A NUMERICAL TRANSMUTATION APPROACH
FOR UNDERWATER SOUND PROPAGATION

By

Robert P. Gilbert
and
Yongzhi Xu

IMA Preprint Series # 907
December 1991
A Numerical Transmutation Approach for Underwater Sound Propagation

Robert P. Gilbert \(^1\) and Yongzhi Xu\(^2\)

1. Introduction

In [8] we have described a procedure for solving the submersible identification problem in three steps. That is, for the first approximation we shall concentrate on an ocean which may be taken to be a slab of thickness \(h\), \(R^3_h\), and has a constant index of refraction. This is reasonable as the most difficult aspects of our problem come from the way the ocean surface interferes with the propagation of sound off an arbitrary submersible.[6], [8] Because of the difficulties introduced by the ocean boundaries, analytical computation of the far field must be based on some sort of approximation scheme such as using a parabolic approximation, which tends to destroy vertical resolution; or using a truncated form of the modal expansion for the Green's function (propagator):

\[
G(r, \theta, z; \rho, \phi, \zeta) = \frac{i\pi}{2} \sum_{n=0}^{\infty} \frac{\phi_n(z)\phi_n(\zeta)}{\|\phi_n\|^2} H_0(ka_n | re^{i\theta} - \rho e^{i\phi}|). \tag{1.1}
\]

Here the \(a_n = [1 - (n + 1/2)^2(\pi/kh)^2]^{-1/2}\) are the modal eigenvalues and \(\phi_n(z) = \sin(k(1 - a_n^2)^{1/2}z)\).[1] By using the propagator \(G(r, z; \rho, \zeta)\) (1.1), the Green's integral representation, and the asymptotic behavior of the Hankel functions, Gilbert and Xu[5], [6] have shown that the acoustic pressure has the asymptotic expansion

\[
p(r, z, \theta) = \frac{1}{r^{1/2}} \sum_{n=0}^{\infty} e^{ikn} \sum_{m=0}^{\infty} \frac{f_{nm}(z, \theta)}{r^m}.
\]

---

\(^1\)Department of Mathematical Sciences, University of Delaware, Newark, DE 19716

The first author's research was supported in part by the National Science Foundation through grant BCS-9002868 and DMS-9002837

\(^2\)Institute for Mathematics and its Applications, University of Minnesota, 514 Vincent Hall, 206 Church Street SE, Minneapolis, MN 55455

The second author's research was supported in part by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation, the Minnesota Supercomputer Institute and the Alliant Techsystem Co.
\[
\sum \frac{1}{r^{1/2}} \sum_{m=0}^{\infty} \frac{F_m(r, z, \theta)}{r^m},
\]

where
\[
F_m(r, z, \theta) := \sum_{n=0}^{N} e^{i k_0 r} F_{nm}(z, \theta).
\]

The modes \(\phi_n(z)\) for \(n > N\) do not propagate, but rather decay exponentially and hence are not included in the sum (1.3). We refer to the term
\[
F_0(r, z, \theta) := \sum_{n=0}^{N} e^{i k_0 r} F_{n0}(z, \theta)
\]
as the finite-ocean far field pattern, which we also write in an array form as
\[
F(z, \theta) := [f_{00}(z, \theta), f_{10}(z, \theta), \ldots, f_{N0}(z, \theta)].
\]

Gilbert and Xu [5],[6] show that it has a modal representation of form
\[
\nu(x, z) := \sum_{n=0}^{N} \frac{\phi_n(\zeta)}{\|\phi_n\|^2 \sqrt{a_n}} \int_{\partial Z_1} g(z, \phi) \phi_n(z) e^{i k_0 \phi} d\sigma_\xi,
\]

where \(g(z, \phi)\) is called a propagating Herglotz kernel.

Having obtained an approximate far field pattern for various wave numbers \(k\) and various \(z\) dependencies in the incident "plane waves", in step two we may solve the inverse problem. Recall that the "plane waves" have modal components in the \(z\) direction. If we are not considering axially symmetric solutions then for each modal component and for each \(k\) we use \(2n+1\) incoming waves with directions in the range coordinates
\[
\alpha_2 = [\cos(2\pi j/(2n+1)), \sin(2\pi j/(2n+1))], \quad j = 0, 1, 2, \ldots, 2n
\]

Now let \(F_{nm}^j \quad 0 \leq n \leq N, \quad -n \leq m \leq n\) be the coefficients of the spherical harmonic approximation of far field pattern \(F^j\) generated by the plane wave with the direction \(\alpha_j\). By expanding the propagating Herglotz kernels, and the parametric representation of the submersible's surface \(\rho = f(\theta, \phi)\) in terms of surface harmonics we are led to consider a minimization problem of the form
\[
\mu(F) = \min_{(k^2, g, \rho)} \left\{ \sum_{j=1}^{2n} \left| \int_{\partial Z_1} F(z, \phi; k, \alpha_2^{(j)}) g(z, \phi) dz d\phi \right|^2 \right\}
\]
\[ + \int_0^{2\pi} \int_0^\pi \left| T^{-1}V(f(\phi, \theta), \phi, \theta) \right|^2 \sin^2 \theta d\theta d\phi, \tag{1.7} \]

where
\[ V(y) := \sqrt{\frac{2}{ik\pi}} \int_{\partial Z_\delta} g(z, \phi) G(x, z, \xi, \zeta) d\sigma_\xi \]

is the propagating (entire) Herglotz function, and \( T^{-1} \) is the inverse transformation which relates the coefficients of the starting field in spherical coordinates to those in cylindrical coordinates.

Other minimization problems might be considered instead, for example see Xu [10].

In step three, we consider the case with an index of refraction which is depth dependent. We must make certain alterations in step one and step two. In this step we need to find a complete system of the solutions to replace the complete family (1.9) by another one, which must be a solution of the depth-dependent Helmholtz equation

\[ \Delta u + k^2 n^2(z) u = 0. \tag{1.8} \]

Such a family may be generated by means of the transmutation

\[ \Xi \Omega := \Omega(r, z, \phi) + \int_{z=h}^{z} K(z, s) \Omega(r, s, \phi) ds, \]

where the kernel \( K(z, s) \) satisfies the Gelfand-Levitan equation

\[ \frac{\partial^2 K}{\partial z^2} - \frac{\partial^2 K}{\partial s^2} + k^2[n^2(z) - 1] K = 0, \tag{1.9} \]

and the characteristic conditions [4]

\[ 2 \frac{\partial}{\partial z} K(z, z) + k^2[n^2(z) - 1] = 0, \tag{1.10} \]

\[ 2 \frac{\partial}{\partial z} K(z, -z + 2h) + k^2[n^2(z) - 1] = 0, \tag{1.11} \]

that characterize a hard ocean boundary at \( z=h \). The functions \( \Omega(r, z) \) are solutions of (1.8) with \( n(z)=1 \) and boundary conditions. In order to obtain the propagating far-field patterns, we represent the propagating solution by

\[ \sum_{n=0}^{N_0} \sum_{m=-M}^{M} \beta_{mn} \phi_n(z) H^{(1)}_m(\kappa a_n r) e^{im\theta}, \tag{1.12} \]
where the $\phi_n(z)$ are the modal solutions (eigenfunctions of the separated z-equation) for the variable index $n(z)$:[4], [6]

$$q_{zz} + k^2(n^2(z) - a^2)q = 0,$$  \hspace{1cm} (1.13)

and the boundary conditions

$$q_z(ka, h) = 0,$$  \hspace{1cm} (1.14)

and

$$q(ka, 0) = 0.$$  \hspace{1cm} (1.15)

It is clear that if we want to study the wave propagation and its far-field patterns, we must know the normal modes in the stratified ocean. Therefore, as an important component of step three, we need to construct the $\phi_n(z)$, $n = 0, 1, 2, ...$ numerically when the refraction index $n$ is a function of depth $z$.

2 Numerical Transmutation Method

As discussed in [4], a solution of (1.13) \sim (1.15) is given by

$$q_1(ka, z) = p_1(ka, z) + \int_{h}^{z} K(z, s)p_1(ka, s)ds$$  \hspace{1cm} (2.1)

where $K(z, s)$ is a solution of (1.9) \sim (1.11) and $p(ka, z)$ satisfies

$$p_{zz} + k^2(1 - a^2)p = 0,$$  \hspace{1cm} (2.2)

and the boundary condition

$$p_z(ka, h) = 0.$$  \hspace{1cm} (2.3)

Notice that $q_1(ka, z)$ will satisfy the boundary condition (1.14) at $z=h$ regardless of the value of $a$ because $p_1(ka, z)$ has the same property and our transmutation preserves boundary conditions at $z=h$. The boundary condition (1.15) at $z=0$ is then used to determine the values of $a_n$ that are roots of

$$p_1(ka, 0) + \int_{h}^{0} K(0, s)p_1(ka, s)ds = 0.$$  \hspace{1cm} (2.4)
Since \( n(z) \) may be \( \geq 1 \) or \( \leq 1 \) in general, we may have that \( a_n^2 \geq 1 \) or \( a_n^2 \leq 1 \). In fact, there is a documented example [13] where

\[
n^2(z) = 1 - 0.1\cos\left(\frac{2\pi z}{h}\right),
\]

where \( h = 2000, k = 8\pi/2000 \), for which we have

\[
(ka_0)^2 = 1.6555156 \times 10^{-4},
\]

\[
(ka_1)^2 = 1.5101910 \times 10^{-4},
\]

that is,

\[
a_0^2 = 1.0483675 > 1,
\]

\[
a_1^2 = 0.9563396 < 1.
\]

Therefore, \( p_1(ka, z) \) may generally be given by either

\[
p_1(ka, z) := \cos[k(1 - a^2)^{1/2}(h - z)], \text{ if } a < 1,
\]

(2.5)

or

\[
p_1(ka, z) := \cosh[k(a^2 - 1)^{1/2}(h - z)], \text{ if } a > 1.
\]

(2.6)

For \( a^2 < 1 \), let \( \lambda = k(1 - a^2)^{1/2}h \), (2.4) becomes

\[
\cos(\lambda) + \int_h^0 K(0, s)\cos[\lambda(1 - \frac{s}{h})]ds = 0.
\]

(2.7)

For \( a^2 > 1 \), let \( \lambda = k(a^2 - 1)^{1/2}h \), (2.4) becomes

\[
\cosh(\lambda) + \int_h^0 K(0, s)\cosh[\lambda(1 - \frac{s}{h})]ds = 0.
\]

(2.8)

Obviously, in order to find all propagating eigenvalues we have to find all \( \lambda \geq 0 \) such that either (2.7) or (2.8) is satisfied. To avoid this inconvenience, we can rewrite the equation (1.13) as

\[
q_{zz} + k^2[(n^2(z) - n_0^2) - (a^2 - n_0^2)]q = 0,
\]

(2.9)

where \( n_0^2 > 0 \) can be chosen so that \( a^2 < 1 + n_0^2 \).
Let $\hat{n}^2(z) = n^2 - n_0^2$, $\hat{a}^2 = a^2 - n_0^2$, the solution of (2.9) can be written as

$$q(k\hat{a}, z) = p(k\hat{a}, z) + \int_h^z \hat{K}(z, s)p(k\hat{a}, s)ds$$  \hspace{1cm} (2.10)

where $\hat{K}(z, s)$ satisfies

$$\frac{\partial^2 \hat{K}}{\partial z^2} - \frac{\partial^2 \hat{K}}{\partial s^2} + k^2[n^2(z) - n_0^2 - 1]\hat{K} = 0,$$  \hspace{1cm} (2.11)

and the characteristic conditions [4]

$$2\frac{\partial}{\partial z}\hat{K}(z, z) + k^2[n^2(z) - n_0^2 - 1] = 0,$$  \hspace{1cm} (2.12)

$$2\frac{\partial}{\partial z}\hat{K}(z, -z + 2h) + k^2[n^2(z) - n_0^2 - 1] = 0.$$  \hspace{1cm} (2.13)

The eigenvalues are the roots of equation (2.7) with $\lambda = k(1 - \hat{a}^2)^{1/2}h$. We can normalize the problem by letting $\hat{z} = z/h$, $\hat{s} = s/h$, $\xi = (\hat{z} + \hat{s} - 2)/2$, and $\eta = (\hat{z} - \hat{s})/2$, $k = k_0h$; then (2.11) \sim (2.13) can be reduced to

$$\frac{\partial^2 M}{\partial \xi \partial \eta} + k^2[n^2(1 - \xi - \eta) - n_0^2 - 1]M = 0, \hspace{1cm} 0 \leq \xi + \eta \leq 1,$$  \hspace{1cm} (2.14)

$$M(\xi, 0) = \frac{1}{2}k^2\int_0^\xi [n^2(1 - t) - n_0^2 - 1]dt, \hspace{1cm} 0 \leq \xi \leq 1,$$  \hspace{1cm} (2.15)

$$M(0, \eta) = \frac{1}{2}k^2\int_0^\eta [n^2(1 - t) - n_0^2 - 1]dt, \hspace{1cm} 0 \leq \eta \leq 1.$$  \hspace{1cm} (2.16)

We use a finite difference method developed in [2] to compute $M(\xi, \eta)$ as well as $K(z, s)$. Following Aziz and Hubbard [2], we define for $0 \leq m + n \leq L$, where $L$ is the number of interpolation points and $dz = 1/L$, we see that

$$M(mdz - \frac{1}{2}dz, ndz) = \frac{1}{2}[M(mdz, ndz) + M(mdz - dz, ndz)],$$

$$M(mdz, ndz - \frac{1}{2}dz) = \frac{1}{2}[M(mdz, ndz) + M(mdz, ndz - dz)],$$

$$M(mdz - \frac{1}{2}dz, ndz - \frac{1}{2}dz) =$$

$$\frac{1}{2}[M(mdz - \frac{1}{2}dz, ndz) + M(mdz - \frac{1}{2}dz, ndz - dz)]$$

$$= \frac{1}{2}[M(mdz, ndz - \frac{1}{2}dz) + M(mdz - dz, ndz - \frac{1}{2}dz)]$$
\[
\frac{1}{4}[M(mdz, ndz) + M(mdz - dz, ndz)] + M(mdz, ndz - dz) + M(mdz - dz, ndz - dz),
\]
\[
M_{\xi\eta}(mdz - \frac{1}{2}dz, ndz - \frac{1}{2}dz)
\]
\[
= (dz)^{-2}[M(mdz, ndz) - M(mdz - dz, ndz)
- M(mdz, ndz - dz) + M(mdz - dz, ndz - dz)],
\]

where
\[
c_{mn} = k^2[n^2(Ndz - (m + n - 1)dz) - n_0^2 - 1],
\]

Now (2.14) may be approximated by
\[
(dz)^{-2}[M(mdz, ndz) - M(mdz - dz, ndz) - M(mdz, ndz - dz)
+ M(mdz - dz, ndz - dz)] + c_{mn}\frac{1}{4}[M(mdz, ndz) + M(mdz - dz, ndz)
+ M(mdz, ndz - dz) + M(mdz - dz, ndz - dz)] = 0.
\]

Denote \( u_{mn} = M(mdz, ndz) \), then
\[
u_{mn} = \frac{4 - c_{mn}(dz)^2}{4 + c_{mn}(dz)^2}u_{m(n-1)} + \frac{4 - c_{mn}(dz)^2}{4 + c_{mn}(dz)^2}u_{(m-1)n} - u_{(m-1)(n-1)}, \tag{2.17}
\]

with
\[
u_{m0} = f(mdz - \frac{1}{2}dz), \tag{2.18}
\]

and
\[
u_{0n} = g(ndz - \frac{1}{2}dz), \tag{2.19}
\]

where
\[
f(\xi) = \frac{k^2}{2} \int_0^\xi [n^2(1 - t) - n_0^2 - 1]dt, \quad 0 \leq \xi \leq 1,
\]
\[
g(\eta) = \frac{k^2}{2} \int_0^\eta [n^2(1 - t) - n_0^2 - 1]dt, \quad 0 \leq \eta \leq 1.
\]

Similar to [2], we may prove that the global error of this scheme is \( O((1/L)^2) \).
We can also use finite difference approximation for the equation (2.1) as well as (2.12)(2.13). Let \( dz = ds = 1/l \),

\[
K_{zz}(idz, ids) = \frac{1}{dz^2}(K_{i+1,j} - 2K_{i,j} + K_{i-1,j}),
\]

\[
K_{ss}(idz, ids) = \frac{1}{dz^2}(K_{i,j+1} - 2K_{i,j} + K_{i,j+1}),
\]

\[
K(idz, ids) = \frac{1}{2}(K_{i,j+1} + K_{i,j-1}),
\]

then (2.11) follows

\[
K_{i+1,j} + K_{i-1,j} - K_{i,j+1} - K_{i,j-1} + (dz)^2[n^2(i) - n_0 - 1](K_{i,j+1} + K_{i,j-1}) = 0, \quad (2.20)
\]

\[
0 \leq i + j \leq 2L, \quad j \geq i \geq 0
\]

and

\[
K_{i,i} = -\frac{k^2(dz)}{2} \left[ \frac{n^2(i) + n^2(0) - 2}{2} + \sum_{j=1}^{i-1}(\hat{n}^2(j) - 1) \right],
\]

\[
K_{i,2L-i} = -\frac{k^2(dz)}{2} \left[ \frac{n^2(i) + n^2(0) - 2}{2} + \sum_{j=1}^{i-1}(\hat{n}^2(j) - 1) \right],
\]

for \( i = 1, 2, \ldots, 2L \).

3 Numerical examples of normal modes

In this section we present two examples and compare them with typical normal mode computation results.

Example 1: The results presented in this example are based on an idealized ocean model with a symmetric sound channel. The index of refraction is

\[
n^2(z) = 1 - 0.1\cos\left(\frac{2\pi z}{h}\right),
\]

where the depth of ocean \( h = 2000 \), and \( k = 8\pi/2000 \).
Table 1: Eigenvalues ($\times 10^{-3}$) for index of refraction $n^2(z) = 1 - 0.1\cos(2\pi z/h)$.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Documented results</th>
<th>Transmutation (fortran)</th>
<th>Transmutation error(f)</th>
<th>Transmutation (MC-Matlab)</th>
<th>Transmutation error(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.165552</td>
<td>0.165543</td>
<td>0.000009</td>
<td>0.165553</td>
<td>0.000001</td>
</tr>
<tr>
<td>2</td>
<td>0.151019</td>
<td>0.151031</td>
<td>0.000012</td>
<td>0.151062</td>
<td>0.000043</td>
</tr>
<tr>
<td>3</td>
<td>0.139981</td>
<td>0.139976</td>
<td>0.000005</td>
<td>0.140015</td>
<td>0.000034</td>
</tr>
<tr>
<td>4</td>
<td>0.126508</td>
<td>0.126466</td>
<td>0.000042</td>
<td>0.126436</td>
<td>0.000072</td>
</tr>
<tr>
<td>5</td>
<td>0.107284</td>
<td>0.107263</td>
<td>0.000021</td>
<td>0.107259</td>
<td>0.000025</td>
</tr>
<tr>
<td>6</td>
<td>0.082841</td>
<td>0.082835</td>
<td>0.000006</td>
<td>0.082869</td>
<td>0.000028</td>
</tr>
<tr>
<td>7</td>
<td>0.053359</td>
<td>0.053364</td>
<td>0.000005</td>
<td>0.053436</td>
<td>0.000077</td>
</tr>
<tr>
<td>8</td>
<td>0.018893</td>
<td>0.018910</td>
<td>0.000017</td>
<td>0.019030</td>
<td>0.000137</td>
</tr>
</tbody>
</table>

Figure 1: The first eight normal modes for example 1.

Figure 1: First eight normal modes for example 1
Example 2 This example is an idealized ocean model for an ocean sound channel with its axis at one-fifth the depth of the ocean. The index of refraction is

\[ n^2(z) = 1 + 0.1 \left[ \sin\left(\frac{5\pi}{4h} z\right) - \frac{2\sqrt{2}}{5\pi} \right] \]

where the depth of ocean \( h = 5000 \), and \( k = 8\pi/5000 \).

Table 2: Eigenvalues \((\times10^{-4})\) for index of refraction \(n^2(z) = 1 + 0.1\sin(5\pi z/4h + \pi/4) - 0.2\sqrt{2}/5\pi\).

<table>
<thead>
<tr>
<th>Mode</th>
<th>Documented results</th>
<th>Transmutation (fortran)</th>
<th>Transmutation error(f)</th>
<th>Transmutation (MC-Matlab)</th>
<th>Transmutation error(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.260430</td>
<td>0.260608</td>
<td>0.000178</td>
<td>0.260870</td>
<td>0.000440</td>
</tr>
<tr>
<td>2</td>
<td>0.237756</td>
<td>0.238109</td>
<td>0.000353</td>
<td>0.238656</td>
<td>0.000900</td>
</tr>
<tr>
<td>3</td>
<td>0.221614</td>
<td>0.221957</td>
<td>0.000343</td>
<td>0.222472</td>
<td>0.000858</td>
</tr>
<tr>
<td>4</td>
<td>0.202254</td>
<td>0.202403</td>
<td>0.000149</td>
<td>0.202586</td>
<td>0.000332</td>
</tr>
<tr>
<td>5</td>
<td>0.171646</td>
<td>0.171741</td>
<td>0.000095</td>
<td>0.171872</td>
<td>0.000226</td>
</tr>
<tr>
<td>6</td>
<td>0.132547</td>
<td>0.132620</td>
<td>0.000073</td>
<td>0.132760</td>
<td>0.000213</td>
</tr>
<tr>
<td>7</td>
<td>0.085379</td>
<td>0.085445</td>
<td>0.000066</td>
<td>0.085621</td>
<td>0.000242</td>
</tr>
<tr>
<td>8</td>
<td>0.030234</td>
<td>0.030303</td>
<td>0.000069</td>
<td>0.030540</td>
<td>0.000306</td>
</tr>
</tbody>
</table>

Figure 2: The first eight normal modes for example 2.
4 Construction of propagating waves in perturbed stratified ocean

In this section we combine the normal modes produced by the numerical transmutation and a finite element method to obtain the propagating waves in a perturbed stratified ocean. Namely, we want to solve the following problem

\[ \Delta_2 u + k^2 n(x, z)u = \delta(z - z_0)\delta(x - x_0), \text{ in } \mathbb{R}_0^2 := \{(x, z) \in \mathbb{R}^2 | 0 < z < h\}, \quad (4.1) \]

\[ u = 0, \text{ as } z = 0, \quad (4.2) \]

\[ \frac{\partial u}{\partial z} = 0, \text{ as } z = h, \quad (4.3) \]

\[ u \text{ outgoing as } |x| \to \infty, \quad (4.4) \]

where \( \Delta_2 \) is the two-dimensional Laplacian operator, \( k > 0 \) the wave number, \( h > 0 \) the depth of the ocean and \( n(x, z) > 0 \) the refraction index. Here we assume that \( n(x, z) \) is bounded and piecewise continuous, and furthermore, that it depends only on the depth \( z \) for \((x, z)\) outside of a bounded region. A coupled finite element and normal mode methods for scattering in such a perturbed stratified ocean has been presented in [15]; however, there were only numerical examples for a perturbed homogeneous ocean presented there. In what follows, we will briefly present the idea of this method with the emphases on the role of transmutation. Then some numerical results for propagating waves in a perturbed stratified ocean will be presented.

It is well-known that the Green's function for a stratified ocean can be obtained by separation of variables, i.e.,

\[ G_0(z, x; z_0, x_0) = \sum_{n=0}^{\infty} \frac{e^{ik_a |x - x_0|}}{2ik_a n} \phi_n(z)\phi_n(z_0). \quad (4.5) \]

In order to construct the Green's function for (4.1)-(4.4), we consider the following two boundary value problems.

1. Interior problem:

\[ \Delta_2 u_i + k^2 n(x, z)u_i = 0, \text{ in } \Omega_a := \{(x, z) \in \mathbb{R}^2 | 0 < z < h, |x| < a\}, \quad (4.6) \]

\[ u_i = 0, \text{ as } z = 0, \quad (4.7) \]

\[ \frac{\partial u_i}{\partial z} = 0, \text{ as } z = h, \quad (4.8) \]

\[ u_i \text{ outgoing as } |x| \to \infty, \quad (4.9) \]
\[ \frac{\partial u_i}{\partial z} = 0, \text{ as } z = h, \]  
\[ (4.8) \]
\[ \frac{\partial u_i}{\partial |x|} + i\eta u_i = \lambda, \text{ as } (x, z) \in \Gamma_a := \{-a, a\} \times [0, h]. \]  
\[ (4.9) \]

2. Exterior problem:
\[ \Delta_2 u_o + k^2 n(z) u_o = 0, \text{ in } \mathbb{R}_b^2 \setminus \Omega_a, \]  
\[ (4.10) \]
\[ u_o = 0, \text{ as } z = 0, \]  
\[ (4.11) \]
\[ \frac{\partial u_o}{\partial z} = 0, \text{ as } z = h, \]  
\[ (4.12) \]
\[ \frac{\partial u_o}{\partial |x|} + i\eta u_o = \lambda, \text{ as } (x, z) \in \Gamma_a := \{-a, a\} \times [0, h]. \]  
\[ (4.13) \]
\[ u_o \text{ outgoing as } |x| \to \infty, \]  
\[ (4.14) \]

Where \( a \) and \( \eta \) are chosen positive constants. \( \lambda \) is a complex function to be determined.

The interior problem defines a mapping \( G^i \) from \( H^{-\frac{1}{2}}(\Gamma_a) \) to \( H^1(\Omega_a) \), and the exterior problem defines a mapping \( G^o \) from \( H^{-\frac{1}{2}}(\Gamma_a) \) to \( H^1(\mathbb{R}_b^2 \setminus \Omega_a) \), where \( H^1(\mathbb{R}_b^2 \setminus \Omega_a) \) is defined as that \( u \in H^1_{loc}(\mathbb{R}_b^2 \setminus \Omega_a) \) and satisfies the outgoing radiation condition. Symbolically, we can denote
\[ u_i = G^i\lambda, \quad u_o = G^o\lambda. \]  
\[ (4.15) \]

Define
\[ G(x, z, x_0, z_0) = \begin{cases} G^i\lambda + G^i \left( \frac{\partial G_o}{\partial |x|} + i\eta G_o \right) & (x, z) \in \Omega_a, \\ G^o\lambda + G_0(x, z, x_0, z_0) & (x, z) \in \mathbb{R}_b^2 \setminus \Omega_a, \end{cases} \]  
\[ (4.16) \]
where \( \lambda \) satisfies a boundary operator equation
\[ (G^i - G^o)\lambda = G_0 - G^i \left( \frac{\partial G_o}{\partial |x|} + i\eta G_o \right), \text{ on } \Gamma_a. \]  
\[ (4.17) \]

**Theorem 4.1:** \( G(x, z, x_0, z_0) \) defined by (4.16) is the Green's function of (4.1)-(4.4).

Proof: A straight-forward calculation shows that \( G(x, z, x_0, z_0) \) satisfies (4.1) for \( (x, z) \in \mathbb{R}_b^2 \setminus \Gamma_a \). But (4.17) implies that \( G(x, z, x_0, z_0) \) is smooth as \( (x, z) \) across \( \Gamma_a \). The other conditions are obviously satisfied.

In a waveguide, there may be some wavenumbers which are eigenvalues of the boundary value problem. For such wavenumbers, the solution of (4.17) may not be unique. However, we have
Theorem 4.2: If the problem (4.1)-(4.4) admits a unique Green's function, then the solution of (4.17) is unique.

To approximate \( G(x, z, x_0, z_0) \) numerically, we construct the interior solution by the Finite Element Method, and the outer solution by a normal mode solution. For details about the FEM procedure, the reader could refer to [15]. For the exterior solution, [15] discussed the construction for a homogeneous medium where the eigenfunctions are trigonometric functions. The numerical transmutation provides a convenient approach to construct the eigenfunctions for a stratified medium. Therefore, using the eigenfunctions obtained in the above, we can solve (4.17) in the same way as we did in [15].

The following graphs show a numerical example from our computations.

**Figure 3:** The sound source is located at \( (x_0, z_0) = (-60, 0.1) \), and the refraction index is stratified:

\[
    n^2(z) = 1 - 0.1\cos(2\pi z/h).
\]

(4.18)

The Green's function, computed from (4.5) with truncation in \( n = 30 \), is plotted in figure 3.

![Figure 3: Point-source propagating wave in stratified ocean](image)
Figure 4: The sound source is located at \((x_0, z_0) = (-60, 0.1)\), and the refraction index is a perturbed stratified function:

\[
    n(x, z) = \begin{cases} 
        1 - 0.1 \cos(2\pi z/h) & \text{if } x^2 + (z - 0.6)^2 > 0.04^2 \\
        2 & \text{if } x^2 + (z - 0.6)^2 \leq 0.04^2
    \end{cases} \tag{4.19}
\]

The Green's function, constructed by (4.17), is plotted in figure 4.

Figure 4: Propagating wave in perturbed stratified ocean
Figure 5: This figure shows the difference between the fields in figure 3 and figure 4, i.e., the scattered wave by the inclusion. This scattered wave contains information from the inhomogeneity which is important in the further study of inverse problems.

Figure 5: Scattered wave by the inclusion
References


<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>818</td>
<td>Avner Friedman and Bei Hu</td>
<td>The Stefan problem with kinetic condition at the free boundary</td>
</tr>
<tr>
<td>819</td>
<td>M.A. Grinfeld</td>
<td>The stress driven instabilities in crystals: mathematical models and physical manifestations</td>
</tr>
<tr>
<td>820</td>
<td>Bei Hu and Lihe Wang</td>
<td>A free boundary problem arising in electrophotography: solutions with connected toner region</td>
</tr>
<tr>
<td>821</td>
<td>Yongzhi Xu, T. Craig Poling, and Trent Brundage</td>
<td>Direct and inverse scattering of time harmonic acoustic waves in an inhomogeneous shallow ocean</td>
</tr>
<tr>
<td>822</td>
<td>Steven J. Altschuler</td>
<td>Singularities of the curve shrinking flow for space curves</td>
</tr>
<tr>
<td>823</td>
<td>Steven J. Altschuler and Matthew A. Grayson</td>
<td>Shortening space curves and flow through singularities</td>
</tr>
<tr>
<td>824</td>
<td>Tong Li</td>
<td>On the Riemann problem of a combustion model</td>
</tr>
<tr>
<td>825</td>
<td>L.A. Peletier &amp; W.C. Troy</td>
<td>Self-similar solutions for diffusion in semiconductors</td>
</tr>
<tr>
<td>827</td>
<td>Minkyu Kwak</td>
<td>Finite dimensional description of convective reaction-diffusion equations</td>
</tr>
<tr>
<td>828</td>
<td>Minkyu Kwak</td>
<td>Finite dimensional inertial forms for the 2D Navier–Stokes equations</td>
</tr>
<tr>
<td>829</td>
<td>Victor A. Galaktionov and Sergey A. Posashkov</td>
<td>On some monotonicity in time properties for a quasilinear parabolic equation with source</td>
</tr>
<tr>
<td>830</td>
<td>Victor A. Galaktionov</td>
<td>Remark on the fast diffusion equation in a ball</td>
</tr>
<tr>
<td>831</td>
<td>Hi Jun Choe and Lihe Wang</td>
<td>A regularity theory for degenerate vector valued variational inequalities</td>
</tr>
<tr>
<td>832</td>
<td>Vladimir I. Oliker and Nina N. Uraltseva</td>
<td>Evolution of nonparametric surfaces with speed depending on curvature, II. The mean curvature case.</td>
</tr>
<tr>
<td>833</td>
<td>S. Kamin and W. Liu</td>
<td>Large time behavior of a nonlinear diffusion equation with a source</td>
</tr>
<tr>
<td>834</td>
<td>Shoshana Kamin and Juan Luis Vazquez</td>
<td>Singular solutions of some nonlinear parabolic equations</td>
</tr>
<tr>
<td>835</td>
<td>Bernhard Kawohl and Robert Kersner</td>
<td>On degenerate diffusion with very strong absorption</td>
</tr>
<tr>
<td>836</td>
<td>Avner Friedman and Fernando Reitich</td>
<td>Parameter identification in reaction-diffusion models</td>
</tr>
<tr>
<td>837</td>
<td>E.G. Kalnins, H.L. Manocha and Willard Miller, Jr.</td>
<td>Models of $q$-algebra representations I. Tensor products of special unitary and oscillator algebras</td>
</tr>
<tr>
<td>838</td>
<td>Robert J. Sacker and George R. Sell</td>
<td>Dichotomies for linear evolutionary equations in Banach spaces</td>
</tr>
<tr>
<td>839</td>
<td>Oscar P. Bruno and Fernando Reitich</td>
<td>Numerical solution of diffraction problems: a method of variation of boundaries</td>
</tr>
<tr>
<td>840</td>
<td>Oscar P. Bruno and Fernando Reitich</td>
<td>Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain</td>
</tr>
<tr>
<td>841</td>
<td>Victor A. Galaktionov and Juan L. Vazquez</td>
<td>Asymptotic behaviour for an equation of superslow diffusion. The Cauchy problem</td>
</tr>
<tr>
<td>842</td>
<td>Josephus Hulshof and Juan Luis Vazquez</td>
<td>The Dipole solution for the porous medium equation in several space dimensions</td>
</tr>
<tr>
<td>843</td>
<td>Shoshana Kamin and Juan Luis Vazquez</td>
<td>The propagation of turbulent bursts</td>
</tr>
<tr>
<td>844</td>
<td>Miguel Escobedo, Juan Luis Vazquez and Enrike Zuazua</td>
<td>Source-type solutions and asymptotic behaviour for a diffusion-convection equation</td>
</tr>
<tr>
<td>845</td>
<td>Marco Biroli and Umberto Mosco</td>
<td>Discontinuous media and Dirichlet forms of diffusion type</td>
</tr>
<tr>
<td>846</td>
<td>Stathis Filippas and Jong-Shenq Guo</td>
<td>Quenching profiles for one-dimensional semilinear heat equations</td>
</tr>
<tr>
<td>847</td>
<td>H. Scott Dumas</td>
<td>A Nekhoroshev-like theory of classical particle channeling in perfect crystals</td>
</tr>
<tr>
<td>848</td>
<td>R. Natalini and A. Tesei</td>
<td>On a class of perturbed conservation laws</td>
</tr>
<tr>
<td>849</td>
<td>Paul K. Newton and Shinya Watanabe</td>
<td>The geometry of nonlinear Schrödinger standing waves</td>
</tr>
<tr>
<td>850</td>
<td>S.S. Sritharan</td>
<td>On the nonsmooth verification technique for the dynamic programming of viscous flow</td>
</tr>
<tr>
<td>851</td>
<td>Mario Taboada and Yuncheng You</td>
<td>Global attractor, inertial manifolds and stabilization of nonlinear damped beam equations</td>
</tr>
<tr>
<td>852</td>
<td>Shigeru Sakaguchi</td>
<td>Critical points of solutions to the obstacle problem in the plane</td>
</tr>
<tr>
<td>853</td>
<td>F. Abergel, D. Hilhorst and F. Issard-Roch</td>
<td>On a dissolution-growth problem with surface tension in the neighborhood of a stationary solution</td>
</tr>
<tr>
<td>854</td>
<td>Erasmus Langer</td>
<td>Numerical simulation of MOS transistors</td>
</tr>
<tr>
<td>855</td>
<td>Haim Brezis and Shoshana Kamin</td>
<td>Sublinear elliptic equations in $\mathbb{R}^n$</td>
</tr>
<tr>
<td>856</td>
<td>Johannes C.C. Nitsche</td>
<td>Boundary value problems for variational integrals involving surface curvatures</td>
</tr>
<tr>
<td>857</td>
<td>Chao–Nien Chen</td>
<td>Multiple solutions for a semilinear elliptic equation on $\mathbb{R}^N$ with nonlinear dependence on the gradient</td>
</tr>
<tr>
<td>858</td>
<td>D. Brochet, X. Chen and D. Hilhorst</td>
<td>Finite dimensional exponential attractor for the phase field model</td>
</tr>
<tr>
<td>859</td>
<td>Joseph D. Fehribach</td>
<td>Mullins-Sekerka stability analysis for melting-freezing waves in helium-4</td>
</tr>
<tr>
<td>860</td>
<td>Walter Schempp</td>
<td>Quantum holography and neurocomputer architectures</td>
</tr>
<tr>
<td>861</td>
<td>D.V. Anosov</td>
<td>An introduction to Hilbert's 21st problem</td>
</tr>
<tr>
<td>862</td>
<td>Herbert E Huppert and M Grae Worster</td>
<td>Vigorous motions in magma chambers and lava lakes</td>
</tr>
</tbody>
</table>
Robert L. Pego and Michael I. Weinstein, A class of eigenvalue problems, with applications to instability of solitary waves

Mahmoud Affouf, Numerical study of a singular system of conservation laws arising in enhanced oil reservoirs

Darin Beigle, Anthony Leonard and Stephen Wiggins, The dynamics associated with the chaotic tangles of two dimensional quasiperiodic vector fields: theory and applications

Gui-Qiang Chen and Tai-Ping Liu, Zero relaxation and dissipation limits for hyperbolic conservation laws

Gui-Qiang Chen and Jian-Guo Liu, Convergence of second-order schemes for isentropic gas dynamics

Aleksander M. Simon and Zbigniew J. Grzywna, On the Larché-Cahn theory for stress-induced diffusion

Jerzy Luczka, Adam Gadomski and Zbigniew J. Grzywna, Growth driven by diffusion

Mitchell Luskin and Tsorng-Whay Pan, Nonplanar shear flows for nonaligning nematic liquid crystals

Mahmoud Affouf, Unique global solutions of initial-boundary value problems for thermodynamic phase transitions

Richard A. Brualdi, Keith L. Chavey and Bryan L. Shader, Rectangular L-matrices

Xinfu Chen, Avner Friedman and Bei Hu, The thermistor problem with zero-one conductivity II

Raoul LePage, Controlling a diffusion toward a large goal and the Kelly principle

Raoul LePage, Controlling for optimum growth with time dependent returns

Marc Hallin and Madan L. Puri, Rank tests for time series analysis a survey

V.A. Solonnikov, Solvability of an evolution problem of thermocapillary convection in an infinite time interval

Horia I. Ené and Bogdan Vernescu, Viscosity dependent behaviour of viscoelastic porous media

Kaushik Bhattacharya, Self-accommodation in martensite

D. Lewis, T. Ratiu, J.C. Simo and J.E. Marsden, The heavy top: a geometric treatment

Leonid V. Kalachev, Some applications of asymptotic methods in semiconductor device modeling

David C. Dobson, Phase reconstruction via nonlinear least-squares

Patricio Aviles and Yoshihazu Giga, Minimal currents, geodesics and relaxation of variational integrals on mappings of bounded variation

Patricio Aviles and Yoshihazu Giga, Partial regularity of least gradient mappings

Charles R. Johnson and Michael Lundquist, Operator matrices with chordal inverse patterns

B.J. Bayly, Infinitely conducting dynamos and other horrible eigenproblems

Charles M. Elliott and Stefan Luckhaus, 'A generalised diffusion equation for phase separation of a multi-component mixture with interfacial free energy'

Christian Schmeiser and Andreas Unterreiter, The derivation of analytic device models by asymptotic methods

LeRoy B. Beasley and Norman J. Pullman, Linear operators that strongly preserve the index of imprimitivity

Jerry Donato, The Boltzmann equation with lie and cartan

Thomas R. Hoffend Jr., Peter Smereka and Roger J. Anderson, A method for resolving the laser induced local heating of moving magneto-optical recording media

E.G. Kalnins, Willard Miller, Jr. and Sanchita Mukherjee, Models of q-algebra representations: the group of plane motions

T.R. Hoffend Jr. and R.K. Kaul, Relativistic theory of superpotentials for a nonhomogeneous, spatially isotropic medium

Reinhold von Schwerin, Two metal deposition on a microdisk electrode

Vladimir I. Oliker and Nina N. Uraltseva, Evolution of nonparametric surfaces with speed depending on curvature, III. Some remarks on mean curvature and anisotropic flows

Wayne Barrett, Charles R. Johnson, Raphael Loewy and Tamir Shalom, Rank incrementation via diagonal perturbations

Thierry Dombre, Alain Pumir, and Eric D. Sigga, On the interface dynamics for convection in porous media

Mingxiang Chen, Xu-Yan Chen and Jack K. Hale, Structural stability for time-periodic one-dimensional parabolic equations

Hong-Ming Yin, Global solutions of Maxwell's equations in an electromagnetic field with the temperature-dependent electrical conductivity

Robert Grone, Russell Merris and William Watkins, Laplacian unimodular equivalence of graphs

Miroslav Fiedler, Structure-ranks of matrices

Miroslav Fiedler, An estimate for the nonstochastic eigenvalues of doubly stochastic matrices

Miroslav Fiedler, Remarks on eigenvalues of Hankel matrices

Charles R. Johnson, D.D. Olesky, Michael Tsatsomeros and P. van den Driessche, Spectra with positive elementary symmetric functions

Pierre-Alain Gremaud, Thermal contraction as a free boundary problem

K.L. Cooke, Janos Turi and Gregg Turner, Stabilization of hybrid systems in the presence of feedback delays

Robert P. Gilbert and Yongzhi Xu, A numerical transmutation approach for underwater sound propagation

LeRoy B. Beasley, Richard A. Brualdi and Bryan L. Shader, Combinatorial orthogonality

Richard A. Brualdi and Bryan L. Shader, Strong hall matrices

Håkan Wennerström and David M. Anderson, Difference versus Gaussian curvature energies; monolayer versus bilayer curvature energies applications to vesicle stability

Shmuel Friedland, Eigenvalues of almost skew symmetric matrices and tournament matrices