THE RIEMANN PROBLEM FOR SYSTEMS
OF CONSERVATION LAWS OF MIXED TYPE

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0. Introduction. The purpose of this paper is to review some recent results on the Riemann problem for the $p$-system of conservation laws of mixed type:

\begin{align*}
(0.1a) & \quad u_t + p(w)_x = 0, \\
(0.1b) & \quad w_t - u_x = 0, \quad x \in \mathbb{R}, \ t > 0, \\
(0.1c) & \quad (u(0, x), w(0, x)) = \begin{cases} 
(u_-, w_-), & \text{if } x < 0, \\
(u_+, w_+), & \text{if } x > 0.
\end{cases}
\end{align*}

In (0.1a), we assume $p \in C^1(\mathbb{R})$ and

\begin{align*}
(0.1d) & \quad p'(w) < 0 \quad \text{if } w \not\in [\alpha, \beta], \\
& \quad p'(w) > 0 \quad \text{if } w \in (\alpha, \beta).
\end{align*}

Figure 1

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Equations (0.1a,b,d) consist of a system of conservation laws of hyperbolic-elliptic mixed type. Hyperbolic regions \( w < \alpha \) and \( (w > \beta) \) correspond, for van der Waals fluids, liquid and vapor phase region respectively. The Riemann problem (0.1) serves as a prototype model for the dynamics of phase transitions in van der Waals like materials.

It is well known that (0.1) generally admits many solutions and not everyone of them is physically relevant. Therefore, we need some admissibility criteria to single out the “physically correct” solutions, or better yet, to lead to the well posedness for Cauchy problems of the system (0.1a,b,d).

One admissibility criterion we can borrow from the theory of scalar conservation laws is that a solution of (0.1) is admissible if each shock of the solution can be connected by a travelling wave solution of the system

\[
\begin{align*}
  u_t + p(w)_x &= \varepsilon u_{xx}, \\
  w_t - u_x &= 0.
\end{align*}
\]

(0.2)

More precisely, a shock solution of (0.1a,b,d) \((u_1, w_1), (u_2, w_2)\) is admissible by viscosity travelling wave criterion if

\[
\begin{align*}
  \frac{dw}{d\zeta} &= -s^2(w(\zeta) - w_1) - p(w(\zeta)) + p(w_1), \\
  w(-\infty) &= w_1, \ w(+\infty) = w_2.
\end{align*}
\]

(0.3)

has a solution. System (0.2) arises from the theory of viscoelastic bar [cf. Dafermos [11], Pago [56]]. Slemrod [66] observes that this viscosity admissibility criterion is equivalent to the “chord condition” which states that

\[
\begin{align*}
  \frac{p(w) - p(w_1)}{w - w_1} &\geq \frac{p(w_2) - p(w_1)}{w_2 - w_1} \quad \text{if } s \geq 0, \quad \text{or} \\
  \frac{p(w) - p(w_1)}{w - w_1} &\leq \frac{p(w_2) - p(w_1)}{w_2 - w_1} \quad \text{if } s \leq 0
\end{align*}
\]

(0.4a)

for any \( w \) between \( w_1 \) and \( w_2 \), where

\[ s^2 = -\frac{p(w_2) - p(w_1)}{w_2 - w_1}. \]

Condition (0.4) is analogous to Oleinik’s Condition E for scalar conservation laws. Epitomizing the experiences accumulated from detailed studies of special systems, Liu [52, 53] proposed Liu’s criterion which can be viewed as a generalization of the chord condition. Liu’s criterion yields a satisfactory solution of the Riemann problem for strictly hyperbolic
systems when the waves are of moderate strength. Attempts for solving the Riemann problem for mixed type system (0.1) by using viscosity travelling wave criterion or variations of the chord conditions were carried out by James [46] and later by Shearer [61] as well as Hsiao [44, 45].

A satisfactory admissibility criterion for mixed type system (0.1a,b,d) should satisfy physical principles governing phenomena under study. For materials which are viscously dominated, e.g., viscoelastic fluids and solids exhibiting phase transitions, a reasonable model may be the viscosity admissibility criterion [56]. For van der Waals fluids, the interfacial energy and hence the capillarity which corresponds to it is not negligible. Therefore, the viscosity criterion or chord conditions which only count viscosity are inadequate for van der Waals fluids. For example, the chord condition and viscosity criterion fail to comply with the well known Maxwell equal area rule, which states that stationary shock connecting \( m \) and \( M \) is the only possible stationary phase boundary. In order to take capillarity into consideration, we see that we need more high gradient terms. Based on Korteweg's theory of capillarity [50], Slemrod [67] proposed the viscosity-capillarity criterion which states that a solution of (0.1) is admissible if it is an \( \epsilon \to 0+ \) limit of solutions of

\[
\begin{align*}
u_t + p(w)_x &= -\epsilon^2 Aw_{xx} + \epsilon u_{xx} \\
w_t - u_x &= 0
\end{align*}
\]

where \( A > 0 \) is constant. A localized version of this criterion is the viscosity-capillarity travelling wave criterion which says a shock solution \((u_1, w_1), (u_2, w_2)\) of (0.1a, b,d) is admissible if the boundary value for the travelling wave equations of (0.5)

\[
\begin{align*}A\frac{d^2 \hat{w}}{d\zeta^2} &= -s \frac{d \hat{w}}{d\zeta} - s^2 (\hat{w} - w_1) + p(\hat{w}) - p(w_1) \\
\hat{w}(-\infty) &= w_1, \quad \hat{w}(+\infty) = w_2, \quad \hat{w}'(\pm\infty) = 0
\end{align*}
\]

A solution of (0.1) is said admissible according to viscosity-capillarity travelling wave criterion if each shock in the solution is admissible by this criterion.

The viscosity-capillarity travelling wave criterion not only admits the Maxwell line as the only admissible stationary phase boundary but also presents shock splitting phenomena [40] observed in experiments [49,72-76]. For shocks with two sides in hyperbolic region \( w < \alpha \) (or \( w > \beta \)), the viscosity-capillarity travelling wave criterion admits classical compressive shocks.

There are also some investigation of the appropriateness of the entropy rate admissibility criterion, which was proposed by Dafermos [12], for the mixed type system (0.1.a,b) (cf. Hattori [41-43], Abeyaratne and Knowles [1] and Pence [57]). The entropy rate admissibility criterion, which dubs admissible those solutions that maximize the rate of entropy production, also admits the Maxwell line as an admissible shock solutions (cf [41]) and leads to the unique solution to the Riemann problem (cf [1] and [56]).
The program of this survey is follows: In §1, we shall derive the equations of motion of a viscous, heat conducting fluid possessing a Korteweg-van der Waals contribution to the stress [50]. We also derive the travelling wave equations corresponding to these equations. In §2, we review some results on the boundary value problem of these travelling wave equations. In §3, we shall first list some results on the existence of solutions of Riemann problem (0.1) admissible by the travelling wave criterion via the wave and shock curve construction approach. Then, we shall, in §3.0, §3.1 and §3.2, give details of the proof of the existence of solutions for (0.1) by the similarity viscosity approach, As a consequence of this, we get better results, in §3.3, on the existence of solutions satisfying the travelling wave criterion. In §3.4, we state results on the uniqueness and stability of the solution of (1.1) which is admissible according to the travelling wave criterion.

§1. The equations of motion. We consider the one dimensional motion of fluid processing a free energy

\begin{equation}
(1.1) \quad f(w, \theta) = f_0(w, \theta) + \frac{\epsilon^2 A}{2} \left( \frac{\partial w}{\partial x} \right)^2 .
\end{equation}

Here \( w \) is the specific volume, \( \theta \) the absolute temperature, \( A > 0 \) a constant, and \( x \) the Lagrangian coordinate. The term

\[
\frac{\epsilon^2 A}{2} \left( \frac{\partial w}{\partial x} \right)^2 ,
\]

where \( \epsilon > 0 \) is a small parameter, is the specific interfacial energy. The graph of \( f_0 \) as a function of \( w \) for fixed \( \theta \) will vary smoothly from a single well potential for \( \theta > \theta_{\text{crit}} \) to double well potential for \( \theta < \theta_{\text{crit}} \). The \( \theta_{\text{crit}} \) is called the critical temperature. Discussions of such free energy formulations may be found in [2-10, 20, 28-32, 59-60, 78]

The stress corresponding to the free energy (1.1) is given by

\begin{equation}
(1.2) \quad T = \frac{\partial f}{\partial w} = \frac{\partial f_0}{\partial w} (w, \theta) - \epsilon^2 A \frac{\partial^2 w}{\partial x^2} .
\end{equation}

Note that there is no viscous forces in (1.2). Adding a viscous stress term gives us the stress of the form

\begin{equation}
(1.3) \quad T = -p(w, \theta) + \epsilon \frac{\partial u}{\partial x} - \epsilon^2 A \frac{\partial^2 w}{\partial x^2} ,
\end{equation}

suggested by Korteweg’s theory of capillarity [50]. In (1.3), \( u(x, t) \) denotes the velocity of the fluid, \( \epsilon > 0 \) is the viscosity and \( p = \frac{\partial f_0}{\partial w} \) is the pressure.

The one dimensional balance laws of mass and linear momentum are easily written down:

\begin{align*}
(1.4a) \quad & \frac{\partial w}{\partial t} = \frac{\partial u}{\partial x} \quad \text{(mass balance)} , \\
(1.4b) \quad & \frac{\partial u}{\partial t} = \frac{\partial T}{\partial x} \quad \text{(linear momentum balance)} .
\end{align*}
The equation for balance of energy is more subtle. While a thorough examination of the energy equation appears in Dunn & Serrin [20] it is the conceptually simple approach of Felderhof [26] we recall here. Let \( \varepsilon(w, \theta) \) denote the internal energy. Felderhof’s postulate is that the internal energy is influenced only by the component of internal stress \( \tau = -p(w, \theta) + \varepsilon \frac{\partial w}{\partial x} \), i.e., the balance of energy is given by

\[
\frac{\partial e}{\partial t} = \tau \frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} ,
\]

(1.4c)

where \( h \) is the heat flux. Unlike equations (1.4a, b), equation (1.4c) is not in divergence form. To alleviate this difficulty we consider the specific total energy

\[
E = \frac{u^2}{2} + \varepsilon(w, \theta) + \frac{\varepsilon^2 A}{2} \left( \frac{\partial w}{\partial x} \right)^2
\]

made up the specific kinetic, internal, and interfacial energy. Now compute the time rate of change of \( E \):

\[
\frac{\partial E}{\partial t} = u \frac{\partial u}{\partial t} + \varepsilon^2 A \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x \partial t} = u \frac{\partial T}{\partial x} + T \frac{\partial u}{\partial x} + \varepsilon^2 A \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} + \varepsilon^2 A \frac{\partial w}{\partial x} \frac{\partial^2 u}{\partial x^2}
\]

where we have used the relation \( T = \tau - \varepsilon^2 A \frac{\partial w}{\partial x} \). We easily see that the balance of energy can be written as

\[
\frac{\partial E}{\partial t} = \frac{\partial}{\partial x} \left( uT \right) + \varepsilon^2 A \frac{\partial}{\partial x} \left( u \frac{\partial w}{\partial x} \right) + \frac{\partial h}{\partial x} .
\]

(1.5)

The term \( \varepsilon^2 A \frac{\partial w}{\partial x} \frac{\partial w}{\partial x} \) represents the “interstitial working” [20]. For simplicity we constitute \( h \) by Fourier’s law: \( h = \kappa \varepsilon \frac{\partial \theta}{\partial x} \) where \( \kappa \varepsilon > 0 \) is the (assumed constant) thermal conductivity. Then we may collect the balance laws and write them as

(1.6a)

\[
\frac{\partial w}{\partial t} = \frac{\partial u}{\partial x} \quad (\text{mass}) ,
\]

(1.6b)

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left\{ -p(w, \theta) + \varepsilon \frac{\partial u}{\partial x} - \varepsilon^2 A \frac{\partial^2 w}{\partial x^2} \right\} \quad (\text{linear momentum}) ,
\]

(1.6c)

\[
\frac{\partial E}{\partial t} = \frac{\partial}{\partial x} \left\{ u(-p + \varepsilon \frac{\partial u}{\partial x} - \varepsilon^2 A \frac{\partial^2 w}{\partial x^2} + \varepsilon^2 A \left( \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} \right) + \kappa \varepsilon \frac{\partial \theta}{\partial x} \right\} \quad (\text{energy}) .
\]

The travelling wave equations corresponding to (1.6) with \( \xi = (x - st)/\varepsilon \), \( w = w(\xi) \),
\[ u = u(\xi), \theta = \theta(\xi) \text{ are} \]

\[
\begin{align*}
(1.7a) & \quad \frac{dw}{d\zeta} = v, \\
(1.7b) & \quad A \frac{dv}{d\zeta} = -s^2(w - w_1) - p(w, \theta) + p(w_1, \theta_1) - sv, \\
(1.7c) & \quad \kappa \frac{d\theta}{d\zeta} = -s\{e(w, \theta) - e(w_1, \theta_1)) \\
& \quad - \frac{s^2}{2} (w - w_1)^2 - \frac{Av^2}{2} - p(w_1, \theta_1)(w - w_1)\}
\end{align*}
\]

where \( s \) is the speed of the travelling wave.

The most up to date results on the full system (1.7) with boundary values

\[
(1.8) \quad (w, u, \theta)(-\infty) = (w_1, u_1, \theta_1), \quad (w, u, \theta)(+\infty) = (w_2, u_2, \theta_2)
\]

were given by Grinfeld [33–35] and Mischiakow [55]. In this survey, however, we shall confine ourself to the isothermal case, i.e., \( \theta = \text{const} \), for (1.7):

\[
(1.9a) \quad \frac{dw}{d\zeta} = v, \\
(1.9b) \quad A \frac{dv}{d\zeta} = -s^2(w - w_1) + p(w_1) - p(w) - sv.
\]

The solvability and qualitative behavior of the boundary value problem of (1.9a,b) with

\[
(1.9c) \quad (w, u)(-\infty) = (w_1, u_1), \quad (w, u)(+\infty) = (w_2, u_2)
\]

were studied by Slemrod [66-68, 70] and Hagan & Slemrod [40], Hagan & Serrin [39] and Shearer [62-64].

§2. The Viscosity-Capillarity Travelling Wave Criterion. We first review some results about the boundary value problem of the travelling wave equation (1.9). We are particularly interested in the case \( w_1 < \alpha, w_2 > \beta \). The study of the solvability and qualitative behavior of (1.9) with \( w_1 < \alpha \) and \( w_2 > \beta \) is not only important, but also inevitable in the sense that each solution of (1.9) must have a shock which jumps over the spinodal region \((\alpha, \beta)\), at least for the case \( A \geq 1/4 \), (cf Fan [23]).

Let \( w_1 \leq \alpha \) and \( s \geq 0 \). For simplicity, we assume the ray starting from \((w_1, p(w_1))\), with slope \(-s^2\), to the right can intersect the graph of \( p \) at most at three points (cf Figure 2).
We denote the $w$-coordinates of these points by

$$w_2(w_1, s), \ w_3(w_1, s) \text{ and } w_4(w_1, s)$$

respectively. $w_1$ and $w_k(w_1, S), \ k = 2, 3, 4,$ are equilibrium points of (1.9). $w_1$ and $w_3(w_1, s)$ are saddle points of (1.9) while $w_4(w_1, s)$ is a node of (1.9). By Fan [23], there is no travelling wave connecting $w_1$ and $w_2(w_1, s)$, at least for the case $A \geq 1/4$.

Now we consider the existence of a travelling wave solution of (1.9) connecting $w_1$ and $w_3(w_1, s)$. i.e. $\dot{w}(-\infty) = w_1, \dot{w}(+\infty) = w_3(w_1, s)$. For $w_1 \in [\gamma, m]$, if there is a $\bar{s} \geq 0$ such that the signed area between the graph of $p$ and the chord connecting $(w_1, p(w_1))$ and $(w_3(w_1, \bar{s}), p(w_3(w_1, \bar{s})))$ is 0 (cf Figure 2), then there is a speed $s^* \geq 0$ such that

$$0 \leq s^* \leq \bar{s}$$

and the problem (1.9) with $s = s^*, w_2 = w_3(w_1, s^*)$ has a solution, which satisfies $\dot{w}'(\zeta) > 0$ and is a saddle-saddle connection, i.e.

$$(2.2) \quad 0 \leq s^* < \sqrt{-p'(w_1)}, \ s^* < \sqrt{-p'(w_3(w_1, s^*))}.$$  

In (2.1), equality hold if and only if $\bar{s} = 0$. Furthermore, for any $0 < s < s^*$ the trajectory of (1.9) emanating from $(w_1, 0)$ will overshoot $w_3(w_1, s)$ and flow to $(w_4(w_1, s), 0)$ as $\zeta \to \infty$. In other words, for all $w_2 > w_4(w_1, s^*)$, there is a travelling wave solution of (1.9). Furthermore, this travelling wave solution is a saddle-node connection, i.e.

$$(2.3) \quad \sqrt{p'(w_1)} > s > \sqrt{-p'(w_4(w_1, s))}.$$  

These statements were proved in Hagan & Slemrod's paper [40].

If $p(w)$ further satisfies

$$(2.5) \quad p''(w)(w - w_0) > 0 \quad \text{for } w \neq w_0,$$

for some $w_0 \in (\alpha, \beta)$ then, for $\gamma \leq w_1 \leq m$, there is a unique speed $s^* \geq 0$ such that $w_1$ can be connected to $w_3(w_1, s^*)$ by a travelling wave solution of (1.9) with $w_2 = w_3(w_1, s^*)$, which is a saddle-saddle connection ( We notice that when (2.5) holds there is no $w_4(w_1, s)$ for $w_1 \in (\gamma, m]$.)). This fact was shown by Shearer [62, 63]. In fact, by arguments similar to that of Lemma 4.4 of Hagan & Slemrod [40], we can show more:

**Theorem 2.1.** Let (2.4) hold and $A > 1/8$. Then, for each fixed $w_1 < \alpha$, (1.9) can have at most one solution with $s \geq 0$ satisfying $w_2 = w_3(w_1, s)$, which is a saddle-saddle connection.

**Proof.** Let $w(\zeta)$ be solution of (1.9) with $s \geq 0$ and $w(+\infty) = w_3(w_1, s)$ which is a saddle-saddle connection. Then, $w'(\zeta) > 0$. We rewrite (1.9a,b) as

$$(2.5) \quad Av \frac{dv}{dw} = -sv - s^2(w - w_1) - p(w) + p(w_1),$$

7
where $v = w' (\zeta)$. We can parametrize by $v(w, s)$ the trajectory of $w(\zeta)$ in the $v > 0$ half-plane which connects to $(w_1, 0)$ and $(w_3(w - 1, s))$. Assume, for contradiction, that there is a $0 < \bar{s} \neq s$ such that (1.9) with $s$ replaced by $\bar{s}$ has a solution $\bar{w}(\zeta)$ which is also a saddle-saddle connection, i.e. $\bar{w}(+\infty) = w_3(w_1, \bar{s})$. Without loss of generality, we assume that

\begin{equation}
\bar{s} > s.
\end{equation}

We parametrize the trajectory of $\bar{w}(\zeta)$ in the upper $(w, v)$-plane as $v(w, \bar{s})$. Clearly, $v(w, \bar{s})$ satisfies

\begin{equation}
A v(w, \bar{s}) \frac{dv(w, \bar{s})}{dw} = -\bar{s} v(w, \bar{s}) - \bar{s}^2(w - w_1) - p(w) + p(w_1),
\end{equation}

\textit{Fig.3}

A calculation shows that

\[
\frac{dv(w, s)}{dw} \bigg|_{w=w_1+} = \frac{1}{2A} \left( -s + \sqrt{(1 - 4A)s^2 - 4Ap'(w_1)} \right) > 0
\]

which decreases as $s$ increases when $A > 1/8$. Thus, for $w - w_1 > 0$ and small,

\begin{equation}
v(w, s) > v(w, \bar{s}).
\end{equation}

Since $w_3(w_1, s) < w_3(w_1, \bar{s})$ and the trajectory $v(w, s)$ connects $w_1$ and $w_3(w_1, s)$ on the $v > 0$ half-plane, (2.8) implies that the trajectory $v(w, \bar{s})$ has to intersect the trajectory $v(w, s)$ at some $w \in (w_1, w_3(w_1, s))$. In other words,

\begin{equation}
v(w^*, s) = v(w^*, \bar{s}), \quad \frac{dv(w, \bar{s})}{dw} \bigg|_{w=w^*} \leq \frac{dv(w, \bar{s})}{dw} \bigg|_{w=w^*}
\end{equation}

for some $w^* \in (w_1, w_3(w_1, s))$. Subtracting (2.6) from (2.7) yields

\[A v(w, \bar{s}) \frac{dv(w, \bar{s})}{dw} - A v(w, s) \frac{dv(w, s)}{dw} = sv(w, s) - \bar{s} v(w, \bar{s}) + (s^2 - \bar{s}^2)(w - w_1).
\]

Applying (2.9) to above equation, we obtain

\[0 \leq A v(w, s) \left( \frac{dv(w, \bar{s})}{dw} - \frac{dv(w, s)}{dw} \right) = v(w, s)(s - \bar{s}) + (s^2 - \bar{s}^2)(w - w_1) < 0
\]

which is a contradiction. \Box

We note when $p(w)$ is a cubic polynomial, we can have explicit solution for (1.9). Let

\begin{equation}
p(w) = p_0 - p_1(w - m)(w - M) \left( w - \frac{m + M}{2} \right)
\end{equation}
where $m$ and $M$ are Maxwell constants. Then a solution of (1.9) is (cf. Truskinovskii [77, 78])

\[
(2.11) \quad w(\zeta) = \frac{w_- + w_+}{2} + \frac{w_+ - w_-}{2} \tanh \left( \sqrt{\frac{p_1}{2A}} \frac{w_+ - w_-}{2} (\zeta - \zeta_0) \right).
\]

For each $w_-$ fixed, $w_+$ in (2.11) is determined by equations:

\[
(2.12) \quad 3(1 - 6A)(2y - z + 1)^2 + z^2 = 1,
\]

\[
y = (M - w_+)/(M - m),
\]

\[
z = (w_+ - w_-)/(M - m).
\]

The number of solutions of (2.12) ranges from zero to two. When (2.12) has two solutions, we get two solutions of (1.9) of the form (2.11); one of them has positive speed and the other negative. This is, of course, consistent with Theorem 2.1.

§3. The Riemann problem.

Now, we return to the Riemann problem (0.1). One method for solving (0.1) is to construct wave and shock curves that are admissible according to some criteria and then construct a wave fan of waves and shocks that matches the initial data. This approach has been pursued by James [46], Shearer [61-64], Hsiao [45] and Fan [23]. Shearer [64] proved that solutions of (0.1) satisfying viscosity-capillarity travelling wave criterion exist for Riemann datum close to the Maxwell line, i.e.,

\[
|w_- - m| < \delta, \quad |w_+ - M| < \delta,
\]

\[
|u_+ - u_-| < \delta \quad \text{for some} \quad \delta > 0.
\]

To establish the existence of solutions of (0.1). Slemrod [69] applied the vanishing similarity viscosity approach to the system (0.1) which will be reviewed in next section. Later, Fan [22] proved, under the assumption

\[
(3.1) \quad p(w) \to -\infty \text{ as } w \to \infty, \quad p(w) \to -\infty \text{ as } w \to \infty,
\]

that there are solutions of (0.1) satisfying the viscosity-capillarity travelling wave criterion. In this section, we shall give details of their proofs. We shall also discuss the uniqueness and stability the solution of (0.1) admissible according to viscosity-capillarity travelling wave criterion.

§3.0. Riemann problems – the similarity viscosity approach. Admissibility criteria we reviewed above are shock admissibility criteria, which are local restrictions on points of jump discontinuity. We hope these local restrictions can characterize completely
admissible solutions of (0.1). At this time, it is known only that this is true for scalar conservation laws (cf Volpert [82]). There are, however, strong indications that shock admissibility criteria may be inadequate for systems that are not strictly hyperbolic (cf Glimm [27] ) or for systems that change type (cf Shearer [63]). For example, in the context of (0.1) (cf Shearer [63]), it is not obvious, even if shock admissibility conditions are known a - priori, in what manner full solution which is a composition of admissible shocks and rarefaction waves is admissible. Thus it is important to experiment with various criteria with global authority, especially those motivated by physics.

One example of such criteria is to dub admissible those solutions which are $\varepsilon \to 0+$ limit of solutions of

\begin{equation}
\begin{aligned}
    u_t + p(w)_x &= \varepsilon u_{xx} - \varepsilon^2 Aw_{xx}, \\
    w_t - u_x &= 0
\end{aligned}
\end{equation}

as proposed in Slemrod [67]. We can also employ the conventional form of viscosity

\begin{equation}
\begin{aligned}
    u_t + p(w)_x &= \varepsilon u_{xx} \\
    w_t - u_x &= \varepsilon w_{xx}
\end{aligned}
\end{equation}

to which (3.1) reduces when $A = 1/4$ (cf Slemrod [68] ). Such an approach seems like an extremely difficult task at present. For the system (3.0.2) written in Eulerian coordinates with $p(\rho) = C\rho^\gamma$, for which the system is hyperbolic, DiPerna [19] and Ding, Chen and Luo [18] constructed solutions of the Cauchy problem, when the states at $\pm\infty$ are the same, by limits of viscous regularization and finite difference schemes. For hyperbolic-elliptic mixed type system (3.0.1), we only have some numerical shemes (cf. Shu [65] and references cited therein ) which are at least visually good for Riemann problems.

Reasonable admissibility criteria should comply with the irreversibility of solutions and thus may impose restrictions on wave sources but not on wave sinks. Also admissibility criteria should be compatible with translations and dilations of coordinates, under which the system is invariant. Based on these requirements, Dafermos [16] argued that admissibility should be tested in the frame work of Riemann problem ( Riemann [58], Lax [51], Liu [52] ), i.e. in the context of solutions of the form $U(x,t) = V(x/t)$ which represent wave fans emanating from the origin at time $t = 0$.

Thus we utilize the similarity viscosity to handle the Riemann problem (0.1). The idea is to replace (0.1 a,b) by

\begin{equation}
\begin{aligned}
    u_t + p(w)_x &= \varepsilon tu_{xx}, \\
    w_t - u_x &= \varepsilon tw_{xx}
\end{aligned}
\end{equation}

which is invariant under dilatation of coordinates, and construct weak solutions of (0.1) as $\varepsilon \to 0+$ limits of solutions of (3.0.3). For convenience, we shall call solutions of (0.1)
constructed in this way as solutions of (0.1) admissible according to similarity viscosity criterion. This approach has been pursued by Kalasnikov [49], Dafermos [13, 14], Tupciev [79, 80], Dafermos and DiPerna [17], Slemrod [70], Slemrod & Tzavaras [71] and Fan [21, 22, 24, 25].

To take the advantage of the invariance of (0.1) under dilatation of coordinates, we make variable change $\xi = x/t$ in (3.0.3). A simple computation shows that (3.0.3) reduces to the following system

\begin{align}
(3.0.4a) & \quad \varepsilon u'' = -\xi u' + p(w)', \\
(3.0.4b) & \quad \varepsilon w'' = -\xi w' - w', \\
(3.0.4c) & \quad (u, w)(-\infty) = (u_-, w_-), \quad (u, w)(+\infty) = (u_+, w_+). 
\end{align}

Slemrod proved that (3.0.4) has a solution $(u(\xi), w(\xi))$ satisfying

\begin{align}
(3.0.5) & \quad w'(\xi) > 0, \quad \text{when} \quad \alpha \leq w(\xi) \leq \beta,
\end{align}

by using Leray–Schauder type fixed point theory. Later, Fan [22] proved that the total variation of solutions of (3.0.4) is bounded uniformly in $\varepsilon$ and hence established the existence of weak solutions of (0.1) satisfying the similarity viscosity criterion under the assumption

\begin{align}
(3.0.6) & \quad p(w) \to \infty \text{ as } w \to -\infty, \quad p(w) \to -\infty \text{ as } w \to \infty.
\end{align}

( In fact, we can prove the existence for (0.1) written in Eulerian coordinate under a weaker assumption on $p$ which includes the van der Waals state function ( cf. Fan [25] ) We will give details of the proof for the existence of these solutions in the following sections §3.1, §3.2. In §3.3, we shall show that weak solutions of (0.1) constructed by the similarity viscosity approach in §3.1 and §3.2 are also admissible according to the viscosity-capillarity travelling wave criterion when $A = 1/4$. As a consequence, solutions of (0.1), admissible by the viscosity-capillarity travelling wave criterion with $A = 1/4$ exist if (3.0.5) holds. It is interesting to see that the travelling wave equation of the common form for most artificial viscosity (3.0.2) is the same as the equation of the shock profile of solutions of (0.1) we constructed by similarity viscosity approach. This is not surprising if we recall that (3.0.1) reduces to (3.0.2) when $A = 1/4$. From the above and the uniqueness and stability results of Fan [23] which we shall state in §3.4, we can see clearly that the $\varepsilon \to 0$ limit of solutions (3.0.4) with (3.0.5) is unique and stable in Lebesgue measure with respect to changes in Riemann datum if $p(w)$ satisfies

\begin{align}
(3.0.7) & \quad p'' < 0 \text{ if } w < \alpha, \quad p'' > 0 \text{ if } w > \beta.
\end{align}
The structure of solutions of (0.1) constructed in Slemrod [70] and Fan [22] by similarity viscosity approach is as follows: Each of these solutions can be imbeded on a continuous curve in \((u, w)\) phase plan. Solutions must have a phase boundary, i.e. \(w(\xi) \notin (\alpha, \beta)\) for any \(\xi \in \mathbb{R}\). Solutions consist of two wave fans: \(\xi < 0\) the first kind wave fan and \(\xi > 0\) the second kind wave fan. A first (second) kind wave fan consists of 1-shocks and (2-shocks) and 1-simple waves (2-simple waves) and possibly the phase boundary and constant states. \(\xi = 0\) is either a constant state or the phase boundary (cf. Fan [22] ).

§3.1. The existence of solutions to (3.0.4). Here, we present the proof, given by Slemrod (1989), of the existence of solutions for the system (3.0.4):

We consider, instead of (3.0.4), the following altered system

\[
\begin{align*}
\varepsilon u'' &= -\xi u' + \mu p(w)' \\
\varepsilon w'' &= -\xi w' - \mu u' \\
(u(\pm L), w(\pm L)) &= (u_\pm, w_\pm)
\end{align*}
\]

where \(L > 1\), \(0 \leq \mu \leq 1\).

For the system (3.1.1) we have the following lemmas:

**Lemma 3.1.1.** Let \((u(\xi), w(\xi))\) be a solution of (3.1.1). Then on any interval \((l_1, l_2) \subset (-L, L)\) for which \(p'(w(\xi)) < 0\), one of the following holds.

(i) \(u(\xi)\) is strictly increasing (or decreasing) with no critical point in \((l_1, l_2)\) while \(w(\xi)\) has at most one critical point in \((l_1, l_2)\) which must be a maximum (or minimum).

(ii) \(w(\xi)\) is strictly increasing (or decreasing) with no critical point in \((l_1, l_2)\) while \(u(\xi)\) has at most one critical point which must be a maximum (or minimum).

**Lemma 3.1.2.** Let \((u(\xi), w(\xi))\) be a solution of (3.1.1) with \(\mu > 0\). Then on any interval \((l_1, l_2) \subset (-L, L)\) for which \(p'(w(\xi)) > 0\) the graph of \(u(\xi)\) versus \(w(\xi)\) is convex at points where \(w'(\xi) > 0\) and concave at points where \(w'(\xi) < 0\).

**Theorem 3.1.3.** If \((u(\xi), w(\xi))\) is a solution of (3.1.1) with \(w'(\xi) > 0\) when \(\alpha \leq w(\xi) \leq \beta\), then

\[
\sup_{-L \leq \xi \leq -L} (|u(\xi)| + |w(\xi)| + |u'(\xi)| + |w'(\xi)|) \leq M_0
\]

where \(M_0\) is a constant independent of \(\mu \in [0, 1]\) and \(L > 1\).

With above preparations, we are ready to prove the existence theorem for (3.4):

**Theorem 3.1.4.** There are solutions of (3.4) which satisfy the constraints \(w'(\xi) > 0\) when \(\alpha \leq w(\xi) \leq \beta\). In other words, the one phase change data connecting orbit problem (3.4) possesses a one phase change solution.
Proof. First notice that when $\mu = 0$, (3.1.1) possesses a unique solution

$$u_0(\xi) = \frac{(u_+ - u_-) \int_{-L}^{\xi} \exp(-\zeta^2/2\varepsilon) d\zeta}{\int_{-L}^{L} \exp(-\zeta^2/2\varepsilon) d\zeta} + u_-,$$

$$w_0(\xi) = \frac{(w_+ - w_-) \int_{-L}^{\xi} \exp(-\zeta^2/2\varepsilon) d\zeta}{\int_{-L}^{L} \exp(-\zeta^2/2\varepsilon) d\zeta} + w_-.$$

Also note that $w_0(\xi) > 0$, $\xi \in [-L, L]$.

Now set $U(\xi) = u(\xi) - u_0(\xi)$, $W(\xi) = w(\xi) - w_0(\xi)$ and impose boundary conditions

(3.1.3) \quad U(-L) = U(L) = W(-L) = W(L) = 0.

If $u, w$ are to solve (3.1.1) we see that $U, W$ must satisfy (1.4) and

(3.1.4) \quad \varepsilon U'' = \mu p(w_0 + W)' - \xi U',

\quad \varepsilon W'' = -U' - \mu u_0' - \xi W'.

Define the vectors

$$y(\xi) = \begin{pmatrix} U(\xi) \\ W(\xi) \end{pmatrix}, \quad f(\xi, y) = \begin{pmatrix} p(w_0 + W) \\ -U(\xi) - u_0 \end{pmatrix}.$$

Then the system (3.1.4) takes the form

(3.1.5) \quad \varepsilon y''(\xi) = \mu f(\xi, y)'(\xi),

(3.1.6) \quad y(-L) = 0, \quad y(L) = 0.

Let $v \in C^1([-L, L]; \mathbb{R}^2)$. Define $T$ to be the solution map that carries $v$ into $y$ where $y$ solves

(3.1.7) \quad \varepsilon y''(\xi) = f(\xi, v)' - \xi y'(\xi),

(3.1.8) \quad y(-L) = 0, \quad y(L) = 0.

A straightforward computation shows that $y(\xi)$ is given by the formula

$$y(\xi) = z^\xi \int_{-L}^{\xi} \exp(-\zeta^2/2\varepsilon) d\zeta + \frac{1}{\varepsilon} \int_{-L}^{\xi} f(\zeta, v(\zeta)) d\zeta$$

(3.1.9) \quad -\frac{1}{\varepsilon^2} \int_{-L}^{\xi} \int_{-L}^{\zeta} \tau f(\tau, v(\tau)) \exp\left(\frac{\tau^2 - \zeta^2}{2\varepsilon}\right) d\tau d\zeta
where
\[
\int_{-L}^{L} \exp(-\xi^2/2\varepsilon) d\xi = -\frac{1}{\varepsilon} \int_{-L}^{L} f(\xi, v(\xi)) d\xi + \frac{1}{\varepsilon^2} \int_{-L}^{L} \int_{0}^{\xi} \tau f(\tau, v(t)) \exp \left( \frac{\tau^2 - \xi^2}{2\varepsilon} \right) d\tau d\xi.
\]
(3.1.10)

Notice the fixed points of $\mu T$ are solutions of (3.1.5), (3.1.6) which in turn yield solutions of (3.1.1).

It is clear that $T$ maps $C^0([-L, L]; \mathbb{R}^2)$ continuously into $C^0([-L, L]; \mathbb{R}^2)$. Of course this implies that $T$ maps $C^1([-L, L]; \mathbb{R}^2)$ continuously into $C^0([-L, L]; \mathbb{R}^2)$. We now show $T$ maps $C^1([-L, L]; \mathbb{R}^2)$ continuously into $C^1([-L, L]; \mathbb{R}^2)$.

For this purpose let $v_1, v_2 \in C^1([-L, L]; \mathbb{R}^2)$, $v_1 = (U_1, W_1), v_2 = (U_2, W_2)$, and $y_1 = \mu T v_1, y_2 = \mu T v_2$. Differentiation of (1.10) shows
\[
y'_1(\xi) - y'_2(\xi) = (z_1 - z_2) \exp(-\xi^2/2\varepsilon) + \frac{f(\xi, v_1(\xi)) - f(\xi, v_2(\xi))}{\varepsilon} - \frac{1}{\varepsilon^2} \int_{0}^{\xi} \tau f(\tau, v_1(\tau)) - f(\tau, v_2(\tau)) \exp \left( \frac{\tau^2 - \xi^2}{2\varepsilon} \right) d\tau,
\]
(3.1.11)

where $z_1, z_2$ are defined in the obvious manner.

Now let $v_1, v_2$ be in a finite ball $B$ in $C^1([-L, L]; \mathbb{R}^2)$. In particular for $v = (U, W)$ in $B$, $w_0 + W$ is uniformly bounded in $R$ and hence $p$ is a uniformly continuous function of the argument $w_0 + w$. But for $\delta > 0$ arbitrary we know from uniform continuity of $p$ that there is $l(\delta) > 0$ such that $p(w_0(\xi) + W_1(\xi)) - p(w_0(\xi) + W_2(\xi)) < \delta$ if $|W_1(\xi) - W_2(\xi)| < l(\delta)$, i.e. if $|W_1(\xi) - W_2(\xi)| < l(\delta)$. Hence $\sup_{-L < \xi < L} |p(w_0(\xi) + W_1(\xi)) - p(w_0(\xi) + W_2(\xi))| < \delta$ if $\sup_{-L < \xi < L} |W_1(\xi) - W_2(\xi)| < l(\delta)$ and so $\sup_{-L < \xi < L} |p(w_0(\xi) + W_1(\xi)) - p(w_0(\xi) + W_2(\xi))| \to 0$ as $\sup_{-L < \xi < L} |W_1(\xi) - W_2(\xi)| \to 0$. But this argument implies by the special nature of $f(\xi, v(\xi))$ that $\sup_{-L < \xi < L} |f(\tau, v_1(\tau)) - f(\tau, v_2(\tau))| \to 0$ as $\sup_{-L < \xi < L} |v_1(\tau) - v_2(\tau)| \to 0$. From (3.1.10), (3.1.11) we see then that $\sup_{-L < \xi < L} |y'_1(\xi) - y'_2(\xi)| \to 0$ as $\sup_{-L < \xi < L} |v_1(\xi) - v_2(\xi)| \to 0$, and so $T$ is a continuous map of $C^1([-L, L]; \mathbb{R}^2)$ into itself.

Now note that (1.8) implies that if $v$ is in a bounded set of $C^1([-L, L]; \mathbb{R}^2)$, $y$ will be in a bounded set of $C^2([-L, L]; \mathbb{R}^2)$. This is because $f(\xi, v(\xi))'$ is uniformly bounded.

Hence $T$ is a continuous compact map of $C^1([-L, L]; \mathbb{R}^2)$ into itself.

Now define $\Omega = \{U, W \in C^1([-L, L]; \mathbb{R}^2) \text{ such that } W(-L) + w_0(-L) < \alpha, W(L) + w_0(L) > \beta; W'(\xi) + w'_0(\xi) > 0 \text{ if } \alpha \geq W(\xi) + w_0(\xi) \leq \beta; \text{ and } \sup_{-L < \xi < L} |U(\xi) + u_0(\xi)| + \sup_{-L < \xi < L} |W(\xi) + w_0(\xi)| < M \}$. Then...
\[ |W(\xi) + w_0(\xi)| + |U'(\xi) + u'_0(\xi)| + |W'(\xi) + w'_0(\xi)| < M_0 + 1 \]. \( \Omega \) is a bounded set in \( C^1([-L,L];\mathbb{R}^2) \).

In addition \( \Omega \) is open. To see this let \( U, W \in \Omega \). Note the definition of \( \Omega \) implies the set \( A \) defined \( \{ \xi \in [-L,L]; \alpha \leq w_0(\xi) + W(\xi) \leq \beta \} \) is a closed interval \([\xi_1, \xi_2]\). For if \( A \neq [\xi_1, \xi_2] \) for some \( \xi_1, \xi_2 \in [-L,L] \) means by the monotonicity of \( w_0 + W \) on \([\xi_1, \xi_2]\) that for some \( \xi \in [\xi_1, \xi_2] \) either \( w_0(\xi) + W(\xi) = \alpha \) or \( w_0(\xi) + W(\xi) = \beta \), with \( w_0(\xi) + W'(\xi) \leq 0 \) in either case. Of course this would imply \( U, W \notin \Omega \), contradiction.

Thus we have \( A = [\xi_1, \xi_2] \), and we set \( \bar{m} = \min_{\xi \in A} (w_0'(\xi) + W'(\xi)) \) which is positive. Since \( w_0 + W \in C^1([-L,L]) \) there is a larger interval \( A_\delta \subseteq [-L,L], \delta > 0, A_\delta = [\xi_1 - \delta, \xi_2 + \delta] \) for some small \( \delta > 0 \), so that \( w_0'(\xi) + W'(\xi) \geq \frac{1}{2} \bar{m} \) for \( \xi \in A_\delta \).

Let \( D = \min_{\xi_2 - \delta \leq \xi \leq \xi_1} w_0(\xi) - W(\xi) - \beta, \min_{-L \leq \xi \leq \xi_1 - \delta} (\alpha - w_0(\xi) - W(\xi)) \). Since \( w_0(\xi) - W(\xi) - \beta > 0 \) on \([\xi_2 + \delta, L]\) and \( \alpha - w_0(\xi) - W(\xi) > 0 \) on \([-L, \xi_1 - \delta]\) we see that \( D > 0 \).

Now let \( \bar{U}, \bar{W} \) be such that
\[ \sup_{-L \leq \xi \leq L} \bar{U}(\xi)| + |\bar{U}'(\xi)| + |\bar{W}(\xi)| + |\bar{W}'(\xi) < v, \]
where \( v = \min(\frac{1}{2} D, \frac{1}{4} \bar{m}) \). Consider \( \xi \in [-L,L] \) for which \( \alpha \leq w_0(\xi) + W(\xi) + \bar{W}(\xi) \leq \beta \). If we can show that \( w_0'(\xi) + \bar{W}'(\xi) + W'(\xi) > 0 \), we will have proven \( \Omega \) is open. But we see in this case that
\[ w_0(\xi) + W(\xi) - \beta \leq -\bar{W}(\xi), \quad \alpha - w_0(\xi) - W(\xi) \leq \bar{W}(\xi), \]
and hence \( w_0(\xi) + W(\xi) - \beta \leq D/2, \alpha - w_0(\xi) - W(\xi) \leq D/2 \). But this implies by the definition of \( D \) that \( \xi \in (\xi_1 - \delta, \xi_2 + \delta) \). Thus we have shown that \( \alpha \leq w_0(\xi) + W(\xi) + \bar{W}(\xi) \leq \beta \) implies \( \xi \in A_\delta \). Now we compute at this value \( \xi \):
\[ w'_0(\xi) + W'(\xi) + \bar{W}'(\xi) \geq \frac{1}{2} \bar{m} + \bar{W}'(\xi) \geq \frac{1}{4} \bar{m} > 0. \]
Hence \( \Omega \) is open.

Now we recall a well known theorem of Leray-Schauder type (see for example Mawhin [54], Corollary IV.7).

**Proposition 3.1.5.** Let \( X \) be a real normed vector space, \( \Omega \) an open bounded subset of \( X \), and \( T \) a compact map of \( X \) into itself. If zero is an interior point of \( \Omega \) and \( \phi + \mu T\phi \) for all \( \phi \in \partial \Omega \), \( 0 < \mu < 1 \), then \( T \) has at least one fixed point in \( \Omega \).

In our problem we take \( X = C^1([-L,L];\mathbb{R}^2) \) and \( T, \Omega \) is as defined above. The origin is an interior point of \( \Omega \) since the constraint \( w_0'(\xi) + \bar{W}'(\xi) > 0 \) is satisfied for all \( \xi \in [-L,L] \) if \( (\bar{U},\bar{W}) \) is a small \( C^1([-L,L];\mathbb{R}^2) \) perturbation. Note \( \phi \in \partial \Omega \), \( \phi = \mu T\phi \), \( \mu \in (0,1) \), means that there is a solution \((u(\xi),w(\xi))\) of (1.1), (1.2), (1.3) which satisfies \( w'(\xi) \geq 0 \) if \( \alpha \leq w(\beta) \leq \beta \) and either
(i) \( w'(\xi_0) = 0, \alpha \leq w(\xi_0) \leq \beta \) for some \( \xi_0 \in (-L, L) \) 

or 

(ii) \( \sup_{-L < \xi < L} \{|u(\xi)| + |w(\xi)| + |u'(\xi)| + |w'(\xi)|\} = M_0 + 1 \) or both (i) and (ii).

Let us first consider possibility (i). In this case either \( \alpha < w(\xi_0) < \beta \), \( w(\xi_0) = \alpha \), or \( w(\xi_0) = \beta \). We consider these cases separately.

**Case 1.** \( \alpha < w(\xi_0) < \beta \), \( w'(\xi_0) = 0 \). In this case there are three possibilities, either \( w''(\xi_0) < 0 \), \( w''(\xi_0) > 0 \), or \( w''(\xi_0) = 0 \). If \( w''(\xi_0) < 0 \) then \( w(\xi_0) \) is a local maximum which implies \( w'(\xi) < 0 \) for some \( \xi < \xi_0, \ |\xi - \xi_0| \) small. But this implies \( \alpha < w(\xi) < \beta \) and violates the requirement that \( w'(\xi) \geq 0 \). An analogous statement holds if \( w''(\xi_0) < 0 \) and now \( w(\xi_0) \) is a local minimum. The case \( w''(\xi_0) = 0 \) is excluded since \( w''(\xi_0) = 0 \), \( w'(\xi_0) = 0 \) implies because of (3.1.1b) that \( u'(\xi_0) = 0 \). But in this case uniqueness of solutions for (3.1.1a.b) as an initial value problem (see [7], Lemma 4.1) \( u'(\xi_0) = 0 \), \( w'(\xi_0) = 0 \) implies \( u(\xi) = u(\xi_0), w(\xi) = w(\xi_0) \) for all \( \xi \in [-L, L] \) and hence we cannot satisfy (3.1.1c) \( w_+ < \alpha \), \( w_+ > \beta \).

**Case 2.** \( w(\xi_0) = \alpha \), \( w'(\xi_0) = 0 \). In this case there are again the three canonical possibilities, \( w''(\xi_0) < 0 \), \( w''(\xi_0) > 0 \), or \( w''(\xi_0) = 0 \). We can immediately dismiss \( w''(\xi_0) > 0 \) and \( w''(\xi_0) = 0 \) for the same reasons as in Case 1. So we need only consider \( w''(\xi_0) < 0 \). In this case \( w(\xi_0) = \alpha \) is a local maximum. Hence if we are to satisfy \( w(L) = w_+ > \beta \) we must proceed through a local minimum at \( \xi_1 > \xi_0 \), i.e. \( w(\xi_1) < \alpha \), \( w'(\xi_1) = 0 \), \( w''(\xi_1) \geq 0 \); \( w(\xi) < \alpha \), \( w'(\xi) < 0, \xi_0 < \xi \leq \xi_1 \). Again \( w''(\xi_1) = 0 \) is impossible since that forces \( u'(\xi) = 0 \) and the uniqueness theorem [13], Lemma 4.1) is contradicted. Thus we need only consider \( w''(\xi_1) > 0 \). From (1.2) (3.1.1b) we see \( u'(\xi_1) > 0 \), \( u'(\xi) < 0 \) which implies \( u \) has a local maximum at a point \( \xi_0 < \xi < \xi_1 \), \( u'(\xi) = 0 \), \( u''(\xi) \leq 0 \), and again Lemma 4.1 of [13] tells us \( u''(\xi) < 0 \). Since \( p'(w) < 0 \) for \( w < \alpha \) this implies by use of (3.1.1a) that \( w'(\xi) > 0 \) which contradicts the fact that \( w \) is decreasing on \( \xi_0, \xi_1 \). Hence \( w''(\xi_0) < 0 \) is excluded as well.

**Case 3.** \( w(\xi_0) = \beta \), \( w'(\xi_0) = 0 \). Here again we see we can exclude \( w''(\xi_0) < 0 \) and \( w''(\xi_0) = 0 \) immediately. If \( w''(\xi_0) > 0 \) it follows that \( w(\xi_0) = \beta \) is a local minimum to satisfy \( w(-L) = w_- < \alpha \) there must be \( \xi_1 < \xi_0 \) where \( w(\xi_1) > \beta \) and \( w \) has a local maximum, \( w(\xi) > \beta \) on \( (\xi_1, \xi_0) \). But the same reasoning as in Case 2 yields a contradiction.

From Cases 1, 2, 3 of (i) we see there is no solution of (3.1.1), \( \mu \in (0, 1) \), \( (u(\xi) - u_0(\xi), \psi(\xi) - u_0(\xi)) \) in \( \Omega \) for which (i) can hold. Thus all solutions of (3.1.1), \( \mu \in (0, 1) \) in \( \overline{\Omega} \) must satisfy \( w'(\xi) > 0 \) in \( \alpha \leq w(\xi) \leq \beta \). But now Theorem 3.1.3 says (ii) cannot hold either. Thus we conclude from Prop. 3.1.5 that (3.1.1) possesses a solution for which \( u(\xi) - u_0(\xi), \psi(\xi) - u_0(\xi) \) is in \( \overline{\Omega} \).
To complete the proof we follow Dafermos [13] and extend the domains of $u, w$: Set
\[
\begin{align*}
    u(\xi; L) &= u_+, \quad w(\xi; L) = w_+, \quad \xi > L, \\
    u(\xi; L) &= u_-, \quad w(\xi; L) = w_-, \quad \xi < -L.
\end{align*}
\]
The extended pair $\{u(\cdot; L), w(\cdot; L)\}$ form a sequence in $C^0((\infty, \infty); \mathbb{R}^2)$ and by virtue of the hypothesis of theorem we know $\sup_{-L < \xi < L} \{|u'(\xi; L)| + |w'(\xi; L)|\} \leq M$. Thus the sequence $\{(u(\cdot; L), w(\cdot; L))\}$ is precompact in $C^0((\infty, \infty); \mathbb{R}^2)$ and so there is a subsequence $L_n \to \infty$ as $n \to \infty$ since that $(u(\xi; L), w(\xi; L)) \to (u(\xi), w(\xi))$ uniformly as $n \to \infty$ on $(-\infty, \infty)$. As in Dafermos (1973) $u(\xi)$ $w(\xi)$ is a solution of (3.0.4) and by its construction $w'(\xi) \geq 0$ if $\alpha \leq w(\xi) \leq \beta$. But by the same reasoning used in Cases 2, 3, this connecting orbit must satisfy the more restrictive requirement $w'(\xi) > 0$ if $\alpha \leq w(\xi) \leq \beta$. This completes the proof of Theorem 3.1.4 \]

§3.2. Solutions for the Riemann problem (0.1) exist. In this section we continue the program carried out in §3.1 to prove the existence for the Riemann problem (0.1). Our strategy is to prove $(u_\varepsilon(\xi), w_\varepsilon(\xi))$, the solutions of (3.0.4), have total variation bounded uniformly in $\varepsilon$ and then to employ Helly's theorem to prove the convergence of these solutions, as $\varepsilon \to 0+$, to a weak solution of (0.1). To this end, it suffices, by Lemmas 3.1.1 and 3.1.2, that $(u_\varepsilon(\xi), w_\varepsilon(\xi))$ is bounded uniformly in $\varepsilon$. This is done by the following theorems supplied by Fan [22].

In this section, we assume $p(w)$ satisfies (3.0.6).

We shall prove, under (3.0.6), that
\[
\sup_{\xi \in \mathbb{R}} (|u_\varepsilon(\xi)|, |w_\varepsilon(\xi)|) \leq C
\]
where $C$ is independent of $\varepsilon$.

**Theorem 3.2.1.** $u_\varepsilon(\xi)$ are bounded from above, uniformly in $\varepsilon$:
\[
(3.2.1) \quad u_\varepsilon(\xi) \leq \max(u_-, u_+) + \max(w_+ - \beta, \alpha - w_-) \max_{w \in [w_-, \alpha] \cup [\beta, w_+]} (\sqrt{-p'(w)}).
\]

**Proof.** Without loss of generality, we assume that each $u_\varepsilon(\xi)$ has a local maximum point $\xi = \theta_\varepsilon$ with $w_\varepsilon(\theta_\varepsilon) \geq \beta$. The proof in the other case is similar. From (1.7) and the chain rule, we have
\[
\begin{align*}
    \varepsilon \frac{d}{d\xi} \left( \frac{du_\varepsilon(\xi)}{dw_\varepsilon(\xi)} \right) &= \left( \frac{du_\varepsilon(\xi)}{dw_\varepsilon(\xi)} - \sqrt{-p'(w_\varepsilon(\xi))} \right) \left( \frac{du_\varepsilon(\xi)}{dw_\varepsilon(\xi)} + \sqrt{-p'(w_\varepsilon(\xi))} \right).
\end{align*}
\]
This implies that, as $\xi$ increases, $\frac{du_\varepsilon(\xi)}{dw_\varepsilon(\xi)}$ is decreasing if $|\frac{du_\varepsilon(\xi)}{dw_\varepsilon(\xi)}| \leq \sqrt{-p'(w_\varepsilon(\xi))}$ and is increasing if $|\frac{du_\varepsilon(\xi)}{dw_\varepsilon(\xi)}| \geq \sqrt{-p'(w_\varepsilon(\xi))}$. Thus the "initial" condition
\[
(3.2.2b) \quad \frac{du_\varepsilon(\xi)}{dw_\varepsilon(\xi)} \big|_{\xi = \theta_\varepsilon} = 0
\]
\[
17
\]
leads to

\[ \left| \frac{du_{\epsilon}(\xi)}{dw_{\epsilon}(\xi)} \right| \leq \max_{w_{\epsilon}(\xi) \geq w \geq \beta} (\sqrt{-p'(w)}). \]

as long as \( w_{\epsilon}(\xi) \geq \beta \). By Lemma 2.1, \( w_{\epsilon}(\xi) \) is increasing when \( w_{\epsilon}(\xi) \geq \beta \). Thus (3.2.1) follows easily. \( \square \)

**Theorem 3.2.2.** \( u_{\epsilon}(\xi) \) are bounded from below, uniformly in \( \epsilon \).

**Proof.** Assume the contrary. Then there exists a sequence of \( \{\epsilon_n\} \) such that each \( u_{\epsilon_n}(\xi) \) has a local minimum point \( \tau_n \) with

\[ u_{\epsilon_n}(\tau_n) \rightarrow -\infty \quad \text{as} \quad n \rightarrow \infty, \]

\[ w_{\epsilon_n}(\tau_n) \in (\alpha, \beta). \]

Without loss of generality, we assume that

\[ \tau_n \geq 0, \quad \text{for} \quad n = 1, 2, \ldots \]

The proof for the case \( \tau_n \leq 0 \) is similar. We denote the maximum point of \( u_{\epsilon_n}(\xi) \) in the region where \( w_{\epsilon_n}(\xi) \geq \beta \) by \( \zeta_n \). By Lemma 3.1.1 and 3.1.2, we know that

\[ u_{\epsilon_n}'(\xi) > 0 \quad \text{for} \quad \xi \in (\tau_n, \zeta_n). \]

By integrating (3.4a) on \( (\tau_n, \theta) \) where \( \theta \in (\tau_n, \zeta_n) \), we obtain

\[ 0 \leq \epsilon u_{\epsilon_n}'(\theta) = \int_{\tau_n}^{\theta} -\xi u_{\epsilon_n}'(\xi) d\xi \]

\[ + p(w_{\epsilon_n}(\theta)) - p(w_{\epsilon_n}(\tau_n)). \]

It follows from (3.2.6) and (3.2.7) that \( -\xi u_{\epsilon_n}'(\xi) < 0 \) for \( \xi \in (\tau_n, \zeta_n) \). Thus, in view of (3.2.5), we have

\[ 0 \leq \epsilon u_{\epsilon_n}'(\theta) \leq p(w_{\epsilon_n}(\theta)) - p(w_{\epsilon_n}(\tau_n)) \]

\[ \leq p(w_{\epsilon_n}(\theta)) - p(\alpha). \]

Therefore,

\[ \alpha < w_{\epsilon_n}(\theta) \leq \delta \quad \text{for} \quad \theta \in (\tau_n, \zeta_n]. \]

Equation (3.2.9) also yields the useful inequality

\[ 0 \leq \epsilon u_{\epsilon_n}'(\theta) \leq p(\beta) - p(\alpha) \]

for \( \theta \in [\tau_n, \zeta_n]. \)
CLAIM. There exist $\eta_n \in (\tau_n, \zeta_n)$ (cf. Fig. 4) such that

\begin{equation}
(3.2.12) \quad u_{\epsilon n}(\eta_n) \geq u_+ - 2(\delta - \gamma) \max_{w \in [\gamma, \delta]} (\sqrt{|p'(w)|}) - 2(p(\beta) - p(\alpha)) \max_{w \in [\gamma, \delta]} (\sqrt{|p'(w)|})
\end{equation}

and

\begin{equation}
(3.2.13) \quad \left| \frac{dw_{\epsilon n}(\xi)}{du_{\epsilon n}(\xi)} \right|_{\xi = \eta_n} \geq \frac{1}{2 \max_{w \in [\gamma, \delta]} (\sqrt{|p'(w)|})}.
\end{equation}

Figure 4

By (3.2.4), we take $\xi_n$ such that

\begin{equation}
(3.2.14) \quad u_{\epsilon n}(\xi_n) = u_{\epsilon n}(\zeta_n) - 2(\delta - \gamma) \max_{w \in [\gamma, \delta]} (\sqrt{|p'(w)|}) \geq u_+ - 2(\delta - \gamma) \max_{w \in [\gamma, \delta]} (\sqrt{|p'(w)|})
\end{equation}

For each $n$, there is a $\theta \in [\xi_n, \zeta_n]$ such that $u_{\epsilon n}(\theta) \geq u_{\epsilon n}(\xi_n)$ and

\[ \frac{dw_{\epsilon n}(\xi)}{du_{\epsilon n}(\xi)} \big|_{\xi = \theta} = \frac{w_{\epsilon n}(\zeta_n) - w_{\epsilon n}(\xi_n)}{u_{\epsilon n}(\zeta_n) - u_{\epsilon n}(\xi_n)}. \]

Substituting the denominator of the above by equation (3.2.14) and noticing that

\[ |w_{\epsilon n}(\zeta_n) - w_{\epsilon n}(\xi_n)| \leq \delta - \gamma, \]

we obtain

\[ \left| \frac{dw_{\epsilon n}(\xi)}{du_{\epsilon n}(\xi)} \big|_{\xi = \theta} \right| \leq \frac{1}{2 \max_{w \in [\gamma, \delta]} (\sqrt{|p'(w)|})}. \]

Thus, the following set is nonempty:

\begin{equation}
(3.2.15) \quad A := \left\{ \eta \in [\tau_n, \theta] \mid \left| \frac{dw_{\epsilon n}(\xi)}{du_{\epsilon n}(\xi)} \right| \leq \frac{1}{2 \max_{w \in [\gamma, \delta]} (\sqrt{|p'(w)|})} \text{ for } \xi \in [\eta, \theta] \right\}.
\end{equation}

A straightforward computation based on (3.4) shows that

\begin{equation}
(3.2.16) \quad \frac{d^2 w_{\epsilon n}(\xi)}{d u_{\epsilon n}^2} = \frac{-1}{\epsilon u_{\epsilon n}'(\xi)} \left[ 1 + p'(w_{\epsilon n}(\xi)) \left( \frac{dw_{\epsilon n}(\xi)}{du_{\epsilon n}(\xi)} \right)^2 \right].
\end{equation}
Hence, if $\xi \in A$, then

$$
(3.2.17) \quad \left| \frac{d^2 w_\epsilon(n)}{d u_\epsilon^2(n)}(\xi) \right| \geq \frac{1}{2 e u_\epsilon'(n)} \geq \frac{1}{2(p(\beta) - p(\alpha))}
$$

where we have used (3.2.11).

Now, we show that (3.2.12) and (3.2.13) hold at $\eta_n = \inf A$ and hence complete the proof of our claim. Indeed, by the definition of the set $A$,

$$
\frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) \bigg|_{\xi = \inf A} = \frac{1}{2 \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})}.
$$

In fact, we can show more i.e.

$$
(3.2.18) \quad \frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) \bigg|_{\xi = \inf A} = \frac{1}{2 \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})}.
$$

To see this, we note, from (3.2.16), that as long as $\xi \in A$ or $\xi = \inf A$,

$$
(3.2.19) \quad \frac{d}{d\xi} \left( \frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) \right) < 0.
$$

So, if

$$
\frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) \bigg|_{\xi = \inf A} < 0,
$$

then (3.2.19) implies that, for $\xi \in (\inf A - \mu, \inf A)$ for some $\mu > 0$,

$$
0 > \frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) > \frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) \bigg|_{\xi = \inf A}
$$

$$
= -\frac{1}{2 \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})},
$$

or in other words,

$$
\left| \frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) \right| \leq \frac{1}{2 \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})},
$$

for $\xi \in (\inf A - \mu, \inf A]$. This simply says that $(\inf A - \mu, \inf A] \subset A$ which is impossible. Thus (3.28) holds.

By using (3.2.18) and (3.2.17), we obtain

$$
\frac{1}{2 \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})} = \frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) \bigg|_{\xi = \inf A}
$$

$$
= \frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) \bigg|_{\xi = \theta} + \int_{\theta}^{\inf A} \frac{d^2 w_\epsilon(n)}{du_\epsilon^2(n)}(\xi) \frac{d u_\epsilon(n)}{d d\xi} d\xi
$$

$$
\geq -\left| \frac{d w_\epsilon(n)}{du_\epsilon(n)}(\xi) \bigg|_{\xi = \theta} \right| + \left| \frac{d u_\epsilon(n)}{du_\epsilon(n)}(\xi) \right| \frac{d^2 w_\epsilon(n)}{du_\epsilon^2(n)}(\xi) d\xi
$$

$$
\geq -\frac{1}{2 \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})} + \left| u_\epsilon(\theta) - u_\epsilon(\inf A) \right| \frac{1}{2(p(\beta) - p(\alpha))}.
$$
By virtue of (3.2.7), the above inequalities imply

\[
\begin{align*}
(3.2.20) \quad u_{\varepsilon_n}(\inf A) & \geq u_{\varepsilon_n}(\theta) - \frac{2(p(\beta) - p(\alpha))}{\max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})} \\
& \geq u_{\varepsilon_n}(\xi_n) - \frac{2(p(\beta) - p(\alpha))}{\max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})} \\
& \geq u_+ - 2(\delta - \gamma) \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|}) \\
& - \frac{2(p(\beta) - p(\alpha))}{\max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})}.
\end{align*}
\]

(3.2.18) and (3.2.20) prove our claim.

Similar analysis on (3.2.16) yields that

\[
\frac{dw_{\varepsilon_n}(\xi)}{du_{\varepsilon_n}(\xi)} \geq \frac{1}{2 \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})}
\]
or

\[
(3.2.21) \quad \frac{du_{\varepsilon_n}(\xi)}{dw_{\varepsilon_n}(\xi)} \leq 2 \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})
\]

for $\xi \in [\tau_n, \eta_n]$. Thus an estimation on the equation

\[
u_{\varepsilon_n}(\eta_n) - u_{\varepsilon_n}(\tau_n) = \int_{w_{\varepsilon_n}(\tau_n)}^{w_{\varepsilon_n}(\eta_n)} \frac{du_{\varepsilon_n}(\xi)}{dw_{\varepsilon_n}(\xi)} dw_{\varepsilon_n}(\xi)
\]

based on (3.2.21), (3.2.5) and (3.2.10) leads to

\[
u_{\varepsilon_n}(\tau_n) \geq u_{\varepsilon_n}(\eta_n) - 2(\delta - \gamma) \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})
\]

\[
\geq u_+ - 4(\delta - \gamma) \max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|}) - \frac{2(p(\beta) - p(\alpha))}{\max_{w \in [\gamma, \delta]}(\sqrt{|p'(w)|})},
\]

which is a contradiction to (3.2.4). \(\square\)

Remark. Above proof for Theorem 3.2.2 only needs the existence of numbers $\delta$ and $\gamma$ (cf. Fig. 1).

With the help of Theorem 3.1.1, 3.1.2, we can prove the following theorem by slightly modifying the idea of Dafermos [13].

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Theorem 3.2.3. If \( p(w) \) satisfies (3.0.6), then \( w_\varepsilon(\xi) \) are bounded uniformly in \( \varepsilon \).

Proof. We only prove that \( w_\varepsilon(\xi) \) are bounded from below uniformly in \( \varepsilon \). The uniform boundedness of \( w_\varepsilon(\xi) \) from above can be proved similarly.

Assume the contrary. Then there is a sequence of \( \{\varepsilon_n\} \) such that each \( w_{\varepsilon_n}(\xi) \) has a local minimum point at \( \xi = \tau_n \) with

\[
(3.2.22) \quad w_{\varepsilon_n}(\tau_n) \to -\infty \quad \text{as} \quad n \to \infty.
\]

Without loss of generality, we assume that

\[
(3.2.23) \quad \tau_n \leq 0.
\]

We know from Lemma 2.1 that \( w_{\varepsilon_n}(\xi) \) and \( u_{\varepsilon_n}(\xi) \) are decreasing on \( (-\infty, \tau_n) \). By integrating (3.4b), we obtain

\[
(3.2.25) \quad 0 \leq -\varepsilon w_{\varepsilon_n}'(-\infty) = -\int_{-\infty}^{\tau_n} \xi w_{\varepsilon_n}'(\xi)d\xi + u_- - u(\tau_n)
\]

It is easy to see from (3.2.23) that \( \xi w_{\varepsilon_n}'(\xi) \geq 0 \) on \( (-\infty, \tau_n) \) and hence

\[
(3.2.25) \quad 0 \leq \int_{-\infty}^{\tau_n} \xi w_{\varepsilon_n}'(\xi)d\xi \leq u_- - u(\tau_n) \leq u_- - u_*.
\]

For any \( \theta \leq \min(-1, \tau_n) \), we have

\[
\int_{-\infty}^{\theta} \xi w_{\varepsilon_n}'(\xi)d\xi \geq -\int_{-\infty}^{\theta} w_{\varepsilon_n}'(\xi)d\xi = w_- - w_{\varepsilon_n}(\theta)
\]

Thus (3.2.25) leads to

\[
(3.2.26) \quad w_{\varepsilon_n}(\theta) \geq w_- + u_* - u_-.
\]

It now remains to consider the case \( \tau_n \in (-1, 0) \). Using the mean value theorem, we can choose, for each \( n \), a \( \theta \in (-2, -1) \) such that

\[
(3.2.27) \quad |u_{\varepsilon_n}'(\theta)| \leq u^* - u_*.
\]

By integrating (3.4a) on \([\theta, \tau_n]\), we obtain

\[
(3.2.28) \quad p(w_{\varepsilon_n}(\tau_n)) = \varepsilon u_{\varepsilon_n}'(\tau_n) - \varepsilon u_{\varepsilon_n}'(\theta) + p(w_{\varepsilon_n}(\theta)) - \int_{\theta}^{\tau_n} \xi u_{\varepsilon_n}'(\xi)d\xi
\]

\[
\leq -\varepsilon u_{\varepsilon_n}'(\theta) + p(w_{\varepsilon_n}(\theta)) - \int_{\theta}^{\tau_n} \xi u_{\varepsilon_n}'(\xi)d\xi
\]

\[
\leq -|\varepsilon u_{\varepsilon_n}'(\theta)| + p(w_{\varepsilon_n}(\theta)) - \tau_n u_{\varepsilon_n}(\tau_n) + \theta u_{\varepsilon_n}(\theta) + \int_{\theta}^{\tau_n} u_{\varepsilon_n}(\xi)d\xi.
\]

In view of (3.2.26), (3.2.27) and the uniform boundedness of \( u_*(\xi) \) and \( \tau_n, \theta \), it follows that the right hand side of (3.2.28) is bounded uniformly in \( \varepsilon \). Thus, by virtue of Assumption 1, \( w_{\varepsilon_n}(\tau_n) \) are bounded from below uniformly in \( \varepsilon \) which contradicts (3.2.22). \( \square \)
Theorem 3.2.4. If (3.0.6) holds, then there exist solutions of (0.1) which are admissible according to the similarity viscosity admissibility criterion.

Proof. In view of Lemmas 3.1.1, 3.1.2, Theorem 3.2.1, 3.2.2, 3.2.3. \((u_\epsilon(\xi), w_\epsilon(\xi))\) have total variation bounded independently of \(\epsilon\). Helly’s theorem implies that \(u_\epsilon(\xi), w_\epsilon(\xi)\) possesses a subsequence which converges a.e. on \((-\infty, \infty)\) to a function \((u(\xi), w(\xi))\) of bounded variation, By Theorem 3.2 of Dafermos [13] or Proposition 3.2 of Slemrod [70], \((u(x/t), w(x/t))\) is a weak solution of (1.1). □

§3.3. Solutions Constructed in §3.0, §3.1, §3.2 are also Admissible by Viscosity-Capillarity Travelling Wave Criterion.

In this section, we study the relation between the similarity viscosity admissibility criterion and the viscosity-capillarity travelling wave criterion. Results in this section are contained in Fan [22, 24].

Consider \(u_\epsilon_n(\xi)\) (or \(w_\epsilon_n(\xi)\)) as a multivalued function of \(w\) (or \(u\)). Denote these functions by \(U_\epsilon_n(w)\) (or \(W_\epsilon_n(u)\)). An analysis based on Lemma 3.1.1, 3.1.2 shows that \(U_\epsilon_n(w)\) may be a two-valued function. For details about this, see Lemma 2.4 of Slemrod [70].

Lemma 3.3.1 [22]. \(\{U_\epsilon_n(w)\}\) has a subsequence which converges to a continuous curve \(U(w)\). Furthermore, \((u(\xi), w(\xi))\) lies on the curve \(U(w)\) for every \(\xi \in \mathbb{R}\).

Remark. \(U(w)\), like \(U_\epsilon_n(w)\), may be a two valued function.

For convenience, we parametrize the curve \(u = U(w)\) by \((U(s), W(s))\) where \(s\) is the length of the arc of \(u = U(w)\) joining \((u_-, w_-)\) and the point \((U(s), W(s))\). Since the curve \(u = U(w)\) does not intersect itself, the parametrization is bijective. In this kind of parametrization, \(s\) increases when \(\xi\) increases. We call the curve \((U(s), W(s))\) the base curve of the solution \((u(\xi), w(\xi))\).

Now, we study the discontinuities of \((u(\xi), w(\xi))\). Let \(\xi_0\) be a point of discontinuity of \((u(\xi), w(\xi))\). We use \(C_{\xi_0}\) to denote the portion of the base curve in the \((u, w)\)-plane that connects points \((u(\xi_0-), w(\xi_0-))\) and \((u(\xi_0+), w(\xi_0+))\). We fix \((\bar{u}, \bar{w})\in C_{\xi_0}\). We define, for \(n\) large, \(\xi_\epsilon_n(w; \bar{u}, \bar{w})\) to be the branch of the inverse function of \(w = w_\epsilon_n(\xi)\) for which

\[
(3.3.1) \quad u_\epsilon_n(\xi_\epsilon_n(w; \bar{u}, \bar{w})) \to \bar{u}
\]
as \(n \to \infty\). We further define, for \(n\) large, \(\xi_\epsilon_n, \bar{u}_\epsilon_n, \bar{w}_\epsilon_n\) by the relations

\[
(3.3.2) \quad \xi_\epsilon_n := \xi_\epsilon_n(\bar{w}) + \epsilon \zeta,
\]
\[
(3.3.3) \quad \bar{u}_\epsilon_n(\xi) := u_\epsilon_n(\xi_\epsilon_n),
\]
\[
(3.3.4) \quad \bar{w}_\epsilon_n(\xi) := w_\epsilon_n(\xi_\epsilon_n).
\]

After a modification of the proof of Proposition 3.4 of Dafermos [14], we obtain the following lemma:
Lemma 3.3.2. Fan [22]. Let $\xi_0$ be a point of discontinuity of $(u(\xi), w(\xi))$. For $(\hat{u}_{\epsilon_n}(\zeta), \hat{w}_{\epsilon_n}(\zeta))$ defined above, there is a subsequence of $\{\epsilon_n\}$, also denoted by $\{\epsilon_n\}$, such that

\begin{equation}
(3.3.5) \quad (\hat{u}_{\epsilon_n}(\zeta), \hat{w}_{\epsilon_n}(\zeta)) \to (\hat{u}(\zeta), \hat{w}(\zeta)) \in C^1(\mathbb{R}; \mathbb{R}^2) \quad \text{as } n \to \infty
\end{equation}

uniformly for $\zeta$ in a compact subset of $\mathbb{R}$. $(\hat{u}(\zeta), \hat{w}(\zeta))$ satisfies the following initial value problem:

\begin{align}
(3.3.6a) \quad & \frac{d\hat{u}(\zeta)}{d\zeta} = -\xi_0(\hat{u}(\zeta) - u(\xi_0-)) + p(\hat{w}(\zeta)) - p(w(\xi_0-)) \\
(3.3.6b) \quad & \frac{d\hat{w}(\zeta)}{d\zeta} = -\xi_0(\hat{w}(\zeta) - w(\xi_0-)) - [\hat{u}(\zeta) - u(\xi_0-)], \\
(3.3.6c) \quad & \hat{u}(0) = \bar{u} \quad \hat{w}(0) = \bar{w}.
\end{align}

Furthermore, $(\hat{u}(\zeta), \hat{w}(\zeta))$ lies on $C_{\xi_0}$.

Remark. By the definitions (3.3.1-3.3.4), we can see easily that as $\zeta$ increases, $(\hat{u}(\zeta), \hat{w}(\zeta))$ moves along $C_{\xi_0}$ in the direction from $(u(\xi_0-), w(\xi_0-))$ to $(u(\xi_0+), w(\xi_0+))$.

Remark. The travelling wave equations of the most common form of artificial viscosity (3.2) for shocks with speed $\xi_0$ is the same as (3.3.6a,b).

Lemma 3.3.3 [24]. The boundary value problem of (3.3.6a,b) and

\begin{align}
(3.3.7a) \quad & (\hat{u}(-\infty), \hat{w}(-\infty)) = (u_1, w_1), \\
(3.3.7b) \quad & (\hat{u}(+\infty), \hat{w}(+\infty)) = (u_2, w_2)
\end{align}

has a solution if and only if

\begin{equation}
(3.3.8) \quad u_2 = u_1 - \xi_0(w_2 - w_1)
\end{equation}

and the following boundary value problem has a solution:

\begin{align}
(3.3.9a) \quad & \frac{d^2\hat{w}}{d\zeta^2} = -2\xi_0 \frac{d\hat{w}(\zeta)}{d\zeta} - \xi_0^2(\hat{w}(\zeta) - w(\xi_0-)) - (p(\hat{w}(\zeta)) - p(w(\xi_0-))). \\
(3.3.9b) \quad & (\hat{w}(-\infty) = w_1, \quad \hat{w}(+\infty) = w_2.
\end{align}
Proof. Suppose \( w_1 \) and \( w_2 \) can be connected by a shock of speed \( \xi_0 \). Then (3.3.8), which is one of the the Rankine-Hugoniot conditions, is satisfied. Eliminating \( \hat{u}(\zeta) \) in (3.3.6a, b), we obtain (3.3.9a).

Suppose (3.3.9) has a solution \( \hat{w}(\zeta) \). We define

\[
(3.3.10) \quad \hat{u}(\zeta) = -\frac{d\hat{w}(\zeta)}{d\zeta} - \xi_0(\hat{w}(\zeta) - w_1) + u_1.
\]

(3.3.9) and (3.3.10) imply (3.3.6a,b) and (3.3.7). □

Comparing (3.3.6) with the traveling wave equation (0.6):

\[
(3.3.11) \quad A \frac{d^2 \hat{w}}{d\zeta^2} = -\xi_0 \frac{d\hat{w}(\zeta)}{d\zeta} - \xi_0^2 (\hat{w}(\zeta) - w_1) - (p(\hat{w}(\zeta)) - p(w_1)).
\]

we can see that (3.3.6a) is a special case of (3.3.11) when \( A = 1/4 \), In[Slemrod (1983), Slemrod proposed the viscosity-capillarity travelling wave criterion. We state his criterion in a more general setting as follows:

**Definition 3.3.4.** (i) We say \( w_1 \) and \( w_2 \) can be connected by a shock with speed \( \xi_0 \) if (3.3.11) and (3.3.7) has a solution.

(ii) A shock of speed \( \xi_0 \) with \((u_1, w_1)\) and \((u_2, w_2)\) on its sides, where \((u_1, w_1), (u_2, w_2)\) satisfy the Rankine-Hugoniot conditions at \( \xi_0 \), is admissible by traveling wave criterion if there are \( v_k, k = 1, 2, ..., n \in \mathbb{N}, \) and \( w_1 = v_1, v_n = w_2 \) such that \( v_k \) can be connected by a shock with speed \( \xi_0 \) to \( v_{k+1}, k=1,2, ..., n-1 \).

(iii) We say a solution \((u(\xi), w(\xi))\) of (1.1) is admissible by the traveling wave criterion if every discontinuity of \((u(\xi), w(\xi))\) is admissible in the sense of (ii).

**Corollary 3.3.5 [24].** If \( p(w) \) has the property that any straight line in \((w, p)\)-plane intersects the graph of \( p(w) \) at finite many points, then the solutions of (0.1) given by Theorem 3.2.4, which are admissible by the similarity viscosity criterion, are also admissible by the traveling wave criterion with \( A = 1/4 \). Hence, solutions of (0.1) admissible by the traveling wave criterion with \( A = 1/4 \) always exist.

Proof. Let \( \xi_0 \) be a point of discontinuity of \((u(\xi), w(\xi))\) given in Theorem 3.2.4. Without loss of generality, we assume \( w(\xi_0-) < w(\xi_0+) \). By Lemma 3.3.2, \( C_{\xi_0} \) satisfies (3.3.6). By the property of \( p(w) \) assumed in this theorem, we know that there are only finitely many points \((u, w)\) satisfy the Rankine-Hugoniot conditions at \( \xi_0 \)

\[
(3.3.11a) \quad -\xi_0(u(\xi_0-) - u) + p(w(\xi_0-)) - p(w) = 0,
\]

\[
(3.3.11b) \quad -\xi_0(w(\xi_0-) - w) - u(\xi_0-) + u = 0.
\]
Since $C_{ξ_0}$ can be oriented in the direction from $(u(ξ_0−), w(ξ_0−))$ to $(u(ξ_0+), w(ξ_0+))$, we can assume that the points on $C_{ξ_0}$ satisfying (3.3.11) are $(u_1, w_1), (u_2, w_2), ..., (u_n, w_n)$, which are ordered in the direction of $C_{ξ_0}$. Let $(u_1, w_1) ∈ C_{ξ_0}$ in (3.3.6c) be on the portion of $C_{ξ_0}$ between $(u_1, w_1)$ and $(u_2, w_2)$ and $(w_1(ξ), w_1(ξ))$ to be the corresponding solution of (3.3.6). By Lemma 3.3.2, 3.3.3, we can see that $w_1$ is connected to some $w_{j_1}$, $1 < j_1 ≤ n$. If $j_1 = n$ then our theorem is proved. If otherwise, we repeat the above procedure to see that $w_{j_1}$ is connected to $w_{j_2}$, $j_1 < j_2 ≤ n$. Repeating this process finite times, we can prove that $w_{j_k}$ can be connected to $w_{j_{k+1}}$, $k = 0, 1, ..., m ≤ n$, where $w_{j_0} = w(ξ_0−)$ and $w_{j_m} = w(ξ_0+)$. Thus, the first statement is proved. The last statement is a consequence of Theorem 3.2.4 and the first statement of this theorem.

§3.4. The uniqueness and stability of the solution of (0.1). In §3.0-§3.3, we established the existence of solutions of the Riemann problem (0.1) which are admissible according to the viscosity-capillarity travelling wave criterion. Now, it is natural to ask questions about the uniqueness and stability of these solutions, i.e., solutions of (0.1) which are admissible by viscosity-capillarity travelling wave criterion. Shearer [63] proved that for each $(u_−, w_−)$ with $w_− < m (w_− > M)$, there are $(u_+, w_+)$ with $w_+ < α (w_+ > β)$ such that (0.1) has two admissible centered solutions. The assumptions he used in [63] is

\begin{equation}
 p'' < 0 \text{ if } w < α , p'' > 0 \text{ if } w > β .
\end{equation}

Note that the initial data in Shearer’s results is in the same phase. If the initial datum are in different phase, i.e., $w_− < α$, $w_+ > β$, this kind of nonuniqueness does not hold, as claimed by the following theorem supplied by Fan [23]:

**Theorem 3.4.1:** Suppose $p(w)$ satisfies (3.4.1) and $w_− < α < β < w_+$. Then (ii) (0.1) has a unique solution within the class of centered wave solutions satisfying the viscosity-capillarity travelling wave criterion.

(ii) Let $(u(ξ), w(ξ))$ be the solution of (0.1) satisfying the viscosity-capillarity travelling wave criterion. For any $ε > 0$ and $γ > 0$, there is a $δ > 0$ such that if

\[
|u_− - u_+| + |u_+ - u_+| + |w_− - w_+| + |w_+ - w_+| < δ
\]

then

\[
\text{meas}\{ξ ∈ R \mid |u(ξ) - u(ξ)| + |w(ξ) - w(ξ)| ≥ ε\} < γ
\]

where $(u(ξ), w(ξ))$ is the solution of (0.1a, b), admissible by the viscosity-capillarity travelling wave criterion, with Riemann initial values $(u_−, w_−)$ and $(u_+, w_+)$, and ‘meas’ denotes the Lebesgue measure.

In §3.0-3.2, we constructed solutions of (0.1) as the $ε_n \to 0+$ limits of solutions of (3.0.4). Is it possible that different sequences $ε_n$ give us different limits and hence different
solutions of (0.1)? Since solutions we constructed in §3.0-3.3 satisfy the viscosity-capillarity travelling wave criterion also, we can see from Theorem 3.4.1 that the answer to above question is almost negative: The \( \epsilon \to 0^+ \) limit of solutions of (3.0.4) with (3.0.5) is unique, if \( p(w) \) satisfies (3.4.1). In this case, of course, the (ii) of Theorem 3.4.1 also holds for this solution.

§4. **Mixture State Solutions of (0.1).** For Van der Waals fluids, the region \( w < \alpha \) and \( w > \beta \) corresponds to the liquid and vapor phase respectively. We know that the only stationary phase boundary admissible according to the viscosity-capillarity travelling wave criterion is \((u_-, m), (u_-, M)\), where \( m \) and \( M \) are the Maxwell constants and \( u_- \) is arbitrary. Based on this fact, we shall construct a weak solution of (0.1) with a region of mixture of liquid and vapor as follows: Choose an arbitrary subset \( A \) of \( \mathbb{R}_- := \{ x \in \mathbb{R} : x < 0 \} \). We define

\[
(u(x,t), w(x,t)) := \begin{cases} 
(u_-, m) & \text{if } x \in A, \\
(u_-, M) & \text{if } x \in \mathbb{R} \setminus A, \\
(u_-, M) & \text{if } 0 < x/t < s, \\
(u_+, w_+) & \text{if } s < x/t, 
\end{cases}
\]

where \((u_-, M)\) and \((u_+, w_+)\) satisfy the Rankine-Hugoniot conditions with speed \( s > 0 \). We claim that \((u(x,t), w(x,t))\) defined by (4.1) is a weak solution of (0.1). We assume that \( p(w) \) is convex for \( w \geq M \). Then, by Theorem 3.3 of [40], the jump discontinuity at \( \xi = s > 0 \) is admissible by the viscosity-capillarity travelling wave criterion For \( x < 0 \), \( u(x,t) \) and \( p(w(x,t)) \) are constants, and \( w_t(x,t) = 0 \). Thus, \((u(x,t), w(x,t))\) defined by (4.1) is indeed a weak solution of (0.1). We notice that the region \( x < 0 \) consists of mixture of liquid and vapor. Similarly, we can construct a weak solution of (0.1) with \( x > 0 \) being the region of mixture.

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