

**SOLUTIONS TO EVOLUTION EQUATIONS WITH
NEAR-EQUILIBRIUM INITIAL VALUES**

By

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Abstract

We consider a general class of evolution equations and show that in a neighborhood of an equilibrium point the semiflow is qualitatively similar to the semiflow obtained when the equation is linearized about the equilibrium. In particular, we prove the existence of an invariant manifold which corresponds to the fastest growing solutions of the linearized equation, and show that most orbits beginning near the equilibrium point enter a small neighborhood of the union of the ω -limit sets of the points on this manifold. As an application, we rigorously justify a conjecture concerning spinodal decomposition for the Cahn-Hilliard model of phase separation.

1 Introduction

We consider evolution equations which can be represented in the form

$$(1) \quad u_t = Au + f(u), \quad u(0) = u_0, \quad t > 0$$

where

- u is an element of a Hilbert space X ;

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- A is a sectorial operator on X which generates an analytic semigroup $S(t)$;
- for some $\alpha \in [0, 1)$, f maps X^α into X , where $X^\alpha \equiv \mathcal{D}(A^\alpha)$ is a Hilbert space with an inner product equivalent to the graph norm (Both the inner product and the induced norm will be identified with a subscript α);
- $f(0) = 0$, $f : X^\alpha \rightarrow X$ is C^1 , and $Df(0) = 0$.

Furthermore, we are interested in those operators A which induce a splitting of X , $X = X^- \oplus X^+$ such that

- X^- and X^+ are invariant under A ;
- X^+ is finite-dimensional and $X^+ \subset \mathcal{D}(A)$;
- $\text{Re}(\sigma(A^-)) < a < b < \text{Re}(\sigma(A^+))$, where $A^- : X^- \cap \mathcal{D}(A) \rightarrow X^-$ and $A^+ : X^+ \rightarrow X^+$ are the restrictions of A to X^- and X^+ , respectively, and $\sigma(L)$ represents the spectrum of the linear operator L .

Extrapolating from the known behavior of ordinary differential equations suggests that there is an invariant manifold for (1) which is tangent to X^+ at 0 and which attracts, in some sense, the forward-time flow of nearby solutions. Similarly, there should be an invariant manifold tangent to X^- at 0. In the special case where $a < 0 < b$, these invariant manifolds are the unstable and stable manifolds whose existence is proved in [7]. In the special case when $\alpha = 0$, i.e., when f maps all of X into X , Bates and Jones [1] have an elegant proof for the existence of these *pseudo-unstable* and *pseudo-stable* manifolds for arbitrary a and b . This special case is also the subject of some powerful results recently obtained by Kening Lu [9].

Here we prove the existence of these invariant manifolds in a slightly more general context. We do this by constructing estimates similar to those in [1], making use of Gronwall-like inequalities for singular integrals found in [7], and then applying Hirsch, Pugh, and Shub's theory for invariant manifolds for *maps* [8]. These same estimates allow us to conclude, in a well-defined sense, that for arbitrarily long finite time, "most" solutions beginning near 0 are drawn along close to the pseudo-unstable manifold.

Finally, we apply this abstract result to the one-dimensional Cahn-Hilliard equation for phase separation [2]. It was the need to justify intuitive notions

of how spinodal decomposition should proceed in the Cahn-Hilliard equation that led to this work [4].

2 Growth Estimates near an Equilibrium Point

It will be useful to consider the problem obtained when the nonlinearity f in (1) is modified so that it is very well-behaved outside of a small neighborhood of 0. From our assumptions we have the following

Lemma 1 *Given $\varepsilon > 0$, there exists $\hat{f} : X^\alpha \rightarrow X$ such that \hat{f} is C^1 , is globally Lipschitz continuous with constant less than ε , and agrees with f in some neighborhood of 0.*

Proof. Let $\psi : \mathbf{R} \rightarrow [0, 1]$ be a C^∞ function which is 1 on $[-1, 1]$ and is 0 on $(-\infty, -2] \cup [2, \infty)$, and satisfies $|\psi'| \leq 2$ on \mathbf{R} . Choose $\delta > 0$ so small that on the ball of radius δ centered at 0, f has a Lipschitz constant less than $\varepsilon/17$. If we define $\hat{f}(u) = \psi\left(\frac{4\|u\|_\alpha^2}{\delta^2}\right) f(u)$ then $\hat{f}(u) = f(u)$ for $\|u\|_\alpha \leq \delta/2$ and $\hat{f}(u) = 0$ for $\|u\|_\alpha > \delta$. Also,

$$D\hat{f}(u)h = \frac{8}{\delta^2} \psi' \left(\frac{4\|u\|_\alpha^2}{\delta^2} \right) \langle u, h \rangle_\alpha f(u) + \psi \left(\frac{4\|u\|_\alpha^2}{\delta^2} \right) Df(u)h,$$

so

$$\begin{aligned} \|D\hat{f}(u)\|_{\mathcal{L}(X^\alpha, X)} &\leq \frac{8}{\delta^2} \left| \psi' \left(\frac{4\|u\|_\alpha^2}{\delta^2} \right) \right| \|u\|_\alpha \|f(u)\| \\ &\quad + \left| \psi \left(\frac{4\|u\|_\alpha^2}{\delta^2} \right) \right| \|Df(u)\|_{\mathcal{L}(X^\alpha, X)} \leq \varepsilon \end{aligned}$$

if $\|u\|_\alpha \leq \delta$. \square

The equation that results from modifying the nonlinearity is

$$(2) \quad u_t = Au + \hat{f}(u), \quad u(0) = u_0, \quad t > 0.$$

Unless we explicitly state otherwise, when we speak of solutions, semiorbits, etc., we are referring to (2). The letters v and w will represent elements of X^- and X^+ , respectively. Also, where the particular topology used is not specified, we are implicitly using the X^α topology.

Let $S^-(t)$ and $S^+(t)$ be the analytic semigroups generated by A^- and A^+ , respectively. From [7, Theorems 1.5.3-4] we have the following estimates on these semigroups:

$$(3) \quad \|S^+(t)w\|_\alpha \leq C_1 e^{bt} \|w\|_\alpha, \quad w \in X^+, \quad t \leq 0$$

$$(4) \quad \|S^+(t)w\|_\alpha \leq C_2 e^{bt} \|w\|, \quad w \in X^+, \quad t \leq 0$$

$$(5) \quad \|S^-(t)v\|_\alpha \leq C_3 e^{at} \|v\|_\alpha, \quad v \in X^- \cap X^\alpha, \quad t \geq 0$$

$$(6) \quad \|S^-(t)v\|_\alpha \leq C_4 e^{at} t^{-\alpha} \|v\|, \quad v \in X^-, \quad t \geq 0$$

Solutions of (2) satisfy the variation-of-constants formula (see, e.g., [6]),

$$(7) \quad u(t) = S(t)u_0 + \int_0^t S(t-s)\hat{f}(u(s))ds.$$

If we write $u = v + w$ with $v \in X^-$ and $w \in X^+$, then projecting onto the two subspaces gives the pair of integral equations

$$(8) \quad v(t) = S^-(t)v(0) + \int_0^t S^-(t-s)\hat{f}^-(v(s), w(s))ds \quad t \geq 0$$

$$(9) \quad w(t) = S^+(t)w(0) + \int_0^t S^+(t-s)\hat{f}^+(v(s), w(s))ds \quad t \geq 0,$$

where \hat{f}^- and \hat{f}^+ are the projections of \hat{f} onto X^- and X^+ , respectively. Note that $S^+(\tau)$ makes sense for negative τ since X^+ is finite-dimensional, so we can make a change of variable in (9) to get

$$(10) \quad \begin{aligned} w(t+\tau) &= S^+(\tau)w(t) \\ &+ \int_0^\tau S^+(\tau-s)\hat{f}^+(v(t+s), w(t+s))ds, \quad \tau \geq -t. \end{aligned}$$

Taking norms in (8) and (10) and using the estimates in (3) through (6) we have

$$(11) \quad \begin{aligned} \|v(t)\|_\alpha &\leq \|v(0)\|_\alpha C_3 e^{at} \\ &+ C_4 \varepsilon \int_0^t (\|v(s)\|_\alpha + \|w(s)\|_\alpha) \frac{e^{a(t-s)}}{(t-s)^\alpha} ds, \quad t \geq 0 \end{aligned}$$

$$(12) \quad \begin{aligned} \|w(t+\tau)\|_\alpha &\leq C_1 \|w(t)\|_\alpha e^{b\tau} \\ &+ C_2 \varepsilon \int_\tau^0 (\|v(t+s)\|_\alpha + \|w(t+s)\|_\alpha) e^{b(\tau-s)} ds, \quad 0 \geq \tau \geq -t \end{aligned}$$

Lemma 2 *Let $U : [-t, 0] \rightarrow [0, \infty)$ be continuous. Suppose that for some positive M and N ,*

$$U(\tau) \leq MU(0) + N \int_{\tau}^0 U(s) ds$$

for $-t \leq \tau \leq 0$. Then $U(0) \geq U(-t)e^{-Nt}/M$.

Proof. This is the standard Gronwall inequality in an unusual form. Let $B(\tau) = \int_{\tau}^0 U(s) ds$. Then $B'(\tau) = -U(\tau) \geq -MU(0) - NB(\tau)$. Adding $NB(\tau)$ to both sides, multiplying both sides by $e^{N\tau}$, and integrating from $-t$ to 0 gives $-B(-t)e^{-Nt} \geq MU(0)(e^{-Nt} - 1)/N$. Thus, $U(-t) \leq MU(0) + NB(-t) \leq MU(0)e^{Nt}$. Rearranging this, we have the claimed result. \square

Now suppose $\|w(s)\|_{\alpha} \geq \mu\|v(s)\|_{\alpha}$ for $0 \leq s \leq t$. Then we can eliminate v from (12) and apply Lemma 2 with $U(\tau) = \|w(t + \tau)\|_{\alpha} e^{-b\tau}$ to get

$$(13) \quad \|w(t)\|_{\alpha} \geq C_1^{-1} \|w(0)\|_{\alpha} \exp((b - C_2\varepsilon(1 + \mu^{-1}))t).$$

Suppose, on the other hand, that $\|w(s)\|_{\alpha} \leq \mu\|v(s)\|_{\alpha}$ for $0 \leq s \leq t$. Then we can eliminate w from (11) and apply the Gronwall-like inequality in [7, Lemma 7.1.1] to get

$$(14) \quad \|v(t)\|_{\alpha} \leq C_5 \|v(0)\|_{\alpha} \exp((a + (C_4\varepsilon(1 + \mu)\Gamma(1 - \alpha))^{\frac{1}{1-\alpha}})t).$$

These two estimates hold for the evolution with the modified nonlinearity \hat{f} , but note that they also hold in a neighborhood of 0 for the original nonlinearity.

3 Pseudo-stable and Pseudo-unstable Manifolds

According to [7, Theorem 3.4.4] the time- t map $T(t)$ induced by (2) is continuously differentiable and the derivative of this map at 0 is $S(t)$. Let $a < \gamma < b$. Then the estimates (3) and (5) imply that for t sufficiently large $S(t)$ is $e^{\gamma t}$ -pseudo hyperbolic, using the terminology of Hirsch, Pugh, and Shub [8]. Also, the canonical spectral decomposition of X^{α} corresponding to this pseudo hyperbolic endomorphism is compatible with the decomposition $X^{-} \oplus X^{+}$ of X .

Lemma 3 *For fixed t , the global Lipschitz constant of $T(t) - S(t)$ can be made arbitrarily small by making the Lipschitz constant of \hat{f} arbitrarily small.*

Proof. Because of (5) and (6) since X^+ is finite-dimensional, we know that there must exist estimates of the form

$$(15) \quad \|S(t)u\|_\alpha \leq C_6 e^{kt} t^{-\alpha} \|u\|$$

$$(16) \quad \|S(t)u\|_\alpha \leq C_7 e^{kt} \|u\|_\alpha.$$

Let $u_1(t)$ and $u_2(t)$ be two solutions of (2). Then from (7) we know

$$(17) \quad \begin{aligned} \|(T(t) - S(t))(u_1(0) - u_2(0))\|_\alpha &= \left\| \int_0^t S(t-s)(\hat{f}(u_1(s)) - \hat{f}(u_2(s))) ds \right\|_\alpha \\ &\leq C_6 \varepsilon \int_0^t \frac{e^{k(t-s)}}{(t-s)^\alpha} \|u_1(s) - u_2(s)\|_\alpha ds. \end{aligned}$$

Also by (7) we have

$$(18) \quad \begin{aligned} \|u_1(t) - u_2(t)\|_\alpha &\leq C_7 e^{kt} \|u_1(0) - u_2(0)\|_\alpha \\ &\quad + C_6 \varepsilon \int_0^t \frac{e^{k(t-s)}}{(t-s)^\alpha} \|u_1(s) - u_2(s)\|_\alpha ds. \end{aligned}$$

If we again apply Henry's Gronwall-like estimate [7, Theorem 7.1.1], this time to (18) with $\|u_1(t) - u_2(t)\|_\alpha e^{-kt}$ as $u(t)$, we find that (if $\varepsilon < 1$)

$$(19) \quad \|u_1(t) - u_2(t)\|_\alpha \leq C_8 \exp(C_{10}t) \|u_1(0) - u_2(0)\|_\alpha,$$

with C_8 independent of ε . Substituting (19) into (17) shows that there is C_9 depending on t but not on ε , $u_1(0)$, or $u_2(0)$, such that

$$\|(T(t) - S(t))(u_1(0) - u_2(0))\|_\alpha \leq C_9 \varepsilon \|u_1(0) - u_2(0)\|_\alpha.$$

This completes the proof of the lemma. \square

Lemma 3 completes the verification of all the hypotheses of Hirsch's Theorem 5.1 and Corollary 5.3 [8]. Using his results we know that if t is sufficiently large and if ε is sufficiently small the sets

$$\overline{W^+} \equiv \{u \in X^\alpha : T(t)^{-n}u \text{ exist and } \|T(t)^{-n}u\|_\alpha e^{\gamma n t} \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

and

$$\overline{W^-} \equiv \{u \in X^\alpha : \|T(t)^n u\|_\alpha e^{-\gamma n t} \rightarrow 0 \text{ as } n \rightarrow \infty\}$$

are, respectively, graphs of C^1 maps $X^+ \rightarrow X^- \cap X^\alpha$ and $X^- \cap X^\alpha \rightarrow X^+$ which are tangent to X^+ and $X^- \cap X^\alpha$ at 0.

These are the *pseudo-stable* and *pseudo-unstable manifolds* of the time- t map of (2) (corresponding to the particular splitting of our space under consideration). We can define similar sets for the semiflow of (2):

$$W^+ \equiv \{u \in X^\alpha : T(t)^{-1}u \text{ exists for all } t > 0, \|T(t)^{-1}u\|_\alpha e^{\gamma t} \rightarrow 0 \text{ as } t \rightarrow \infty\}$$

$$W^- \equiv \{u \in X^\alpha : \|T'(t)u\|_\alpha e^{-\gamma t} \rightarrow 0 \text{ as } t \rightarrow \infty\}.$$

Clearly $W^+ \subseteq \overline{W^+}$ and $W^- \subseteq \overline{W^-}$. By the proof of Lemma 3, there is a maximum factor by which the norm of a solution $\|u(t)\|_\alpha$ can grow *within* a fixed length of time. This implies that in fact $W^+ = \overline{W^+}$ and $W^- = \overline{W^-}$, so we have pseudo-stable and pseudo-unstable manifolds for the semiflow of (2). The intersection of these with a small neighborhood of 0 gives local versions of these invariant manifolds for (1).

4 Flow near the Pseudo-Unstable Manifold

From now on, we assume $b > 0$. Let $K_\lambda = \{v + w \in X^\alpha : \lambda \|v\|_\alpha \leq \|w\|_\alpha\}$.

Lemma 4 *Let $\mu > 0$ be given. If the Lipschitz constant ε of \hat{f} is sufficiently small then any semiorbit starting in $K_\mu \setminus \{0\}$ remains in $K_{\mu/(C_1 C_5)}$ for all positive time.*

Proof. Choose $\varepsilon > 0$ so small that $\gamma_0 \equiv (b - C_2 \varepsilon (1 + C_1 C_5 \mu^{-1}))$ and $\gamma_1 \equiv (a + (C_4 \varepsilon (1 + \mu) \Gamma(1 - \alpha))^{\frac{1}{1-\alpha}})$ satisfy $\gamma_0 > \gamma_1$ and $\gamma_0 > 0$. Let $v(t) + w(t)$ be a semiorbit starting at $v(0) + w(0) \in K_\mu \setminus \{0\}$. We claim it never reaches the lateral surface of $K_{\mu/(C_1 C_5)}$. Suppose our claim is false. Then without loss of generality we may assume $v(0) + w(0) \in \partial K_\mu$, $v(t) + w(t) \in \partial K_{\mu/(C_1 C_5)}$, and for $0 < \tau < t$, $v(\tau) + w(\tau) \in K_{\mu/(C_1 C_5)} \setminus K_\mu$. Using (13) and (14) we have

$$\frac{\|v(t)\|_\alpha}{\|w(t)\|_\alpha} \leq \frac{C_1 C_5 \|v(0)\|_\alpha e^{\gamma_1 t}}{\|w(0)\|_\alpha e^{\gamma_0 t}} = \frac{C_1 C_5}{\mu} e^{(\gamma_1 - \gamma_0)t} < \frac{C_1 C_5}{\mu}.$$

This means $v(t) + w(t) \notin \partial K_{\mu/(C_1 C_5)}$, contrary to assumption. This contradiction shows that our claim is, in fact, true, so our proof is complete. \square

Let K_λ^c represent the closure of the complement of K_λ . Considering the implications of the preceding lemma for backwards evolution, we have the following.

Lemma 5 *Let $\mu > 0$ be given. If the Lipschitz constant ε of \hat{f} is sufficiently small, then any backwards semiorbit starting in $K_{\mu/(C_1 C_5)}^c \setminus \{0\}$, remains in K_μ^c as long as it exists.*

The same method that gave us the estimates (13) and (14) on the growth and decay of a solution can be used to give similar estimates on the growth and decay of the *difference* of two solutions. Thus, we can center the cones described above at points other than the origin, and let the centers move with the flow, and the corresponding generalizations of Lemmas 4 and 5 hold.

The following theorem says roughly that a semiorbit starting at a random point near 0 will blowup as $t \rightarrow \infty$, and as it does so it will stay close to W^+ .

Theorem 1 *Let $\mu > 0$ be given. If the Lipschitz constant ε of \hat{f} is sufficiently small then the following holds: For any $\delta > 0$ and for $R > 0$ sufficiently small there exists $r > 0$ such that any semiorbit $v(t) + w(t)$ starting at $v(0) + w(0) \in K_\mu$ and satisfying $\|w(0)\|_\alpha \leq r$ will leave the cylinder $(X^- \cap X^\alpha) \times \{w \in X^+ : \|w\|_\alpha \leq R\}$. Furthermore, at the point on the semiorbit where $\|w(t)\|_\alpha = R$ the distance of the semiorbit from W^+ is less than δ .*

Proof. Define the truncated cones $K_\lambda(\rho) \equiv \{v + w \in K_\lambda : \|w\|_\alpha \leq \rho\}$ and $K_\lambda^c(\rho) \equiv \{v + w \in K_\lambda^c : \|v\|_\alpha \leq \rho\}$. Since W^+ is tangent to X^+ at 0, the intersection of W^+ with $K_\mu(R)$ avoids its lateral surface if we take R small enough. Take ε so small that Lemma 4 holds and that the exponents γ_0 and γ_1 defined in the proof of Lemma 4 satisfy $\gamma_0 > \gamma_1$ and $\gamma_0 > 0$.

Let $u_2(0) = v_2(0) + w_2(0)$ be a point on the base of $K_{\mu/(C_1 C_5)}(R)$, and let $u_1(0) = v_1(0) + w_1(0)$ be the point lying on W^+ such that $w_1(0) = w_2(0)$. The analogue of Lemma 5 for cones centered at $u_1(-t)$ tells us that as long as the backwards orbit $u_2(-t)$ exists and remains within $K_{\mu/(C_1 C_5)}(R)$, $u_2(-t) \in u_1(-t) + K_\mu^c$. Also, we have an estimate similar to (14):

$$\|v_2(0) - v_1(0)\|_\alpha \leq C_5 e^{\gamma_1 t} \|v_2(-t) - v_1(-t)\|_\alpha.$$

We claim that there is $t^* > 0$ dependent only on δ such that if $\|v_2(0) - v_1(0)\|_\alpha \geq \delta$ then $u_2(-t^*)$ either does not exist or lies outside of $K_{\mu/(C_1 C_5)}(R)$. To see this, note that as long as $u_2(-t) \in K_{\mu/(C_1 C_5)}(R)$,

$$\|v_2(-t) - v_1(-t)\|_\alpha \geq \frac{\delta}{C_5} e^{-\gamma_1 t}$$

so by the triangle inequality

$$\begin{aligned} \|v_2(-t)\|_\alpha &\geq \frac{\delta}{C_5} e^{-\gamma_1 t} - \|v_1(-t)\|_\alpha \\ &\geq \frac{\delta}{C_5} e^{-\gamma_1 t} - \mu^{-1} \|w_1(-t)\|_\alpha \\ &\geq \frac{\delta}{C_5} e^{-\gamma_1 t} - C_1 \mu^{-1} R e^{-\gamma_0 t}. \end{aligned}$$

On the other hand $\|w_2(-t)\|_\alpha \leq C_1 R e^{-\gamma_0 t}$, so in order to have $\|w_2(-t)\|_\alpha \geq \mu/(C_1 C_5) \|v_2(-t)\|_\alpha$ it is necessary that

$$C_1 R e^{-\gamma_0 t} \geq \frac{\delta \mu}{C_1 C_5^2} e^{-\gamma_1 t} - R e^{-\gamma_0 t} / C_5.$$

It is clear that for large enough t this last inequality does not hold, since $\gamma_0 > \gamma_1$. This verifies our claim.

Now we have that as long as $u_2(-t)$ exists and remains within $K_{\mu/(C_1 C_5)}(R)$ it is contained in

$$\left(\bigcup_{0 \leq t \leq t^*} (u_1(-t) + K_\mu^c) \right) \cap K_{\mu/(C_1 C_5)}(R).$$

It is not hard to see that there is a smaller cone $K_\mu(r)$ which is disjoint from this set. Furthermore, r can be chosen independent of $w_2(0)$. If a (forward) semiorbit starts in $K_\mu(r)$ it must exit $K_{\mu/(C_1 C_5)}(R)$ through its base. But by the way we chose $K_\mu(r)$ it cannot exit through the base more than δ units away from W^+ . This completes the proof. \square

Corollary 1 *Let Ω be a neighborhood of the union of the ω -limit sets (under (1)) of the points on the local pseudo-unstable manifold corresponding to (1), and let $\mu > 0$ be given. Then there exists $r > 0$ such that any semiorbit $v(t) + w(t)$ starting at $v(0) + w(0) \in K_\mu(r)$ enters Ω .*

Proof. Choose ε and R so small that Theorem 1 holds. Choose R possibly smaller so that f and \hat{f} agree on $K_\mu(R)$. Let S_R consist of those points $v + w \in X^+ \oplus X^-$ which lie on the local pseudo-unstable manifold and satisfy $\|w\|_\alpha = R$. Note that any semiorbit beginning on S_R enters Ω . By continuous dependence [13], some open subset V of $X^- \times S_R$ containing S_R has this same property. Since S_R is compact, there exists $\delta > 0$ such that $\{v \in X^- \cup X^\alpha : \|w\|_\alpha \leq \delta\} \times S_R$ is contained in V (see [10, Lemma 5.8]). Apply Theorem 1 and the corollary follows immediately. \square

Theorem 2 *Let Ω be as in Corollary 1. Then there exists $\hat{r} > 0$ and $F : \mathbf{R} \rightarrow \mathbf{R}$ satisfying $F(0) = F'(0) = 0$ such that if $\|v + w\|_\alpha \leq \hat{r}$ and $\|w\|_\alpha > F(\|v\|_\alpha)$ then the semiorbit of (1) beginning at $v + w$ enters Ω .*

Proof. Pick some $\lambda > 0$ and apply Corollary 1 with $\mu = \lambda$. Using the resulting value of r , let $\hat{r} = \min(r, r/\lambda)$. Define F as follows: F is even and for $x \geq 0$, $F(x)$ is the supremum of all $y \leq \hat{r}$ such that there exists $v + w \in X^- \oplus X^+$ satisfying $\|v\|_\alpha = x$, $\|w\|_\alpha = y$, and such that the semiorbit under (1) beginning at $v + w$ never enters Ω . Then by definition F has all the desired properties except possibly $F(0) = F'(0) = 0$. We verify that property now. Clearly, the graph of F lies below the graph of $x \mapsto \lambda|x|$ on $(-\hat{r}, \hat{r})$. Now let $\mu > 0$ be any number smaller than λ . Applying Corollary 1 again produces a “flatter” truncated cone $K_\mu(\tilde{r})$. The base of $K_\mu(\tilde{r})$ hits the lateral surface of $K_\lambda(\hat{r})$ at some positive distance $d(\mu)$ away from the X^+ axis. Then it is clear that on $(-d(\mu), d(\mu))$ the graph of F lies below the graph of $x \mapsto \mu|x|$. Since this holds for all $\mu \in (0, \lambda)$, it must be true that $F(0) = F'(0) = 0$. \square .

5 Application to the Cahn-Hilliard Equation

The standard formulation of the one-dimensional Cahn-Hilliard equation is

$$(20) \quad \begin{aligned} u_t &= (-\varepsilon^2 u_{xx} + \psi(u))_{xx} & x \in [0, 1] \\ u_x &= u_{xxx} = 0 & x = 0, 1 \\ u(x, 0) &= u_0(x) \end{aligned}$$

where ψ is a C^4 function that has three simple zeros at $\alpha, 0, \beta$, with $\alpha < 0 < \beta$, and two critical points, a local maximum between α and 0 and a local

minimum between 0 and β . In typical applications u measures the concentration of one component of a binary mixture. Because the boundary conditions are equivalent to no-flux conditions, the *total mass* $\equiv \int_0^1 u(x, t) dx$ is equal to the initial mass $M \equiv \int_0^1 u_0(x) dx$. If u_0 is constant and lies in the *spinodal region* between the two critical points of ψ then a linearization of (20) about u_0 shows it to be unstable to one or more modes, assuming ε is small. The results of numerical and physical experiments indicate that for most initial conditions near u_0 a fine-grained separation called *spinodal decomposition* occurs, during which u grows rapidly to have large amplitude and approximately the same oscillatory symmetry as the fastest-growing mode of the linearization. That this should occur is rather intuitive, and the abstract results we have developed in the preceding sections allow us to provide a rigorous justification.

Given M lying in the spinodal region, we restrict our attention to solutions with mass M . For convenience, we change variables $\hat{u} \equiv u - M$ and define $\varphi(\hat{u}) \equiv \psi(\hat{u} + M) - \psi(M)$ and $\beta \equiv -\varphi'(0) = -\psi'(M) > 0$. Then dropping the caret, the equation is

$$(21) \quad \begin{aligned} u_t &= \left(-\varepsilon^2 u_{xx} + \varphi(u) \right)_{xx} & x \in [0, 1] \\ u_x &= u_{xxx} = 0 & x = 0, 1 \\ u(x, 0) &= u_0(x) \end{aligned}$$

and we are now concerned with initial conditions near 0. We can rewrite (21) as

$$u_t = Au + [g(u)]_{xx},$$

where $Au \equiv -\varepsilon^2 u_{xxxx} - \beta u_{xx}$ and $g(u) \equiv \varphi(u) + \beta u$. Let X consist of those elements of $L^2[0, 1]$ which have mean value 0, and let $\mathcal{D}(A)$ be those elements of $H^4[0, 1] \cap X$ which satisfy Neumann boundary conditions. Then it is easy to see that A generates an analytic semigroup $S(t)$ since an explicit spectral representation can be obtained. If $u(x) = \sum_{n=0}^{\infty} S_n(t) \cos(n\pi x)$ is the solution of the linearized PDE $u_t = Au$ with initial condition $u(x, 0) = u_0(x) \equiv \sum_{n=0}^{\infty} S_n(0) \cos(n\pi x)$, then it is easy to see that the coefficients $S_n(t)$ satisfy $S'_n = \lambda_n S_n$, where $\lambda_n \equiv \beta(n\pi)^2 - \varepsilon^2(n\pi)^4$. Thus, $S_n(t) = S_n(0)e^{\lambda_n t}$. Also, it was proved in [5] that $X^{1/2}$ consists of those elements of $H^2[0, 1] \cap X$ which satisfy Neumann boundary conditions.

Note that $\sup_n \lambda_n = \lambda_{n_0}$ for some n_0 . Generically n_0 is unique; split this n_0 -th mode off from the rest of the space as X^+ . This splitting satisfies all

of our hypotheses.

We need to verify that the map

$$u \mapsto [g(u)]_{xx}$$

is a C^1 map from $X^{1/2}$ to X whose derivative at 0 is 0. First note that $[g(u)]_{xx}$ has mean value 0 if u satisfies Neumann boundary conditions, so all we must show is that this map, call it f , is C^1 from H^2 to L^2 and its derivative at 0 is 0.

If $u \in H^2$ then

$$\|g(u)_{xx}\|_{L^2} \leq \|g''(u)\|_{L^\infty} \|u_x^2\|_{L^2} + \|g'(u)\|_{L^\infty} \|u\|_{H^2} < \infty$$

by standard Sobolev embedding theorems, so f has the proper domain and range. We claim that $Df(u)h = [g'(u)h]_{xx}$ for $h \in H^2$. A simple calculation yields

$$\begin{aligned} & \|g(u+h)_{xx} - g(u)_{xx} - [g'(u)h]_{xx}\|_{L^2} \leq \\ & \quad \|(g'(u+h) - g'(u))h_{xx}\|_{L^2} \\ & \quad + \|2(g''(u+h) - g''(u))u_x h_x\|_{L^2} + \|g''(u+h)h_x^2\|_{L^2} \\ & \quad + \|(g'(u+h) - g'(u))u_{xx} - g''(u)u_{xx}h\|_{L^2} \\ & \quad + \|(g''(u+h) - g''(u))u_x^2 - g'''(u)u_x^2 h\|_{L^2} \\ & \leq \|g''(\theta)\|_{L^\infty} \|h\|_{L^\infty} \|h\|_{H^2} + 2\|g'''(\theta)\|_{L^\infty} \|u\|_{H^2} \|h\|_{L^\infty} \|h\|_{H^2} \\ & \quad + \|g''(\theta)\|_{L^\infty} \|h_x\|_{L^\infty} \|h\|_{H^2} + \|g'''(\theta)\|_{L^\infty} \|u\|_{H^2} \|h\|_{L^\infty} \\ & \quad + \|g''''(\theta)\|_{L^\infty} \|u_x\|_{L^\infty}^2 \|h\|_{L^\infty} \|h\|_{H^2}, \end{aligned}$$

where θ represents a generic function between u and $u+h$. Again, standard Sobolev embedding theorems tell us the quantity above is $O(\|h\|_{H^2}^2)$, so our claim holds. Since $g(0) = g'(0) = 0$, we see that $f(0) = Df(0) = 0$.

In [3] it is shown that the steady-state solutions of (21) with mean value 0 are periodic with minimal period of the form $2/n$, $n \in \mathbf{Z}^+$, and have axes of symmetry at $x = k/n$, $k = 0, \dots, n$. Let \mathcal{S} be the set of steady-state solutions that are $2/n_0$ -periodic, and let \mathcal{S}' be the subset of steady-state solutions with minimal period $2/n_0$.

We are now in a position to make use of the general theory developed in the previous sections to prove:

Theorem 3 *Let Ω be a neighborhood of \mathcal{S} . Then there exists $r > 0$ and $F : \mathbf{R} \rightarrow \mathbf{R}$ satisfying $F(0) = F'(0) = 0$ such that if $\|v + w\|_\alpha \leq r$ and $\|w\|_\alpha \geq F(\|v\|_\alpha)$ then the semiorbit of (21) beginning at $v + w$ enters Ω . If, in addition, φ is a cubic polynomial, $\sup(-\varphi') < 2\beta$, and ε^2/β is sufficiently small then the same result holds if Ω is taken to be a neighborhood of \mathcal{S}' alone.*

Proof. Applying the results of section 3 gives the existence of a local pseudo-unstable manifold for (21). Consider the closed subspace Y of X which has as a basis $\{\cos kn_0\pi x : k = 1, 2, \dots\}$. The uniqueness of solutions to (21) implies that the semiflow it generates preserves the translational and reflectional symmetries of the type described above. This means that Y is invariant under (21), so we can apply the results of section 3 to Y instead of X and get a local pseudo-unstable manifold contained in Y . Clearly these two manifolds must coincide near 0. The relevant union of ω -limit sets referred to in Corollary 1 must, therefore, also lie in Y . The ω -limit sets for (21) contain only steady-state solutions [13], and since they must lie in Y , they must have minimal periods of the form $2/(kn_0)$. In particular, they are contained in \mathcal{S} , so the first half of our theorem follows from an application of Theorem 2.

Now steady-state solutions of (21) satisfy $u'' = \hat{\varphi}(u) \equiv \varphi(u) + \sigma$ and $u'(0) = u'(1) = 0$ for some constant σ such that $\hat{\varphi}$ retains 3 zeros. Smoller and Wasserman [12] show that if φ is a cubic polynomial then all solutions of this boundary value problem have minimal period no smaller than $2\pi\varepsilon(-\hat{\varphi}'(u_\sigma))^{-1/2}$, where u_σ is the middle zero of $\hat{\varphi}$. By examining the dispersion relation of the linearized equation it is easy to see that as $\varepsilon \rightarrow 0$

$$\frac{2}{\varepsilon kn_0} \rightarrow \frac{2\sqrt{2}\pi}{k} < \frac{4\pi}{k\sqrt{-\hat{\varphi}'(u_\sigma)}},$$

so for ε sufficiently small, the steady-state solutions in the union of ω -limit sets must, in fact, have minimal period $2/n_0$. This means that the union of ω -limit sets from Corollary 1 is, in fact, contained in \mathcal{S}' . In light of this, the second half of our theorem also follows from Theorem 2. \square

In [11] Renate Schaaf extends Smoller and Wasserman's results to a more generalized class of nonlinearities which meet derivative conditions similar to those satisfied by cubic polynomials. For this class of nonlinearities, the result above is still valid. It is reasonable to conjecture that \mathcal{S} is a finite set,

but at present this has only been proven in certain special cases, (see, e.g., [14]).

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