YOUNG MEASURES AND THE ABSENCE OF FINE MICROSTRUCTURES IN THE $\alpha - \beta$ QUARTS PHASE TRANSITION

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Young Measures and the Absence of Fine Microstructures in the $\alpha$-$\beta$ Quartz Phase Transition

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Recently Young measures have been used to describe, based in energy considerations, the fine structures appearing in crystals and related to phase transitions. See [3, 4, 7, 15] for further references and discussion of the model.

Experimental observations and predictions of the theory have been so far in remarkable agreement. One experimental observation is the absence of fine microstructures in the $\alpha$-$\beta$ quartz phase transition, see e.g. [13, 14] and their references. In particular it is the objective of this note to show that we can justify this when we extend the scope of the classical analysis as in [13, 14] from the consideration of sectionally affine deformations to possibly more complex structures represented by Young measure solutions of a variational problem. We conjecture the same observations should be expected in similar phase transitions provided that condition (1) below is satisfied.

We describe the body using a reference configuration $\Omega \subset \mathbb{R}^3$, and a continuous deformation map $y : \Omega \rightarrow \mathbb{R}^3$ verifying appropriate regularity assumptions and with $F = \nabla y$ constrained by $\det F > 0$. As usual the reference configuration is thought of as the higher temperature phase at the transition temperature. The material is assumed to be characterized by a bulk energy density function $W$, defined in the subset $\mathbb{M}^+$ of $\mathbb{M} \equiv M^{3 \times 3}$ formed by matrices with positive determinant, and satisfying the galilean invariance

$$W(RF) = W(F),$$

for all $R$ in the special orthogonal group $SO(3)$ and all $F \in \mathbb{M}^+$. This implies that minima of $W$ occur in potential wells, i.e., right cosets of $SO(3)$. For each potential well there is a unique $H$ symmetric positive definite such that the well can be written as $SO(3)H$. At the transition temperature $\Theta_0$ we consider $W$ to have minimum values in a set containing $SO(3) \cup SO(3)H$ where the transformation strain $H$ is symmetric positive definite and corresponds to the change of shape associated to the transition from the higher to lower temperature. In general due to loss of symmetry for temperatures $\Theta < \Theta_0$ the minima will also occur in $SO(3)HR_i$ for $R_i$, $i = 1, \ldots, k$, in the point group corresponding to the higher temperature phase. A

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1In this discussion leading to the assumption on the support of the Young measure solution we neglect surface energy densities which arise when discussing the formation of interfaces, multi-valued energy densities arising when discussing hysteresis near the transition, and assume the crystal to be unloaded.

$SO(n)$ being, as usual, the group of all rotations acting in $\mathbb{R}^n$, i.e., $\{R \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) : R^T R = R R^T = 1, \det R = 1\}$.

3We can always reduce the study of Young measures supported in two wells $SO(3)H_1 \cup SO(3)H_2$ to the case $SO(3) \cup SO(3)H$ by a linear change of variables.
lower dimensional diagram suggesting the evolution of the energy density function near the transition temperature is presented in figure 1.

We consider that the low temperature variants present in the fine structure only originate one potential well. There are cases when all low temperature variants originate only one potential well: the α-β quartz phase transition is one such case where we can take the low temperature well defined by a transformation strain of the form

\[
H = \begin{bmatrix}
1 - \epsilon & 0 & 0 \\
0 & 1 - \epsilon & 0 \\
0 & 0 & 1 - \delta
\end{bmatrix}
\]

with \(0 < \delta < \epsilon \ll 1\).

The equilibrium state of the material cannot, in general, be described by a minimizer of

\[
\int_\Omega W(\nabla y(x)) \, dx
\]

under appropriate boundary conditions, due to the physical unreasonableness of assuming \(W\) quasi-convex. Instead we consider a minimizing sequence of deformations and the Young measure defined by the associated sequence of gradients. It is natural to assume that near the transition temperature this Young measure should be supported in \(SO(3) \cup SO(3)H \cup SO(3)H_1 \cup \ldots \cup SO(3)H_k\), where \(SO(3)H_k = SO(3)HR_k\). Our results apply to the case when we further assume that the Young measure is supported in \(SO(3) \cup SO(3)H\).

We prove in Theorem 5.6 that, under rather general technical conditions on \(\Omega\) and the minimizing sequence \((y_j)_{j \in \mathbb{N}}\), and provided

\[
(1) \quad 3 - \text{tr} H - \text{tr} \text{adj} H + 3 \det H > 0,
\]

which is equivalent to

\[
(2) \quad \sum_{i=1}^{3} (1 - \mu_i)(1 - \det H / \mu_i) > 0
\]

where \(H = \sum_{i=1}^{3} \mu_i e_i \otimes e_i\) with the \(\mu_i\)'s positive and the \(e_i\)'s orthonormal, a Young measure supported in \(SO(3) \cup SO(3)H\) with \(H\) satisfying (1) must reduce to a spatially constant Dirac mass. Consequently coexistence of the two phases in a continuous crystal specimen undergoing a phase transition satisfying the previous assumptions should not be observed. This is accomplished using properties independent of the specific bulk energy density and only dependent of the properties of Young measures which generalize the classic Hadamard compatibility condition across an interface, namely, the weak continuity of minors.

It should be emphasized that the methods applied here do not yield results for instance for the austenite-martensite phase transition in InTl. In that case, not only low temperature variants generate extra potential wells, but also we have for the transformation strain

\[
H = \begin{bmatrix}
1 + 2\gamma & 0 & 0 \\
0 & 1 - \gamma & 0 \\
0 & 0 & 1 - \gamma
\end{bmatrix}
\]

with \(0 < \gamma \ll 1\) and consequently \(\sum_{i=1}^{3} (1 - \mu_i)(1 - \mu_{i+1}\mu_{i+1}) < 0\). If the Young measure is assumed to be homogeneous, meaning independent of \(x \in \Omega\), we are able

\[\text{Measured values of the parameters according to [5, 6] are } \epsilon = 0.0027, \delta = 0.0012.\]

\[\text{We use } \text{adj} \text{ for the classical adjugate of a square matrix, i.e., the continuous extension to } M \text{ of the map } A \mapsto (\det A)A^{-T}.\]
Figure 1: Schematic bulk energy density phase transition diagram. For $\alpha$-$\beta$ quartz $H \equiv H_1$. 

\[ W \]
to reach in Propositions 3.7 and 3.8 the same conclusion as in Theorem 5.6, which will hold if \( i \) for some \( i \)

\[(3) \quad (1 - \mu_i)(1 - \mu_{i+1}\mu_{i-1}) > 0.\]

The class of \( H \)’s satisfying (3) is strictly larger than the one corresponding to (1)\(^7\). Nevertheless (3) still does not cover InTl.

1 Young measures

Young measures are a useful tool to record oscillations. The following two results give the necessary technical information in the present context. In the first \( C_0(\mathbb{R}^n) \) denotes the space of continuous functions with limit 0 at \( \infty \).

**Theorem 1.1 (Young, Tartar, Ball, …)** Let \( \Omega \subset \mathbb{R}^m \) be Lebesgue measurable, \( K \subset \mathbb{R}^n \) be closed, and \( z_j : \Omega \to \mathbb{R}^n \), for \( j \in \mathbb{N} \), be a sequence of Lebesgue measurable functions satisfying \( z_j(x) \in K \) for a.e. \( x \in \Omega \) and all \( j \in \mathbb{N} \). Then there exists a subsequence \( (z_{j'})_{j' \in \mathbb{N}} \) of \( (z_j)_{j \in \mathbb{N}} \) and a family of non-negative measures on \( \mathbb{R}^n \), depending measurably on \( x \in \Omega \), such that

i) \( \|\nu_x\| = \int_K d|\nu_x| \leq 1 \) for all \( x \in \Omega \),

ii) \( \text{supp} \nu_x \subset K \) for all \( x \in \Omega \),

iii) \( f(z_j) \rightharpoonup (\nu_x, f) = \int_{\mathbb{R}^n} f(\lambda) d\nu_x(\lambda) \) in \( L^\infty(\Omega) \) for every \( f \in C_0(\mathbb{R}^n) \).

**Proposition 1.2 (Schonbek[22])** Let \( 1 < p < \infty \), \( 1 < r \). If:

i) \( (\tilde{z_j})_{j \in \mathbb{N}} \) is bounded in \( L^p(\Omega) \),

ii) \( \varphi \in C(K) \) satisfies

\[
\lim_{\lambda \to 0} \frac{|\varphi(\lambda)|^r}{|\lambda|^p} = 0,
\]

iii) \( |\Omega| < \infty \),

then \( \varphi \in L^1(\nu_x) \) for a.e. \( x \in \Omega \) and \( \varphi(\tilde{z}_j) \rightharpoonup \int_K \varphi(\lambda) d\nu_x(\lambda) \) in \( L^r(\Omega) \).

For further details and references on Young measures the reader is referred to [2].

2 The two well problem

Let \( \nu \) be a probability\(^6\) Young measure supported in \( SO(n) \cup SO(n)H \), which without loss of generality can be assumed to have the form

\[
(4) \quad \nu = (1 - \lambda(x))\rho_1(x) \otimes \delta_1 + \lambda(x)\rho_2(x) \otimes \delta_H
\]

where the \( \rho_i \)'s are probability densities in \( SO(n) \), \( 1 \) denotes the identity matrix, \( H \) is some \( n \times n \) matrix with \( \det H > 0 \), \( \lambda : \Omega \to [0,1] \) is a measurable function, and \( \delta_A \) denotes a Dirac mass at \( A \).

\(^{6}\)Here and below summation in the indices will be understood mod \( n \). In this case \( n = 3 \).

\(^{7}\)It is easily checked that the converse is not true. Take \( \mu_1 = 1 + \epsilon, \mu_2 = 1 - \epsilon, \epsilon > 0 \) and let \( \mu_3 \equiv 1 \).

\(^{6}\)Meaning \( \|\nu_x\| = \int_K d|\nu_x| = 1 \) a.e. in \( \Omega \). A necessary and sufficient condition for a Young measure to be a probability Young measure can be found in [19].
We assume that $\nu$ was defined by a sequence of gradients $(\nabla y_j)_{j \in N}$ associated to a sequence of deformations $(y_j)_{j \in N}$ with $y_j : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $\Omega$ is bounded and open. We assume that $(y_j)_{j \in N}$ is bounded in $W^{1,p}(\Omega)$, $p > n$. It is an open problem verifying or refuting:

**Conjecture 2.1** Either the wells $SO(n)$ and $SO(n)H$ have a rank-1 connection, i.e.,

$$\text{rank}(RH - 1) \leq 1$$

for some $R \in SO(n)$ or $\lambda = 0$ a.e. in $\Omega$ or $\lambda = 1$ a.e. in $\Omega$.

Very little is known about this problem which was posed by Kinderlehrer [16]. It is known that a general Young measure can be supported in a non-trivial set with no rank-1 connections [1] thus refuting an old conjecture of Tartar [24]. So far all the known cases of the conjecture have been established using the weak continuity of quasi-affine functions and the relatively simple structure of their restrictions to $SO(n)$. Basically this is the reason that makes plausible that the conjecture will hold in general.

### 3 The homogeneous case

We start by considering the homogeneous case of the two well problem. Considerably more is known for this case where we assume $\lambda$, $\rho_1$ and $\rho_2$ to be a.e. constant, and define

$$P_k = \int_{SO(n)} Q d\rho_k(Q)$$

for $k = 1,2$ and the underlying deformation of $\nu$,

$$F = (1 - \lambda)P_1 + \lambda P_2 H.$$

Several particular cases of the conjecture have been checked in the homogeneous case. In particular it has been proved when $n = 2$ [17] and also when $H$ is a multiple of the identity$^9$ [1].

The arguments below simply use the restrictions on a Young measure implied by the weak continuity of quasi-affine functions. Specifically we recall that [21, Theorem 4] and the Sobolev imbedding theorem imply:

**Theorem 3.1** Let $m < p \leq \infty$, $\Omega \subset \mathbb{R}^n$ open and bounded. If $M$ denotes an $m \times m$ minor of $n \times n$ matrices and $y_j \rightharpoonup y$ in $W^{1,p}(\Omega)$ (or $y_j \rightarrow y$ in $W^{1,\infty}(\Omega)$), then $M(\nabla y_j) \rightarrow M(\nabla y)$ in $L^1(\Omega)$.

Note that no further restrictions are implied by the weak lower semicontinuity of polyconvex functions. We can speculate that further restrictions are implied by the existence of quasi-convex functions which are not polyconvex. Known examples are so far restricted to quadratic rank-1 convex functions which are not polyconvex [23, 25]. Unfortunately it is far from obvious how to use such examples to obtain further results.

Note also that the same methods yield information for homogeneous Young measures supported in two rank-1 connected wells. Essentially it can be shown, see [19], that we can reduce the analysis of such Young measures to the study of Young measures defined by sequences of 2-dimensional deformations$^{10}$. For analysis of the structure of Young measures in two dimensions the reader is referred to [20].

$^9$ In fact a more general result is proved, namely, the same conclusion holds when the support of $\nu$ is the cone $(\alpha R : \alpha > 0, R \in SO(n))$.

$^{10}$ If $\det H = 1$ this can already be found in [4].

5
3.1 The convex hull of rotations

If $S$ is a subset of a vector space $E$ its convex hull will be denoted by $\text{conv } S$ and it is is formed by all finite convex combinations of elements of $S$, i.e., an element of $\text{conv } S$ has the form $\sum_{i=1}^{r} t_i x_i$ for some $r \in \mathbb{N}$, $t_i \in \mathbb{R}$, $t_i \geq 0$, $\sum_{i=1}^{r} t_i = 1$.

We will consider $E = M_{n \times n} \cong \mathbb{R}^{n^2}$ and $S = SO(n)$. Carathéodory’s well known result on the convex hull in finite dimensional vector spaces implies

**Proposition 3.2** $\text{conv}(SO(n))$ is closed and, given $P \in \text{conv}(SO(n))$, there are $r \in \mathbb{N}$, $r \leq n^2 + 1$ and $t_i \in \mathbb{R}$, $0 \leq t_i \leq 1$, $R_i \in SO(n)$ for $i = 1, \ldots, r$ such that

$$P = \sum_{i=1}^{r} t_i R_i.$$

This has the simple but important consequence that given a probability measure $\nu$ in $SO(n)$ there are $r \in \mathbb{N}$, $r \leq n^2 + 1$ and $t_i \in \mathbb{R}$, $0 \leq t_i \leq 1$, $R_i \in SO(n)$ for $i = 1, \ldots, r$ such that

$$\int_{SO(n)} Q d\nu = \sum_{i=1}^{r} t_i R_i.$$

Note that as an easy consequence of the definition convex combinations and products of elements of $\text{conv}(SO(n))$ are in $\text{conv}(SO(n))$.

The next proposition groups some simple facts about $\text{conv } SO(n)$.

**Proposition 3.3** Let $P \in \text{conv}(SO(n))$, where $n \geq 2$, and let $0 \leq \alpha \leq 1$. Then

i) $\alpha P \in \text{conv}(SO(n))$,

ii) $\text{adj } P \in \text{conv}(SO(n))$,

iii) $|\det P| \leq 1$.

**Proof.** We start by remarking that the polar decomposition theorem allows us just to consider in the proof the subset of $\text{conv } SO(n)$ formed by diagonal matrices.

We will prove (i) if we show that $0 \in \text{conv } SO(n)$.

For $n$ even we notice that $0 = \frac{1}{2}I + \frac{1}{2}(-1)$. For $n = 3$ we have

$$0 = \frac{1}{4}(1 + \text{diag}(-1, -1, 1) + \text{diag}(1, -1, -1) + \text{diag}(-1, 1, -1))$$

For $n$ odd, $n \geq 3$, we consider $\mathbb{R}^n = \mathbb{R}^3 \oplus \mathbb{R}^{n-3}$ and denote the matrices on the right-hand side of the previous equality by $1, R_i, i = 1, 2, 3$. Then

$$0 = \frac{1}{4}(1 + R_1 \oplus 1 + R_2 \oplus (-1) + R_3 \oplus (-1)).$$

To prove (ii) we can assume that

$$P = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n).$$

Then

$$\text{adj } P = P_1 P_2 \cdots P_{n-1}$$

where $P_i = \text{diag}(\mu_{i+1}, \mu_{i+2}, \ldots, \mu_{n+1})$. Let

$$T = \sum_{i=1}^{n} e_i \otimes e_{i+1}.$$
We obviously have \( T \in SO(n) \) and \( P_i = T_i^T P T_i^T \). Then (ii) follows as all matrices in the right hand side of (5) are in \( \text{conv}(SO(n)) \).

We claim that if \( P = \text{diag}(\mu_1, \mu_2, \ldots, \mu_n) \in \text{conv} \, SO(n) \) then \( |\mu_i| \leq 1 \) for all \( i \).

Indeed we would have

\[
\mu_i e_i = \sum_{i=1}^{n^2+1} \lambda_j R_j e_i
\]

where the \( R_j \)'s are in \( SO(n) \), \( 0 \leq \lambda_j \leq 1 \) for all \( j \) and \( \sum \lambda_j = 1 \). But then

\[
|\mu_i| \leq \sum_{i=1}^{n^2+1} \lambda_j |R_j e_i| \leq \sum_{i=1}^{n^2+1} \lambda_j = 1
\]

This implies \( |\text{det} \, P| \leq 1 \) proving (iii).

It is useful to know when the wells at 1 and at \( H \) are rank-1 connected in terms of the polar decomposition of \( H \), i.e., if \( \text{det} \, H > 0 \) we can write \( H = RS \) where \( R \in SO(n) \) and \( S = S^T \) and we would like to characterize rank-1 connectedness in terms of the eigenvalues of \( S \). The following result is a rewriting of [3, Proposition 4].

**Theorem 3.4** Given a symmetric matrix \( H \in M^{3 \times 3} \) a necessary and sufficient condition for the existence of a matrix \( R \in SO(3) \) and \( a, b \in \mathbb{R}^3 \) such that

\[
R - H = a \otimes b
\]

is that if the eigenvalues of \( H \) are denoted \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \) then \( \lambda_2 = 1 \) and \( \lambda_1 \geq 0 \).

### 3.2 Some known results

Kinderlehrer [17] verified the homogeneous case of the conjecture for dimension 2.

The following argument is a condensed version of his.

Define a function \( G : SO(2) \to \mathbb{R} \) by:

\[
G(R) = \det((1 - \lambda)1 + \lambda RH) - (1 - \lambda) - \lambda \det H
\]

\[
= \det((1 - \lambda)1 + \lambda RH) - (1 - \lambda)^2
+ (1 - \lambda)\lambda(1 + \det H) - \lambda^2 \det H
\]

\[
= \lambda(1 - \lambda)(\text{tr}(RH) - 1 - \det H)
= -\lambda(1 - \lambda) \det(1 - RH)
\]

The proof now reduces to show that \( G \) takes both positive and negative values.

To accomplish this, write, using the polar decomposition theorem and the spectral theorem, \( H = R' \sum_{i=1}^2 \mu_i e_i \otimes e_i \), for \( R' \in SO(2), \mu_i > 0 \) and orthonormal vectors \( e_i \).

Hence, if we choose \( R \) such that \( RR' \) is the 180 degree rotation, we get \( \text{tr}(RH) < 0 \) and, consequently \( G(R) < 0 \). On the other hand weak-\* continuity implies

\[
\det((1 - \lambda)P_i + \lambda P_2 H) = 1 - \lambda + \lambda \det H
\]

or equivalently

\[
(1 - \lambda)^2(1 - \det P_1) + \lambda(1 - \lambda)(1 + \det H - \text{tr}(\text{adj} P_1^T P_2 H)) + \lambda^2(1 - \det P_2) = 0.
\]

As \( \det P_i \leq 1 \) we must have

\[
1 + \det H - \text{tr}(\text{adj} P_1^T P_2 H) \leq 0.
\]

Also

\[
\text{tr}(\text{adj} P_1^T P_2 H) \leq \mu_1 + \mu_2.
\]

Together the last two inequalities imply that for \( R \) such that \( RR' = 1 \) we have \( G(R) \geq 0 \). This completes the proof of the conjecture in dimension 2, i.e.,
Theorem 3.5 A Young measure of the form (4) with \( n = 2 \) must satisfy one of the following:

i) \( \nu = \delta_{R'}, \) for some rotation \( R' \);

ii) \( \nu = \delta_{Q'\mathbf{H}}, \) for some rotation \( Q' \);

iii) \( \text{rank}(R_1 - R_2\mathbf{H}) \leq 1 \) for some rotations \( R_1, R_2. \)

We will also need

Theorem 3.6 A homogeneous Young measure supported in \( SO(n) \) is necessarily a Dirac mass.

Proof. See [1, 18].

### 3.3 Some new results

The trouble with trying a straightforward generalization of the argument leading to Theorem 3.5 to higher dimensions is that dealing with the determinant is replaced with dealing with the adjugate. Consequently searching for a zero of a scalar equation becomes searching for a zero of a system of equations, not allowing the use of simple intermediate value arguments. Nevertheless for a large class of matrices \( \mathbf{H} \) we can still draw some conclusions using a different approach.

Although primarily interested in results applying to \( SO(3) \) we describe most of the argument as applying to \( SO(n) \). At some point we will assume \( n = 3 \) to get slightly sharper results for that case.

First note that the following equalities are a consequence of weak-* continuity

\[
\begin{align*}
\det F &= 1 - \lambda + \lambda \det \mathbf{H} \\
\text{adj } F &= (1 - \lambda)P_1 + \lambda P_2 \text{ adj } \mathbf{H}.
\end{align*}
\]

This implies that

\[
F^{-1} = ((1 - \lambda)P_1 + \lambda P_2 \text{ adj } \mathbf{H})^T / (1 - \lambda + \lambda \det \mathbf{H})
\]

or equivalently

\[
(1 - \lambda + \lambda \det \mathbf{H}) \mathbf{1} = ((1 - \lambda)P_1 + \lambda P_2 \text{ adj } \mathbf{H})^T ((1 - \lambda)P_1 + \lambda P_2 \mathbf{H})
\]

\[
= ((1 - \lambda)P_1 + \lambda P_2 \mathbf{H})((1 - \lambda)P_1 + \lambda P_2 \text{ adj } \mathbf{H})^T
\]

which gives for any unit vector \( u \)

\[
1 - \lambda + \lambda \det \mathbf{H} = ((1 - \lambda)P_1 + \lambda P_2 \text{ adj } \mathbf{H})u \cdot ((1 - \lambda)P_1 + \lambda P_2 \mathbf{H})u.
\]

This can be rewritten as

\[
(1 - \lambda)^2 + \lambda(1 - \lambda)(1 + \det \mathbf{H}) + \lambda^2 \det \mathbf{H}
\]

\[
= (1 - \lambda)^2 |P_1 u|^2 + \lambda(1 - \lambda)(P_2 \text{ adj } \mathbf{H} + \mathbf{H})u \cdot P_1 u + \lambda^2 (P_2 \text{ adj } \mathbf{H} u \cdot P_2 \mathbf{H} u).
\]

Recall that the use of the polar decomposition theorem and a rotation of the coordinate axis allowed us to assume that \( \mathbf{H} \) is diagonal. Denote the eigenvalues of \( \mathbf{H} \) by \( \mu_i, \ i = 1, \ldots, n, \) with \( \mathbf{H}_{ii} = \mu_i. \) Denote the elements of the standard basis by \( e_i, \ i = 1, \ldots, n. \) Then choosing \( u = e_i \) we obtain

\[
(1 - \lambda)^2 + \lambda(1 - \lambda)(1 + \det \mathbf{H}) + \lambda^2 \det \mathbf{H}
\]

\[
= (1 - \lambda)^2 |P_1 e_i|^2 + \lambda(1 - \lambda)(\det \mathbf{H}/\mu_i + \mu_i)(P_2 e_i \cdot P_1 e_i) + \lambda^2 \det \mathbf{H}/|P_2 e_i|^2.
\]
Now we remark that as $P_k \in \text{conv}(SO(n))$ we have $|P_k e_i| \leq 1$. Also note that the inequality $1 + \det H \geq \det H/\mu_i + \mu_i$ is equivalent to

$$\left(1 - \mu_i\right)\left(1 - \det H/\mu_i\right) \geq 0.$$ 

Hence, if we assume (11) for some fixed $i$, comparing the two sides of (10) lets us conclude that either:

i) $\lambda = 0$, or

ii) $\lambda = 1$, or

iii) $\mu_i = 1$, or

iv) $\det H = \mu_i$.

If (i) holds then $\nu = \delta_{R'}$ for some $R' \in SO(n)$ using Theorem 3.6. Similarly if (ii) holds $\nu = \delta_{Q''H}$ for some $Q'' \in SO(n)$.

If (iii) or (iv) holds then also $|P_k e_i| = 1$, $k = 1, 2$. As $P_k \in \text{conv}(SO(n))$ then we may write for some $r_k \in \mathbb{N}$

$$P_k = \sum_{j=1}^{r_k} \alpha_{j,k} R_{j,k}, \quad \text{where } \alpha_{j,k} > 0, \sum \alpha_{j,k} = 1, R_{j,k} \in SO(n).$$

Then

$$\left| \sum_{j=1}^{r_k} \alpha_{j,k} R_{j,k} e_i \right| = 1.$$ 

These equalities can only hold if, for each $k$, the $R_{j,k} e_i$'s are colinear. As we must also have $P_1 e_i \cdot P_2 e_i = 1$ we see that there is $R \in SO(n)$ such that

$$RR_{j,k} e_i = e_i.$$ 

Let $\bar{P}_k$ and $H$ denote the $(i,i)$ minors of $RP_k$ and $H$ respectively.

Then, if (iii) holds, (6) becomes equivalent to

$$\det((1 - \lambda)\bar{P}_1 + \lambda\bar{P}_2 H) = 1 - \lambda + \lambda \det H.$$ 

Now, if $n = 3$, we know from the proof of the two dimensional result that (12) implies that there must be a rank-1 connection between the wells at $H$ and $1$. Consequently in case (iii) with $n = 3$ we established the existence of a rank-1 connection between the wells at $H$ and $1$, i.e., for some $S \in SO(3)$ we have $\text{rank}(SH - 1) \leq 1$.

To deal with case (iv) assume $n = 3$ and notice that (8) is now equivalent to

$$\left(1 - \lambda + \mu_i\right)1 = ((1 - \lambda)\bar{P}_1 + \lambda\mu_i \bar{P}_2 H)^T ((1 - \lambda)\bar{P}_1 + \lambda\bar{P}_2 H)$$

$$= ((1 - \lambda)\bar{P}_1 + \lambda\bar{P}_2 H)((1 - \lambda)\bar{P}_1 + \lambda \mu_i \bar{P}_2 H)^T.$$ 

In particular

$$\left(1 - \lambda + \mu_i\right)1 = \lambda(1 - \lambda)(\mu_i \bar{P}_1 \text{ adj } H \bar{P}_2^T + \bar{P}_1 \bar{P}_2 H \bar{P}_1^T) + \lambda^2 \mu_i \bar{P}_2 \bar{P}_2^T.$$ 

Now note that as $\bar{P}_k \in \text{conv}(SO(2))$ we can write

$$\bar{P}_k = \begin{bmatrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix}$$
and that we must have $\alpha_j^2 + \beta_k^2 \leq 1$. Applying both sides of (14) to $e_j$ and then taking a dot product with $e_i$, where $j \neq l$, and $j, l \neq i$ we get

$$0 = \mu_i \hat{P}_1^T e_j \cdot \text{adj } \hat{H} \hat{P}_1^T e_i + \hat{P}_2^T e_j \cdot \hat{H} \hat{P}_1^T e_i$$

Considering $(j, l) = (i - 1, i + 1)$ and $(j, l) = (i + 1, i - 1)$ we obtain the equalities

$$0 = \mu_i (\alpha_1\beta_2/\mu_{i+1} - \alpha_2\beta_1/\mu_{i+1}) + (\alpha_2\beta_1/\mu_{i+1} - \alpha_1\beta_2/\mu_{i+1})$$

$$0 = \mu_i (\alpha_3\beta_1/\mu_{i+1} - \alpha_1\beta_3/\mu_{i+1}) + (\alpha_1\beta_3/\mu_{i+1} - \alpha_3\beta_1/\mu_{i+1})$$

As all $\mu_j \neq 1$ we must have $\alpha_1\beta_2 = 0$ and $\alpha_3\beta_1 = 0$. Then consideration of all possible cases in equation (13) leads easily to the conclusion that (iv) cannot occur unless $\lambda = 0$ or $\lambda = 1$.

Hence we have proved

**Proposition 3.7** A Young measure of the form (4) and satisfying

$$(1 - \mu_i)(1 - \det H/\mu_i) > 0.$$ 

must satisfy one of the following:

1. $\nu = \delta_{R'}$, for some rotation $R'$;
2. $\nu = \delta_{Q, H}$, for some rotation $Q'$;

We have also shown that the preceding can be slightly improved if $n = 3$ to

**Proposition 3.8** A Young measure of the form (4) and satisfying (11) with $n = 3$ must satisfy one of the following:

1. $\nu = \delta_{R'}$, for some rotation $R'$;
2. $\nu = \delta_{Q, H}$, for some rotation $Q'$;
3. $\mu_i = 1$ for some $i$ and $\text{rank}(R_1 - R_2 H) \leq 1$ for some rotations $R_1, R_2$.

We also established in a roundabout way that, if the inequality in (11) is strict, there cannot be a rank 1 connection between the wells. This was already known as a consequence of Theorem 3.4 in $SO(3)$.

Also we remark that Proposition 3.7 does not yield the full result, Theorem 3.5, in $SO(2)$. This suggests that improvement of Proposition 3.7 is possible. One idea that may seem plausible is trying to use (6) noticing that, if $n = 3$, it can be rewritten in the form

$$0 = (1 - \lambda)^2(1 - \det P_1) + (1 - \lambda)^2 \lambda(2 + \det H - \text{tr(adj } P_1^T P_2 H))$$

$$+ (1 - \lambda)\lambda^2(1 + 2 \det H - \text{tr(adj } P_1^T \text{adj } P_2 \text{adj } H)) + \lambda^3 \det H(1 - \det P_2)$$

Hence if

$$2 + \det H - \text{tr(H)} \geq 0$$

$$1 + 2 \det H - \text{tr(adj H)} \geq 0$$

hold, we would get some conclusion similar to what we obtained in Proposition 3.8. Unfortunately nothing new was obtained as (16) and (17) imply

$$3 + 3 \det H - \text{tr(H)} \geq 0$$

which is easily seen to imply

$$(1 - \mu_i)(1 - \mu_{i+1}\mu_{i-1}) \geq 0$$
for some $i$. Note that, except for the inequality not being strict, (18) is just (1).

It is important to remark that weak$\ast$ continuity of minors can give more detailed information than what is provided by Proposition 3.8 but that it seems unlikely to the author that this method will establish the conjecture for a larger class of wells in the homogeneous case improving the preceding result. The following result somewhat substantiates this statement.

**Proposition 3.9** Let

$$H = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}$$

where $0 < \mu_1 < \mu_2 < \mu_3$ and rank$(Q - RH) > 1$ for all $Q, R \in SO(3)$. Then, for homogeneous Young measures of the form

$$(1 - \lambda)\delta_1 \otimes \delta_1 + \lambda \rho \otimes \delta_H$$

where $\rho$ is a probability density in $SO(3)$, and $0 < \lambda < 1$,

$$P \equiv \int_{SO(3)} Q \, d\rho(Q)$$

is a diagonal matrix.

**Proof.** Write $P = QS$ where $Q \in O(3)$ and $S$ is symmetric non-negative definite. From (8) we get

$$((1 - \lambda)(1 + \det H) + \lambda \det H)1 =$$

$$= (1 - \lambda)(PH + \text{adj } HP^T) + \lambda \det HP^TP$$

$$= (1 - \lambda)(PH + \text{adj } HP^T) + \lambda \text{adj } HP^TPH.$$  

(20)

The first two lines of (20) imply that $PH + \text{adj } HP^T$ is symmetric. Hence the first and third lines of (20) show that $\text{adj } HP^TPH$ is also symmetric. Thus

$$\text{adj } HP^TPH = HP^TP \text{adj } H,$$

implying $S^2H^2 = H^2S^2$. Using $|S^2e_i|^2 = \sum_{k=1}^{3} (S^2 e_i \cdot e_k)^2$ we establish then

$$\sum_{k=1}^{3} (\mu_k - \mu_i)(S^2 e_k \cdot e_i)^2 = 0$$

for every $i$. Considering $i = 1, 3$ we obtain that $S^2 e_i \cdot e_k = 0$ for $i \neq k$, i.e., there are $\alpha_i \in \mathbb{R}$, $i = 1, 2, 3$, such that

$$S^2 = P^TP = \sum_{i=1}^{3} \alpha_i e_i \otimes e_i.$$

Now note that from (20) we have

$$((1 - \lambda)(1 + \det H) + \lambda \det H)e_i =$$

$$= (1 - \lambda)(PH + \text{adj } HP^T)e_i + \lambda \text{adj } HP^TPHe_i$$

$$= (1 - \lambda)(PH + \text{adj } HP^T)e_i + \lambda \det H \alpha_i e_i.$$  

Consequently, we get if $k \neq i$

$$\mu_i Pe_i \cdot e_k + \mu_{k+1} \mu_{k-1} (Pe_k \cdot e_i) = 0.$$
In particular
\[ \mu_1 P e_1 \cdot e_2 + \mu_3 \mu_3 P e_2 \cdot e_1 = 0 \]
\[ \mu_2 P e_2 \cdot e_1 + \mu_2 \mu_3 P e_1 \cdot e_2 = 0 \]

implying
\[ P e_2 \cdot e_1 - \mu_3^2 P e_2 \cdot e_1 = 0. \]

If \( \mu_3 = 1 \) the existence of such a Young measure would be excluded by using Proposition 3.8. This shows that \( P e_2 \cdot e_1 = 0 \) and \( P e_1 \cdot e_2 = 0 \). Similarly excluding \( \mu_1 = 1 \), we have \( P e_2 \cdot e_3 = 0 \) and \( P e_3 \cdot e_2 = 0 \). To exclude \( \mu_2 = 1 \) we use the fact that this would imply the wells to be rank-1 connected according to Theorem 3.4. We conclude that \( P e_k \cdot e_i = 0 \) for \( i \neq k \), i.e., \( P \) is diagonal in the basis \( \{ e_i \} \).

Note that, when the Young measure has the form (19), Proposition 3.9 allows reducing system (6, 7) to a non-linear system of 4 equations in 4 unknowns depending on 3 parameters. Further manipulation allows reducing that system to a single equation. We leave to the reader the task of verifying that this seemingly easier problem is still quite challenging.

4 An elementary construction

We will need solving the following elementary problem in order to tackle our main result.

**Problem 4.1** Given \( H \in M^+ \) determine if there exists a function \( F : M \to \mathbb{R} \), uniformly convex, smooth, verifying for some \( C > 0 \) and all \( P \in M \)

\[ DF = D(\det), \quad \text{in } SO(n) \cup SO(n)H. \]
\[ |DF(P)| \leq C(1 + |P|), \]
\[ |D^2F(P)| \leq C. \]

We claim that

**Lemma 4.2** The answer to Problem 4.1 is affirmative if and only if

\[ \sum_{i=1}^{n} (1 - \mu_i)(1 - \det H/\mu_i) > 0. \]

This lemma will allow us to use results from the regularity theory of elliptic systems that together with the result from the homogeneous case will establish the main result, Theorem 5.6.

Of course we will need to make precise what we mean by a uniformly convex, smooth function. It will mean that \( F \in C^\infty(M) \) and there is some constant \( c > 0 \) such that a uniform ellipticity condition holds, i.e.,

\[ B \cdot D^2F(A)B > c|B|^2 \quad \text{for all } A, B \in M. \]

Nevertheless the reader is warned that a more general concept of uniform convexity for not necessarily smooth functions will be used while solving parts of the problem (see Definition 4.8).

It should be remarked that it is trivial to exhibit functions \( F \) as in Problem 4.1 for \( H = \lambda I, \lambda \in \mathbb{R} \). We can take \( F(A) = n^{-n/2}|A|^n \) if \( n \geq 2 \) or \( F(A) = \frac{1}{2}|A|^2 \) as in [15].
4.1 Necessary conditions

Assume that $F : \mathbb{M} \to \mathbb{R}$ is smooth and satisfies (21). As det is constant when restricted to each of \( SO(n) \) or \( SO(n)H \) the directional derivatives of det tangential to \( SO(n) \) or \( SO(n)H \) are 0, and consequently the same holds for \( F \). Hence, when restricted to one of \( SO(n) \) or \( SO(n)H \), \( F \) is constant, i.e.,

**Proposition 4.3** Let $F : \mathbb{M} \to \mathbb{R}$, $F \in C^1(M)$ and satisfying (21). Then there are two constants $C_1, C_H$, such that

$$
F = C_1, \text{ in } SO(n) \\
F = C_H, \text{ in } SO(n)H.
$$

Now additionally assume that $F : \mathbb{M} \to \mathbb{R}$ is convex. Not all pairs of numbers can be considered for $C_1$ and $C_H$ as the hyperplanes tangent to the graph of $F$ must lie below its graph. As

$$
D(\text{det})(A)(B) = B \cdot \text{adj} \ A
$$

$F$ must satisfy for all $R, Q \in SO(n)$

$$
(24) \quad C_1 + (RH - Q) \cdot Q \leq C_H \\
(25) \quad C_H + (Q - RH) \cdot (R \text{adj} H) \leq C_1 \\
(26) \quad C_1 + (R - Q) \cdot Q \leq C_1 \\
(27) \quad C_H + (R - Q)H \cdot (Q \text{adj} H) \leq C_H.
$$

Note that if $\alpha_i, i = 1, \ldots, n$ denote positive numbers and $f_i, i = 1, \ldots, n$ denote orthonormal vectors

$$
\max_{R \in SO(n)} R \cdot \sum_{i=1}^{n} \alpha_i f_i \otimes f_i = \sum_{i=1}^{n} \alpha_i.
$$

Hence (24, 25) become

$$
C_1 + \text{tr } H - n \leq C_H, \\
C_H + \text{tr adj } H - n \text{ det } H \leq C_1,
$$

and (26, 27) are always satisfied. Hence, given $C_1$, we must satisfy

$$
C_1 + \text{tr } H - n \leq C_H \leq C_1 - \text{tr adj } H + n \text{ det } H.
$$

Hence we have proved

**Proposition 4.4** It is a necessary condition for the existence of $F : \mathbb{M} \to \mathbb{R}$, $F$ differentiable in $SO(n) \cup SO(n)H$ and convex, and such that (21) holds, that

$$
n \text{ det } H - \text{tr adj } H \geq n\text{ det } H - n \geq \text{tr } H - n
$$

which is equivalent to (1) if $n = 3$.

4.2 Convexity

Now we note that the same type of argument allows computing a function $G : \mathbb{M} \to \mathbb{R}$, convex in $\mathbb{M}$, smooth in a neighborhood of $SO(n) \cup SO(n)H$ and satisfying (21). In fact in this way one can obtain minorants of the eventual solutions of Problem 4.1 having the same values and derivatives in $SO(n) \cup SO(n)H$. 

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Proposition 4.5 Let $H$ satisfy (1), and $C_1$, $C_H$ satisfy (29). Then $G : \mathbb{M} \to \mathbb{R}$, defined by

$$G(A) = \max\{G_1, G_2\},$$
$$G_1(A) = \max_{R \in SO(n)} C_1 + (A - R) \cdot R$$
$$= C_1 - n + \max_{R \in SO(n)} A \cdot R$$
$$G_2(A) = \max_{R \in SO(n)} C_H + (A - RH) \cdot (R \text{adj } H)$$
$$= C_H - n \det H + \max_{R \in SO(n)} A \cdot (R \text{adj } H)$$

is convex and, if the inequality in (1) is strict, it is smooth in a neighborhood of $SO(n) \cup SO(n)H$ and satisfies (21). Also, if $\det A > 0$ we have

$$G(A) = \max \left\{ C_1 - n + \text{tr} \sqrt{AAT}, C_H - n \det H + \text{tr} \sqrt{A(\text{adj } H)^T A^T} \right\}.$$

Proof. Being a supremum of affine functions $G$ is indeed convex. We obtain (31) by considering (28) and the polar decomposition theorem. The statement about smoothness follows from inspecting (31) and noting that the set $\{A \in \mathbb{M} : G_1(A) = G_2(A)\}$ is bounded away from $SO(n) \cup SO(n)H$. Finally, the way $G$ was constructed shows that for a fixed $A \in SO(n) \cup SO(n)H$, the desired derivative is among the subgradients at $A$ (see Definition 4.6 below). Together with smoothness this shows (21). $
$

4.3 Uniform Convexity

Finally we note that it is still a simple affair to solve problem (4.1) if we drop the smoothness requirement and redefine uniform convexity in an adequate way. To achieve this we recall and introduce a few relatively general definitions. In these $D$ denotes a convex subset of a normed space $E$ and $f : D \subset E \to \mathbb{R}$ denotes a convex function.

Definition 4.6 $\phi \in \mathcal{L}(E, \mathbb{R})$ is a subgradient of $f$ at $x_0 \in D$ if for every $x \in D$

$$f(x_0) + \phi(x - x_0) \leq f(x).$$
$S(f, x_0)$ will denote the set of all subgradients of $f$ at $x_0$.

It is well known that if $S(f, x)$ has only one element then $f$ is differentiable at $x$.

**Definition 4.7** The uniform convexity constant of $f$ at $x_0$, $\epsilon_{x_0}(f)$, is defined by

$$
\epsilon_{x_0}(f) = \sup \left\{ c \geq 0 : \exists \phi \in S(f, x_0) \forall x \in D \cap B_r(x_0), f(x_0) + \phi(x - x_0) + c|x - x_0|^2 < f(x) \right\}.
$$

**Definition 4.8** A function $f : D \subset E \to \mathbb{R}$ is called uniformly convex if its convexity modulus, $\epsilon(f) = \inf_{x \in D} \epsilon_x(f)$, is positive.

**Proposition 4.9** For $f \in C^2(D)$ and $E = \mathbb{R}^d$ the previous definition of uniform convexity is equivalent to (23).

*Proof.* This follows easily using Taylor's formula.

Now to solve problem (4.1) under no requirements relative to smoothness we start by considering that from now on

$$
C_1 = 0,
$$

$$
C_H = \frac{1}{2}(\text{tr}(H - \text{adj} H) + n(\det H - 1)).
$$

The idea is to use convenient paraboloids instead of hyperplanes to define a solution. To do this let $\delta > 0$ be a small number to be chosen later and define

$$
\tilde{G}(A) = \max\{\tilde{G}_1(A), \tilde{G}_2(A)\}
$$

$$
\tilde{G}_1(A) = \max_{R \in SO(n)} \left\{ (A - R) \cdot R + \delta |A - R|^2 \right\}
$$

$$
\tilde{G}_2(A) = C_H + \max_{R \in SO(n)} \left\{ (A - R H) \cdot (R \text{adj} H) + \delta |A - R H|^2 \right\}.
$$

$\tilde{G}$ is convex. We must check the analogues of (24–27). These are

$$(32) \quad (R H - Q) \cdot Q + \delta |R H - Q|^2 \leq C_H$$

$$(33) \quad C_H + (Q - R H) \cdot R \text{adj} H + \delta |Q - R H|^2 \leq 0$$

$$(34) \quad (R - Q) \cdot Q + \delta |R - Q|^2 \leq 0$$

$$(35) \quad C_H + (R - Q) H \cdot (Q \text{adj} H) + \delta |R - Q H|^2 \leq C_H$$

for all $R, Q \in SO(n)$. Using a simple uniform continuity argument we see that the choice for $C_1, C_H$ guarantees that (32, 33) are indeed verified for small enough $\delta > 0$ and we can even assume that the inequalities are strict. As for (34, 35) these reduce to

$$(36) \quad (R \cdot Q - n)(1 - 2\delta) \leq 0$$

$$(37) \quad \det H(R \cdot Q - n) + 2\delta(|H|^2 - R H \cdot Q H) \leq 0$$

for all $R, Q \in SO(n)$. There is no doubt that (36) will be satisfied for small enough $\delta > 0$. As for (37) it is implied for small enough $\delta > 0$ by the following easy consequence of the spectral theorem. If $\lambda$ denotes the maximum of the absolute values of the eigenvalues of $H$, then for all $S \in SO(n)$

$$n - \text{tr} S \geq \frac{1}{\lambda^2}(|H|^2 - \text{tr}(S H H^T)).$$
This establishes that $\hat{G}$ will have the required values in $SO(n) \cup SO(n)H$ for small enough $\delta > 0$. Assume that such a $\delta > 0$ has been fixed. We claim that outside the set

\[ S = \{ A \in M : \hat{G}_1(A) = \hat{G}_2(A) \text{ or } \det A = 0 \} \]

$\hat{G}$ is a smooth function, which is equivalent to $\hat{G}_1$ and $\hat{G}_2$ being smooth in the complement of $\{ A \in M : \det A = 0 \}$ \(^{11}\). This can easily checked, from the explicit computation

\[
\hat{G}_1(A) = (1 - 2\delta) \text{tr} \sqrt{AA^T} - (1 - \delta)n + \delta |A|^2 \\
\hat{G}_2(A) = C_H + \text{tr} \sqrt{A(\text{adj} H - 2\delta H)^2 A^T} + \delta |A|^2 - n \det H + \delta |H|^2.
\]

Once again we can argue that (21) is satisfied. In the complement of $S$ the explicit form of $\hat{G}_1$ and $\hat{G}_2$ guarantees that $\hat{G}$ is uniformly convex. Nevertheless we will need

**Proposition 4.10** The supremum of a family of uniformly convex functions is uniformly convex.

**Proof.** The subgradient of one of the elements in the family will be a subgradient of the supremum . . .

The previous arguments establish that

**Proposition 4.11** For any $H \in M$ positive definite and satisfying (1), there is $\hat{G} : M \to \mathbb{R}$, uniformly convex, smooth in a neighborhood of $SO(n) \cup SO(n)H$ and satisfying (21).

### 4.4 Smoothing

Let $\varphi \in C^\infty(M)$ be such that $0 \leq \varphi \leq 1$, supp $\varphi$ is contained in the complement of a neighborhood of $SO(n) \cup SO(n)H$, $\varphi = 1$ in a neighborhood $N_S$ of $S$. Let $\psi \in C^\infty_c(M)$, $\psi \geq 0$, supp $\psi \subset B_1(0)$ and $\int_M \psi = 1$. As usual we define for $\epsilon > 0$ a family of mollifiers $(\psi_\epsilon)_{\epsilon > 0}$ by

\[
\psi_\epsilon(A) = \frac{1}{\epsilon^{n/2}} \psi \left( \frac{A}{\epsilon} \right) \quad \text{for all } A \in M.
\]

\(^{11}\)Note that this will prove that $\hat{G}$ is smooth in a neighborhood of $SO(n) \cup SO(n)H$.  

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Now, with $\tilde{G}$ the function in Proposition 4.11, consider $G_\epsilon = \psi_\epsilon \ast \tilde{G}$ where $\epsilon < \epsilon_0 \equiv \text{dist}(\supp D\varphi, S \cup SO(n) \cup SO(n)H)$. Define $H_\epsilon : M \to \mathbb{R}$ by

$$H_\epsilon(A) = (1 - \varphi(A))\tilde{G}(A) + \varphi(A)\tilde{G}_\epsilon(A)$$

We claim that, for sufficiently small $\epsilon > 0$, $H_\epsilon$ has the desired properties. $H_\epsilon$ is obviously $C^\infty$ in $M$. Now we establish the claim through several simple lemmas.

**Lemma 4.12** $H_\epsilon$ is uniformly convex in a neighborhood $N_\delta$ of $S$. In particular there is $c_1 > 0$ such that

$$B \cdot D^2 H_\epsilon(A)B > c_1 |B|^2$$

for all $A \in N_\delta, B \in M$.

**Proof.** The hypothesis on $\varphi$ guarantees that for $A$ in a neighborhood of $S, N_\delta$, we have

$$H_\epsilon(A) = \tilde{G}_\epsilon(A) = \int_M \psi_\epsilon(B)\tilde{G}(A - B) \, dB.$$

Hence, given $A_1 \in M$ with $|A - A_1|$ sufficiently small we have

$$H_\epsilon(A_1) - H_\epsilon(A) = \int_M \psi_\epsilon(B)(\tilde{G}(A_1 - B) - \tilde{G}(A - B)) \, dB$$

$$\geq \int_M \psi_\epsilon(B)D\tilde{G}(A - B)(A_1 - A) \, dB + c|A_1 - A|^2. \tag{38}$$

The integral on the right hand side of (38) is well defined as a convex function is differentiable a.e. and in particular $|S| = 0$. Smoothness of $H_\epsilon$ together with (38) easily implies that the derivative of $H_\epsilon, D H_\epsilon$, exists for every $A$ in $N_\delta$ and is given by

$$D H_\epsilon(A)(C) = \int_M \psi_\epsilon(B)D\tilde{G}(A - B) \, dB(C).$$

But then (38) is the statement of uniform convexity of $H_\epsilon$ in $N_\delta$. \hfill $\blacksquare$

**Lemma 4.13** For sufficiently small $\epsilon > 0$, $H_\epsilon$ is $C^\infty$, uniformly convex and verifies (21).

**Proof.** This follows from Lemma 4.12, and from the fact that we can write

$$H_\epsilon(A) = \tilde{G}_\epsilon(A) + \varphi(A)(\tilde{G}(A) - \tilde{G}_\epsilon(A))$$

and then use the uniform convergence of all derivatives of $\tilde{G}_\epsilon$ to the corresponding derivative of $\tilde{G}$ in $\supp D\varphi$ when $\epsilon \to 0$. \hfill $\blacksquare$
The previous lemma concludes the proof of Lemma 4.2 except for the estimates on $DF$ and $D^2F$.

### 4.5 Growth Estimates

It will be convenient for $F \equiv H$, as in lemma 4.13 to satisfy some additional properties. Namely we want to guarantee the existence of $C > 0$ such that

\[
|DF(A)| \leq C|A|
\]

\[
|D^2F(A)| \leq C.
\]

This can be achieved rather easily by changing the definition of $\tilde{G}$ in (32) as follows

\[
\tilde{G}(A) = \max\{\tilde{G}_1(A), \tilde{G}_2(A), \alpha|A|^2 + \beta\}
\]

where $\alpha$ and $\beta$ are chosen so that there are two balls $B_1, B_2 \subset \mathbb{M}$ such that $SO(n) \cup SO(n)H \subset B_1 \subset B_2$ and

\[
\tilde{G}(A) = \max\{\tilde{G}_1(A), \tilde{G}_2(A)\} \text{ in } B_1
\]

\[
\tilde{G}(A) = \alpha|A|^2 + \beta \text{ in } \mathbb{M} \setminus B_2.
\]

Then we may proceed as before with an appropriate change in the definition of $S$.

### 5 The inhomogeneous case

We follow the reasoning of James and Kinderlehrer in [15, §4] to establish what might be called a regularity result for Young measures. We will need to use results from the general regularity theory for elliptic systems (see e.g. [10]) instead of just dealing with Laplace’s equation. In particular we need to concern ourselves on how to control a possible singular set of the solution of our elliptic system\(^{12}\).

We first remark that results concerning homogeneous Young measures give information on inhomogeneous Young measures through the following localization theorem.

**Theorem 5.1 (Chipot and Kinderlehrer)** Let $(\nu_x)_{x \in \Omega}$ be a Young measure defined by a sequence of gradients $(\nabla y_j)_{j \in \mathbb{N}}$ where $(y_j)_{j \in \mathbb{N}} \subset W^{1,p}(\Omega), \Omega \subset \mathbb{R}^n$ an open set. Let $Q$ be a cube in $\mathbb{R}^n$. Then for a.e. $x_0 \in \Omega$ we have that $(\mu_x)_{x \in \Omega}$, where $\mu_x = \nu_{x_0}$ for each $x \in Q$ is defined by a sequence of gradients of functions in $W^{1,p}(Q)$.

**Proof.** See [7, part of Theorem 7.3]. $$
\]

Consider a Young measure $(\nu_x)_{x \in \Omega}$ defined by a sequence\(^{13}\) $(\nabla y_j)_{j \in \mathbb{N}}$ where $y_j \in W^{1,p}(\Omega, \mathbb{R}^n), \Omega \subset \mathbb{R}^n$ being open, bounded and connected. Assume that $\text{supp} \nu_x \subset SO(n) \cup SO(n)H$ for a.e. $x \in \Omega$, and $H$ satisfies (1). Assume $p > n$ so that Reshetnyak’s results on the weak continuity of minors hold. Then we see using Proposition 3.7 and Theorem 5.1, that a.e. in $\Omega$

\[
\nu_x = (1 - \chi_E(x))\delta_R(x) + \chi_E(x)\delta_Q(x)H
\]

where $\chi_E$ is the characteristic function of some measurable set $E$, and $Q : E \rightarrow SO(n)$, $R : \Omega \setminus E \rightarrow SO(n)$ are measurable functions. We claim that the weak

\(^{12}\)Such control seems unavailable if we try using the results on partial regularity of minimizers of strictly uniform quasiconvex integrands by assuming extra hypothesis on the energy density $W$ (see e.g. [12, 8, 9]).

\(^{13}\)Considering that a subsequence has already been extracted so that $y_j \rightharpoonup y$ and $\nabla y_j \rightharpoonup \nabla y$ (resp. $\nabla y_j \rightharpoonup \nabla y$) in $L^p(\Omega, \mathbb{R}^n)$ (resp. $L^\infty(\Omega, \mathbb{R}^n)$).
limit (resp. weak-* limit) $y$ of $(y_j)_{j \in \mathbb{N}}$ is a solution of the Euler-Lagrange system associated to the functional $\mathcal{F}: W^{1,p}(\Omega, \mathbb{R}^n) \to \mathbb{R}$, $p > n$, defined by

$$\mathcal{F}(u) = \int_{\Omega} F(\nabla u(x)) - \det(\nabla u(x)) \, dx$$

where $F$ is the function in Problem 4.1.

Indeed the Euler-Lagrange system associated to $\mathcal{F}$ is

$$\text{div} \left[ \frac{d}{dA}(F - \det)(\nabla u) \right] = 0. \quad (39)$$

Due to (21) $y$ is indeed a solution. As $\det$ is a null-Lagrangian, (39) is equivalent to

$$\text{div} \, F_A(\nabla u) = 0 \quad (40)$$

which, due to the uniform convexity of $F$ is an elliptic system. It is our objective to show that $\nabla y$ is continuous and consequently a result due to Reshetnyak can be applied to conclude that the only functions defined in $\Omega$ with values in $SO(n) \cup SO(n)\mathbb{H}$ are a.e. constants. This implies, together with the results about Young Measures just quoted, that either $\nu_2 = \delta_R$ a.e. in $\Omega$, or $\nu_2 = \delta_QH$ a.e. in $\Omega$ for some $R, Q \in SO(n)$, provided $p > n$.

To carry out the details we recall some results from the regularity theory of elliptic systems. The statement of these will involve the notation

$$v_{r,x} = \frac{1}{|B_r(x)|} \int_{B_r(x)} v(y) \, dy,$$

and $\mathcal{H}^k(A)$ for the $k$-dimensional Hausdorff measure of a set $A$.

**Theorem 5.2 (Giaquinta and Modica, [11])** Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of

$$\sum_{\alpha=1}^n D_\alpha A_\alpha^\alpha(\nabla u) = 0, \quad i = 1, \ldots, N. \quad (41)$$

where the coefficients satisfy for some positive numbers $\lambda, L$

i) $A_\alpha^\alpha \in C^1(M^{N \times n})$;

ii) $|A_\alpha^\alpha(p)| \leq L(1 + |p|)$;

iii) $|A_\alpha^\alpha(x, p)| \leq L$;

iv) the strong ellipticity condition $A_\alpha^\alpha(x, p)\zeta_\alpha \zeta_\beta^\top \geq \lambda |\zeta|^2$ holds for all $p, \zeta \in M^{N \times n}$.

Then the first order partial derivatives of $u$ are Hölder continuous in an open set $\Omega_0$, such that

$$\Omega \setminus \Omega_0 \subset \Sigma_1 \cup \Sigma_2 \quad (42)$$

where

$$\Sigma_1 = \left\{ x_0 \in \Omega : \lim_{r \to 0} \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |\nabla u(x) - (\nabla u)_{x,r}|^2 \, dx > 0 \right\} \quad (43)$$

$$\Sigma_2 = \left\{ x \in \Omega : \sup_{r > 0} (|u_{x,r}| + |(\nabla u)_{x,r}|) = +\infty \right\}. \quad (44)$$

If $n = 2$ we have $\Omega = \Omega_0$.\footnote{Proposition [21, Corollary of Lemma 3]. Let $\Omega \subset \mathbb{R}^n$ be an open connected set. If $f: \Omega \to \mathbb{R}^n$ is such that $f$ is continuous, $f \in W^{1,n}(\Omega, \mathbb{R}^n)$ and $Df \in O(n)$ for a.e. $x \in \Omega$, then $f(x) = Rz$ for a constant $R \in O(n)$. In particular the result holds if $f \in C^1(\Omega, \mathbb{R}^n)$.}
Proof. See [11] or [10, Chapter 6, Theorem 2.1 and Proposition 3.1].

Theorem 5.3 A solution $u$ of (41) satisfying the same hypothesis of Theorem 5.2 satisfies $u \in W^{2,2}_{\text{loc}}(\Omega, \mathbb{R}^N)$.

Proof. This a well-known consequence of the quotient method. See e.g. [10, Chapter 2, Theorem 1.1].

Theorem 5.4 Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in W^{1,q}_{\text{loc}}(\Omega)$, $q < n$. If

$$G = \{ x_0 \in \Omega : \lim_{\rho \to 0^+} u_{x_0^+} \text{ does not exist} \} \cup \{ x_0 \in \Omega : \lim_{\rho \to 0^+} |u_{x_0^+}| = +\infty \}$$

then for all $\epsilon > 0$

$$\mathcal{H}^{n-\epsilon}(G) = 0.$$

Proof. See [10, Chapter 4, Theorem 2.1].

Theorem 5.5 Let $\Omega \subset \mathbb{R}^n$ be open, $v \in L^1_{\text{loc}}(\Omega)$ and $0 \leq \alpha < n$. If

$$E_{\alpha} = \left\{ x \in \Omega : \lim_{\rho \to 0^+} \frac{1}{\rho^\alpha} \int_{B_\rho(x)} |v(y)| \, dy > 0 \right\}.$$

Then we have $\mathcal{H}^{\alpha}(E_{\alpha}) = 0$.

Proof. See [10, Chapter 4, Theorem 2.2].

Now we can conclude that

Theorem 5.6 Let $H$ denote some positive definite symmetric matrix satisfying (1) and consider a Young measure $(\nu_\varepsilon)_{x \in \Omega}$ defined by a sequence of gradients $(\nabla y_j)_{j \in \mathbb{N}}$ of maps $y_j : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, $y_j \in W^{1,p}(\Omega, \mathbb{R}^n)$ where $\Omega$ is open, bounded and connected, and supp $\nu_\varepsilon \subset SO(n) \cup SO(n)H$ a.e. in $\Omega$. Then if $p > n$ either

$$\nu_\varepsilon = \delta_R \text{ a.e. in } \Omega, \quad R \in SO(n) \text{ constant},$$

$$\nu_\varepsilon = \delta_{QH} \text{ a.e. in } \Omega, \quad Q \in SO(n) \text{ constant}.$$

Proof. Using Theorem 5.3 a solution $u$ of (40), where we assume $F$ altered as described in §4.5 so that if we take $A_\varepsilon^2 = \frac{\partial F}{\partial y}$ in Theorem 5.2 we can satisfy (ii) and (iii), lies in $W^{2,2}_{\text{loc}}(\Omega)$. On the other hand, according to Theorem 5.2, $u$ fails to have continuous partial derivatives at most in a set $\Omega \setminus \Omega_0$ characterized by (42-44). Note that as $p > n$ we always have $\sup_{x \in \Omega} |u_{x^+}| < \infty$. Then we can argue as in [10, pg. 190] that Theorems 5.3, 5.4, 5.5 tell us that $\mathcal{H}^{n-2+}(\Omega \setminus \Omega_0) = 0$ for all $\epsilon > 0$.

If $n > 2$ we have $\mathcal{H}^{n-1}(\Omega \setminus \Omega_0) = 0$. As sets with null $n - 1$ dimensional Hausdorff measure cannot separate an open set we conclude that $\Omega_0$ has only one connected component. For $n = 2$ we have $\Omega = \Omega_0$ from Theorem 5.2. The wells being disconnected this implies that either $\nabla y(x) \in SO(n)$ for a.e. $x \in \Omega$ or $\nabla y(x) \in SO(n)H$ for a.e. $x \in \Omega$. Then Reshetnyak’s result allows us to conclude the proof.

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13 Establishing $\Omega = \Omega_0$ in Theorem 5.2 involves the same estimate of the Hausdorff dimension as for $n \geq 3$ and a reverse Hölder inequality argument.
References


