place.

8. Conclusions

Clearly, the classical Hamiltonian model presented here establishes a general framework for energetic particle motion in crystals and should serve as the foundation for a more comprehensive mathematical theory. This approach includes the first mathematically rigorous results on planar channeling, and, by modifying the mathematical tools presented here, it should be possible to carry out calculations for planar channeling trajectories analogous to those performed for axial channeling in Ref. 15 (this possibility is discussed in Section 5 of Ref. 15). It should also be possible to incorporate relativistic effects and other crystal geometries, as we have done for axial channeling using averaging techniques. In addition to the spatial continuum motions described here, we hope to eventually investigate motions “between” channeling and spatial continuum motions (perhaps including the axial-planar transition region). In the perfect crystal model, it is important to include the effects of lattice vibrations and electron multiple scattering, and work is in progress along these lines using ideas from the modern theory of stochastic processes (e.g., random evolutions, probabilistic limit theorems based on strong mixing, and stochastic averaging techniques). Finally, while the above results and proposals are in the classical domain, it should be possible to extend many of them to the quantum case, as was partly done in Ref. 5.

As it stands, the Channeling Theorem of Section 7 corroborates the picture of perfect-crystal-model channeling which has emerged over the last two decades: solutions are approximated by continuum models on very long time intervals, and transverse energy and longitudinal momentum are nearly conserved. It comes perhaps as no surprise that these results are true, but the mathematical techniques used to deduce them are novel and surprising.

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References

10. R. Cogburn, J.A. Ellison, and J. Tapia, private communication
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