EVOLUTION OF NONPARAMETRIC SURFACES WITH SPEED DEPENDING ON CURVATURE, III. SOME REMARKS ON MEAN CURVATURE AND ANISOTROPIC FLOWS

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Evolution of nonparametric surfaces with speed depending on curvature, III. Some remarks on mean curvature and anisotropic flows

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Dedicated to James Serrin on the occasion of his 65th birthday

1. Introduction

This paper is a sequel to our paper [OU] where we investigated questions concerning solvability and asymptotic behavior of solutions to the mean curvature evolution problem

\begin{align}
\frac{\partial u}{\partial t} &= \sqrt{1 + |Du|^2} H(u) \text{ in } \Omega \times (0, \infty), \\
u(x, t) &= 0 \text{ on } \partial \Omega \times [0, \infty), \\
u(x, 0) &= u_0(x) \text{ in } \overline{\Omega}, \ u_0 \in C^\infty(\overline{\Omega}),
\end{align}

where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $C^\infty$ boundary $\partial \Omega$, $H$ is the mean curvature operator

\begin{align}
H(u) &= \text{div} \frac{Du}{\sqrt{1 + |Du|^2}}, \\
Du &= \text{grad } u, \ |Du|^2 = \langle Du, Du \rangle, \text{ and } \frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}.
\end{align}

In the first part of this paper, we investigate the same equation (1.1) in case of nonhomogeneous Dirichlet boundary condition and in the second part we study the problem (1.1)-(1.3) with $H(u)$ replaced by its "anisotropic" version (see the equation 1.22) below).

In the nonhomogeneous case the mean curvature evolution problem is formulated as follows.

Suppose there exists a function $\phi \in C^\infty(\overline{\Omega})$ such that

\begin{align}
H(\phi) &= 0 \text{ in } \Omega,
\end{align}

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that is, the graph of the function $\phi$ is a minimal surface. Here the graph of $\phi$ is considered in $\mathbb{R}^{n+1}$ in which a Cartesian coordinate system $x_1, \ldots, x_n, x_{n+1}$ is chosen so that $\Omega$ lies in the (hyper)plane $x_{n+1} = 0$ and the graph of $\phi$ is given by the set $(x, \phi(x)), x \in \bar{\Omega}$.

We are interested in solvability of the problem

$$u_t = \sqrt{1 + |Du|^2} \ H(u) \quad \text{in} \quad \Omega \times [0, \infty), \quad (1.5)$$

$$u(x, t) = \phi(x) \quad \text{in} \quad \partial \Omega \times [0, \infty), \quad (1.6)$$

$$u(x, 0) = \phi(x) + u_0(x) \in \bar{\Omega}, \quad u_0 \in C_0^\infty(\bar{\Omega}). \quad (1.7)$$

The equation (1.5) describes a motion of the surface $(x, u(x, t))$ evolving with normal speed equal to the mean curvature $H(u)$. The boundary of the surface remains fixed. At the initial moment $u(x, 0)$ is a smooth bounded perturbation of the minimal surface $x_{n+1} = \phi(x), x \in \Omega$.

We show in this paper that (1.5)-(1.7) admits a "solution" $u$ for all time; however, this solution may develop singularities on $\partial \Omega$ in finite time and, in general, will not satisfy (1.6). After a sufficiently long time the singularities will disappear and the solution will be smooth in $\bar{\Omega}$. Furthermore, the solution $u(x, t) \to \phi(x)$ as $t \to \infty$.

It is known that if the domain $\Omega$ satisfies the condition of H. Jenkins and J. Serrin (see [JS] and [S]) then a smooth in $\bar{\Omega}$ solution of (1.5)-(1.7) exists for all time. This was shown by G. Huisken [H]. The novelty in our case is that we do not assume that $\Omega$ satisfies the Jenkins-Serrin condition.

In order to formulate the results we need the notion of a generalized solution; it is similar to the one in [LT] and [OU].

A function $u(x, t)$ is called a generalized solution of (1.5)-(1.7) if it satisfies the following conditions:

$$u \in C^\infty(\Omega \times [0, \infty)); \quad u \in L_\infty([0, \infty); \quad W^{1,1}(\Omega)); \quad (1.8)$$

$$u_t = L_\infty(\Omega \times [0, \infty)); \quad (1.9)$$

$$Lu := u_t - \sqrt{1 + |Du|^2} \ H(u)$$

$$\equiv u_t - \Delta u + \frac{u_i u_j}{1 + |Du|^2} u_{ij} = 0 \quad \text{in} \quad \Omega \times (0, \infty), \quad (1.10)$$

where $u_i := \partial u / \partial x_i, u_{ij} := \partial^2 u / \partial x_i \partial x_j$, $\Delta$ is the Laplace operator, and the convention about
summation over repeated indices is assumed here and throughout the paper;

\[
- \frac{\langle Du, v \rangle}{\sqrt{1 + |Du|^2}} \in \text{sign}(u - \phi) \ a.e. \ on \ \partial\Omega \times (0, \infty),
\]  

(1.11)

where \( v \) is the exterior unit normal field on \( \partial\Omega \);

\[
u(x, 0) = \phi(x).
\]  

(1.12)

Now we can formulate the first result.

**Theorem 1.** Problem (1.5)-(1.7) admits a generalized solution.

In order to construct this generalized solution, we consider a family of regularized problems:

\[
L_c u^\varepsilon := Lu^\varepsilon - \varepsilon \sqrt{1 + |Du^\varepsilon|^2} \Delta u^\varepsilon = 0 \ in \ \Omega \times (0, \infty),
\]  

(1.13)

\[
u^\varepsilon(x, t) = \phi(x) \ on \ \partial\Omega \times [0, \infty),
\]  

(1.14)

\[
u^\varepsilon(x, 0) = \phi(x) + u_0(x) \ in \ \overline{\Omega}.
\]  

(1.15)

The problems (1.13)-(1.15) are uniformly parabolic for each \( \varepsilon > 0 \) and each of them admits a unique solution \( u^\varepsilon \) from \( C^{\infty}(\overline{\Omega} \times [0, \infty)) \). The generalized solution of (1.5)-(1.7) that we construct is obtained as a limit of a subsequence of \( \{ u^\varepsilon \} \) converging in \( C^k(\overline{\Omega'} \times [0, T]) \) for any \( k \geq 0 \), any \( T > 0 \), and any \( \Omega' \subset \subset \Omega \) to some function \( u \) satisfying (1.8)-(1.12).

The next result shows that \( u(x, t) \) is smooth in \( \overline{\Omega} \), possibly, after a sufficiently long time.

**Theorem 2.** Let \( u(x, t) = \lim_{\varepsilon \to 0} u^\varepsilon(x, t) \) be the generalized solution of (1.5)-(1.7). There exists \( \bar{t} \geq 0 \) such that

\[
u \in C^{\infty}(\overline{\Omega} \times [\bar{t}, \infty)),
\]  

(1.16)

\[
u(x, t) = \phi(x), \ (x, t) \in \partial\Omega \times [\bar{t}, \infty).
\]  

(1.17)

Furthermore,

\[
|u(x, t) - \phi(x)| \leq C e^{-\mu t}, \ x \in \overline{\Omega}, \ t \geq 0,
\]  

(1.18)

where \( C \) and \( \mu \) are positive constants depending on initial data and domain \( \Omega \).

The proofs of Theorems 1 and 2 follow the same plan as the proofs of Theorem A and the first part of Theorem D in [OU]. However, some of the barriers needed here for the \( C^0 \) estimates are different from those in [OU] and new efforts are needed to obtain these estimates. These proofs are given in sections 2 and 3.
Remark. Everywhere in the paper the capitals $C$, $C_1$, ... etc. denote positive constants depending, at most, on the domain $\Omega$, initial data, and function $\phi$.

In the second part of this paper, we show how the techniques in [OU] can be extended to allow investigation of the "anisotropic" version of the evolution flow under mean curvature. For closed surfaces, this evolution was considered by Y.-G. Chen, Y. Giga, and S. Goto in [GG] and [CGG]. In this connection see also the paper by S. Angenent and M. Curtin [AC]. We study here the evolution of graphs and establish the following result.

Let $F(p)$, $p \in \mathbb{R}^n$, be a positive $C^\infty$ function satisfying for all $p$, $\xi \in \mathbb{R}^n$ the following structure conditions:

$$|F_i(p)| \leq \alpha_0,$$  \hspace{1cm} (1.19)

$$F_i(p)p_i \geq \alpha_1 \frac{p^2}{\sqrt{1+p^2}} - \alpha_2,$$ \hspace{1cm} (1.20)

$$\frac{\alpha_4 |\xi_p|^2}{(1+p^2)^{3/2}} + \frac{\alpha_3 |\xi|^2}{\sqrt{1+p^2}} \leq F_{ij}(p)\xi_i\xi_j \leq \frac{\alpha_4 |\xi|^2}{\sqrt{1+p^2}}$$ \hspace{1cm} (1.21)

where

$$F_i := \frac{\partial F}{\partial p_i}, \quad F_{ij} := \frac{\partial^2 F}{\partial p_i \partial p_j},$$

$$\xi_p = \left< \frac{\xi_p}{|p|} \right> \frac{p}{|p|} \text{ for } p \neq 0, \quad \xi' = \xi - \xi_p,$$

$\alpha_0$, $\alpha_1$, $\alpha_3$, $\alpha_4$ are positive constants and $\alpha_2 \geq 0$.

These structure conditions are derived from postulated properties of a function defined in $\mathbb{R}^{n+1}$ which represents physically the interfacial energy; for more details see [AC], [GG], and [CGG]. The special case where $F(p) = \sqrt{1+p^2}$ leads to the mean curvature evolution equation as it can be seen from the equation (1.22) below.

Consider the initial boundary value problem

$$u_t = \sqrt{1+|Du|^2} \frac{d}{dx_i} \left( \frac{\partial F(p)}{\partial p_i} \right) \text{ in } \Omega \times (0, \infty),$$ \hspace{1cm} (1.22)

$$u(x, t) = 0 \text{ on } \partial \Omega \times [0, \infty),$$ \hspace{1cm} (1.23)

$$u(x, 0) = u_0(x) \text{ in } \Omega, \quad u_0 \in C^\infty_c(\Omega),$$ \hspace{1cm} (1.24)
where $d/dx_i$ denotes the total derivative, and $\Omega$, as before, is a bounded domain in $\mathbb{R}^n$ with smooth boundary.

Again, we define the generalized solution of (1.22)-(1.24) as a function $u(x,t)$ with properties:

\begin{align*}
  u &\in C^0(\Omega \times [0,\infty)); \quad u \in L_{\omega}([0,\infty); W^{1,1}(\Omega)); \\
  u_t &\in L_{\omega}(\Omega \times [0,\infty)); \\
  L u : = u_t - \sqrt{1 + |Du|^2} \frac{d}{dx_i} \left( \frac{\partial F(p)}{\partial p_i} \right) = 0 \text{ in } \Omega \times (0,\infty); \\
  -\frac{\langle Du, v \rangle}{\sqrt{1 + |Du|^2}} &\in \text{sign}(u) \text{ a.e. on } \partial\Omega \times (0,\infty); \\
  u(x,0) &= u_0(x) \text{ in } \Omega.
\end{align*}

(1.25) \hspace{1cm} (1.26) \hspace{1cm} (1.27) \hspace{1cm} (1.28) \hspace{1cm} (1.29)

In order to construct a generalized solution, we consider the regularized problem

\begin{align*}
  L^\varepsilon u^\varepsilon : = L u - \varepsilon \sqrt{1 + |Du|^2} \Delta u^\varepsilon &= 0 \text{ in } \Omega \times [0,\infty), \\
  u^\varepsilon &= 0 \text{ on } \partial\Omega \times [0,\infty), \\
  u^\varepsilon &= u_0 \text{ on } \Omega \times \{0\}.
\end{align*}

(1.30) \hspace{1cm} (1.31) \hspace{1cm} (1.32)

**Theorem 3.** The problem (1.22)-(1.24) admits a generalized solution which can be constructed as a limit of solutions of (1.30)-(1.32). For this generalized solution, there exists some $\bar{t} \geq 0$ such that

\[ u \in C^0(\Omega \times [\bar{t},\infty)) \text{ and } u = 0 \text{ on } \partial\Omega \times [\bar{t},\infty). \]

The proof of Theorem 3 is given in section 4. It follows the same basic steps as the proofs of Theorems 1 and 2. In comparison to [OU] essentially new arguments are needed only for construction of $C^0$-barriers. We present these estimates in detail. The local $C^1$-estimates in $x$ and $t$ are obtained by a slight modification of the arguments in [OU].
2. $C^0$-estimates for solutions of (1.13)-(1.15)

**Lemma 2.1.** There exist positive constants $C$ and $\mu$ depending on $\Omega$, $u_0$, and $\phi$ such that for all $(x, t) \in \overline{\Omega} \times [0, \infty)$

$$|u^e(x, t) - \phi(x)| \leq Ce^{-\mu t}. \quad (2.1)$$

**Proof.** The proof is obtained by constructing appropriate barriers and then applying the usual maximum principle.

It will be convenient to use the following notation:

$$\tilde{u}^e := u^e - \phi,$$

$$a_{ij}(D \tilde{u}^e + D \phi) := \left[1 + \epsilon \sqrt{1 + |D \tilde{u}^e + D \phi|^2}\delta_{ij} - \frac{\partial_i(\tilde{u}^e + \phi)\partial_j(\tilde{u}^e + \phi)}{1 + |D \tilde{u}^e + D \phi|^2}\right], \quad \partial_i := \partial / \partial x_i,$$

$$b_{ij}(D \tilde{u}^e) := a_{ij}(D \tilde{u}^e + D \phi).$$

By (1.13) we have in $\overline{\Omega} \times [0, \infty)$

$$L^e \tilde{u}^e = \tilde{u}_i - b_{ij}(D \tilde{u}^e)\tilde{u}_j - b_{ij}(D \tilde{u}^e)\phi_{ij} = 0, \quad (2.2)$$

where $\tilde{u}_{ij} = \partial_i \partial_j \tilde{u}^e / \partial x_i \partial x_j$ and $\phi_{ij} = \partial_i \partial_j \phi / \partial x_i \partial x_j$.

The equation (2.2) is invariant relative to parallel translations of the origin $O$ of the coordinate system in the plane $x_{n+1} = 0$. Therefore, we may assume that $O \in \Omega$. Denote by $B_R$ a ball of radius $R$ centered at $O$. Assume that $R > 1$ and $\Omega \subset B_{R-1}$.

Consider the functions

$$v^\pm(x, t) = \pm \frac{m_0}{m} e^{-\mu t}(e^m - e^{\psi(x)}),$$

where $\psi(x) = -m(R^2 - 1/2 |x|^2 - 1)$ and the constants $m_0$, $m$, and $\mu$ are positive and to be chosen so that

$$L^e(v^+ + \phi) = v^+_i - b_{ij}(D v^+)v^+_j - b_{ij}(D v^+)\phi_{ij} > 0 \text{ in } \overline{\Omega} \times [0, \infty) \quad (2.3)$$

$$L^e(v^- + \phi) < 0 \text{ in } \overline{\Omega} \times [0, \infty), \quad (2.3)'$$

$$v^+(x, t) \geq 0 \text{ in } \partial \Omega \times [0, \infty), \quad (2.4)$$

$$v^+(x, 0) \geq |u_0(x)| \text{ in } \overline{\Omega}. \quad (2.5)$$
Later we show that (2.3), (2.3)', (2.4), and (2.5) imply (2.1).

The inequality (2.4) is satisfied trivially. It is also clear that there exists \( \overline{m} > 0 \) such that for all \( m \geq \overline{m} \), \( \frac{1}{m} (e^m - e^{\psi(x)}) \geq 1 \). Setting

\[
m_0 = \max_{\Omega} |u_0(x)|,\]

we obtain (2.5) with any \( m \geq \overline{m} \).

Consider now (2.3). We have

\[
v_i^+ = -\mu v_i^+, \quad v_i^+ = -m_0 e^{-\psi(x)} x_i, \quad (v_i = \partial_i v)
\]

\[
v_{ij}^+ = -m_0 e^{-\psi(x)} (m x_i x_j + \delta_{ij}), \quad (v_{ij} = \partial^2 v / \partial x_i \partial x_j).
\]

Note next that

\[
 b_y(Dv^+) v_{ij}^+ = -m_0 e^{-\mu} e^{\psi(x)} [m b_y(Dv^+) x_i x_j + b_u(Dv^+)],
\]

\[
 b_y(Dv^+) x_i x_j \geq \frac{|x|^2}{1 + |Dv^+|^2 + |D \phi|^2}, \tag{2.6}
\]

\[
 b_u(Dv^+) \geq n - 1. \tag{2.7}
\]

From the expression for \( v_i^+ \) we get

\[
 |Dv^+| \leq m_0 e^{-\mu} e^{\psi(x)} |x|. \tag{2.8}
\]

Let \( \Phi \) be a constant such that

\[
 \| \phi \|_{C^2(\overline{\Omega})} \leq \Phi.
\]

From (2.6) and (2.8) we obtain

\[
 b_y(Dv^+) x_i x_j \geq \frac{|x|^2}{1 + m_0^2 R^2 + \Phi^2}. \tag{2.9}
\]

We also have,
\[ b_{ij}(Dv^+)\phi_{ij} = a_{ij}(D\phi)\phi_{ij} + \int_0^1 \left( \frac{d}{d\tau} a_{ij}(\tau Dv^+ + D\phi) \right) d\tau \phi_{ij} = \overline{a}_{ij,k}v^*_k\phi_{ij}, \]

where

\[ \overline{a}_{ij,k} = \int_0^1 \frac{\partial}{\partial p_k} a_{ij}(\tau Dv^+ + D\phi) d\tau, \quad p_k = v^*_k. \]

It is straightforward to check that

\[ \max_{i,j} \max_{\Omega} |\overline{a}_{ij,k}| \leq 1. \]

Since \( a_{ij}(D\phi)\phi_{ij} = 0 \), we obtain

\[ |b_{ij}(Dv^+)\phi_{ij}| \leq |Dv^+| \leq m_0 \Phi e^{-\mu|x|} |x|. \]

From this inequality and (2.7), (2.9) we get

\[ L^e(v^+ + \phi) \geq \left\{ -\frac{\mu}{m} + e^{-m(r^2 - \frac{1}{2}x^2)} \left( \frac{m|x|^2}{1 + m_0^2 R^2 + \Phi^2} - \Phi|x| + n - 1 \right) \right\} m_0 e^{-\mu} e^m. \]

Let

\[ m = \max \left\{ \frac{\phi^2(1 + R^2 + \phi^2)}{4(n-3/2)}, \frac{m_0}{m} \right\}, \quad \mu = \frac{1}{4} m e^{-m R^2}. \]

Then \( L^e(v^+ + \phi) > 0 \) in \( \overline{\Omega} \times [0, \infty) \) and (2.3) is satisfied.

Consider now (2.3'). In this case one shows in the same way as above that

\[ L^e(v^- + \phi) \leq \left\{ \frac{\mu}{m} - e^{-m(r^2 - \frac{1}{2}x^2)} \left( \frac{m|x|^2}{1 + m_0^2 R^2 + \phi^2} - \phi|x| + n - 1 \right) \right\} m_0 e^{-\mu} e^m \]

and, consequently, \( L^e(v^- + \phi) < 0 \) in \( \overline{\Omega} \times [0, \infty) \).

It is also clear from (2.4) and (2.5) that

\[ v^+(x, t) + \phi(x) \geq \phi(x) = u^e(x, t) \text{ on } \partial\Omega \times [0, \infty), \]

(2.10)

\[ v^+(x, 0) + \phi(x) \geq u_0(x) + \phi(x) = u^e(x, 0) \text{ on } \overline{\Omega}, \]
\[ v(x,t) + \phi(x) \leq \phi(x) = u^\epsilon(x,t) \text{ on } \partial\Omega \times [0, \infty), \]

\[(2.10)' \]

\[ v(x,0) + \phi(x) \leq u_0(x) + \phi(x) = u^\epsilon(x,0) \text{ on } \overline{\Omega}. \]

By the maximum principle, it follows from (2.3), (2.10), and (2.3)', (2.10)' that in \( \overline{\Omega} \times [0, \infty) \)

\[ |u^\epsilon(x,t) - \phi(x)| \leq v^*(x,t). \]

Letting \( C = m/m \cdot e^m \), we obtain (2.1). The lemma is proved.

**Lemma 2.2.** Let \( d(x) = \text{dist}(x, \partial\Omega), x \in \overline{\Omega}. \) There exist positive constants \( C_1 \) and \( T \) depending on \( \Omega \), the constant \( C \) in (2.1), and \( \Phi = \| \phi \|_{C^2(\overline{\Omega})} \) such that

\[ |u^\epsilon(x,t) - \phi(x)| \leq C_1 d(x) e^{-\mu t}, (x,t) \in \overline{\Omega} \times [T, \infty). \quad (2.11) \]

**Proof.** Let \( \Omega_\delta = \{ x \in \Omega \mid d(x) < \delta \} \) where \( \delta \) is so small that \( d \in C^\infty(\overline{\Omega_\delta}) \). Put \( f(d(x)) = (1/k)(1 - e^{-k\delta(x)}), \) where \( k \) is a positive constant to be chosen later, and \( g(t) = [(T-t)_+]^2 \), where \( (\_)_+ \) denotes the nonnegative part. Consider the functions

\[ w(x,t) = v(x,t) + \phi(x), \]

\[ v(x,t) = C_2[f(d(x)) + g(t)] e^{-\mu t}. \]

We want to show that by choosing appropriately \( T, C_2, k, \) and \( \delta \) we can arrange so that \( w \) is an upper barrier for \( u^\epsilon(x,t) \) in \( \overline{\Omega_\delta} \times [T, \infty) \). For that we need to verify the following inequalities for some \( T \geq 1 \):

\[ v(x,t) \geq 0 \text{ in } \partial\Omega \times [T-1, \infty), \quad (2.12) \]

\[ w(x,t) \geq u^\epsilon(x,t) \text{ for } d(x) = \delta \text{ and } t \geq T - 1, \quad (2.13) \]

\[ w(x,T-1) \geq u^\epsilon(x,T-1) \text{ in } \overline{\Omega_\delta} \quad (2.14) \]

\[ L^\epsilon w \geq 0 \text{ in } \Omega_\delta \times (T-1, \infty). \quad (2.15) \]

The inequality (2.12) is obviously true in \( \partial\Omega \times [0, \infty) \). Further, if
\[ C_2 \delta e^{k \delta} \geq C, \quad (2.16) \]

where \( C \) is as in (2.1), then because of (2.1) and since \( f(\delta) \geq \delta e^{-k \delta} \) we have

\[ v(x, t) \geq C e^{-\mu t} \geq u^c(x, t) - \phi(x) \text{ for } d(x) = \delta \text{ and } t \geq 0. \]

This implies (2.13).

Next we note that by (2.1)

\[ v(x, T - 1) \geq C_2 e^{-\mu(T - 1)} \geq C e^{-\mu(T - 1)} \geq u^c(x, T - 1) - \phi(x) \text{ in } \Omega_\delta, \]

provided

\[ C_2 \geq C, \quad (2.17) \]

and this implies (2.14).

Thus, if (2.16), (2.17), and (2.15) can be satisfied, then \( w \) is indeed an upper barrier. Then by the maximum principle

\[ u^c(x, t) \leq w(x, t) \text{ in } \Omega_\delta \times [T - 1, \infty). \quad (2.18) \]

Since \( g(t) = 0 \) for \( t \geq T \), and because one can choose a constant \( C_1 \) so that

\[ C_2 f(d(x)) \leq C_1 d(x) \text{ in } \Omega, \]

we obtain

\[ u^c(x, t) \leq C_1 d(x) e^{-\mu t} + \phi(x) \text{ in } \Omega \times [T, \infty). \]

The bound from below is obtained by constructing similarly a lower barrier. Thus, we obtain (2.11).

Consider now (2.15). We have

\[ \nu_i = C_2 e^{-\mu t} f' d_i, \quad \nu_j = C_2 e^{-\mu t} [f'' d_i d_j + f' d_{ij}], \]

where \( d_i = \partial d / \partial x_i, \) \( d_j = \partial^2 d / \partial x_i \partial x_j \) and similarly \( \nu_i, \) \( \nu_j. \) Using the same notation \( a_{ij}, b_{ij} \) as in the proof of Lemma 2.1, we get

\[ b_{ij}(D v) \nu_j = f'' \left[ q - \frac{(D (\nu + \phi), D d)^2}{1 + |D (\nu + \phi)|^2} \right] + f' \left[ q \Delta d - \frac{2 \nu_j \phi_j + \phi_i \phi_j}{1 + |D (\nu + \phi)|^2} d_j \right], \quad (2.19) \]
where \( q = 1 + \varepsilon \sqrt{1 + |D(v + \phi)|^2} \). Here, we used the facts that \(|Dd| = 1\) and \(d_{ij}d_{ij} = 0\).

In the same way as in the proof of Lemma 2.1 we get

\[
|b_i(Dv)\phi_{ij}| \leq \Phi |Dv| \leq \Phi C_2 e^{-\mu}.
\]

(2.20)

Now using (2.19), (2.20) and noting that \( f'' < 0\), we obtain

\[
L^\varepsilon w \geq C_2 e^{-\mu} \{ -\mu(f + g) + g_i + [1 + |D(v + \phi)|^2]^{-1} \times
\]

\[
[-f'' - f'[(1 + |D(v + \phi)|)^2](q|\Delta d| + \Phi) + |2v,\phi_j + \phi,\phi_j|d_{ij}] \}
\]

Recall that \( d(x) \in C^2(\overline{\Omega}_\delta) \) for any \( \delta < \delta_0 := \inf_{x}(1/\kappa(x)) \) where \( \kappa(x) \) is the maximum of the absolute value of the normal curvature of \( \partial \Omega \) at \( x \in \partial \Omega \) (see [S], p. 421). Put

\[
\rho := \| d \|_{C^2(\overline{\Omega}_\delta)}.
\]

On the other hand, \( |Dv| \leq C_2 e^{-\mu} \). Thus, if

\[
C_2 e^{-\mu(T-1)} \leq 1
\]

(2.21)

then

\[
L^\varepsilon w \geq C_2 e^{-\mu} \{ -\mu(1 + \delta) - 2 + (2 + \Phi^2)^{-1} [k - M] d^{-kd} \},
\]

where

\[
M = (2 + \Phi^2)[(3 + \Phi^2)\rho + \Phi] + \rho \Phi (2 + \Phi),
\]

and it is assumed that \( \varepsilon \leq 1 \).

Set

\[
k = M + (2 + \Phi^2)[2 + \mu(1 + \delta_0)]e, \quad \delta = \min(\delta_0/2, 1/k, 1), \quad C_2 = C \delta^{-1} e^{k\delta},
\]

and choose \( T \) so that (2.21) is satisfied.

Then (2.16), (2.17) are satisfied and

\[
L^\varepsilon w \geq C_2 \mu \frac{\delta_0}{2} e^{-\mu} > 0 \text{ in } \Omega_\delta \times [T - 1, \infty).
\]

The lemma is proved.
3. Proofs of Theorems 1 and 2

Before we can proceed with the proofs of Theorems 1 and 2, we need to record some preliminary estimates which allow applications of the estimates in x and t from [OU].

Proposition 3.1. Let

\[ M_0 = \max \{ \max_{\Omega} |\phi + u_0|, \max_{\partial\Omega} \phi \} \]

and \( u^\varepsilon(x, t) \) a solution of (1.13)-(1.15). Then

\[ \sup_{\Omega \times [0, \infty)} |u^\varepsilon(x, t)| \leq M_0, \tag{3.1} \]

\[ \sup_{\Omega \times [0, \infty)} |u^\varepsilon_t(x, t)| (x, t) \leq M, \tag{3.2} \]

where \( M \) is a constant depending on the \( C^2 \)-norms of \( u_0 \) and \( \phi \), and

\[ \int_{\Omega} (|Du^\varepsilon|^2 + \varepsilon|Du^\varepsilon|^2) \, dx \leq C_3 \text{ for all } t \geq 0. \tag{3.3} \]

Proof. The inequality (3.1) is a consequence of the maximum principle applied to (1.13) - (1.15). The inequality (3.2) also follows from the maximum principle applied to the differentiated in t equation (1.13); one needs to observe here that \( u^\varepsilon_t = 0 \) on \( \partial\Omega \times [0, \infty) \) and

\[ u^\varepsilon_t = \sqrt{1 + |D(\phi + u_0)|^2}[H(\phi + u_0) + \varepsilon\Delta(u_0 + \phi)] \text{ in } \Omega \times \{0\}. \]

In order to verify (3.3), note first that by (1.13)

\[ \int_{\Omega} \left( \frac{u^\varepsilon_t \eta + u^\varepsilon \eta_t}{\sqrt{1 + |Du^\varepsilon|^2}} + \varepsilon u^\varepsilon \eta_t \right) \, dx = 0 \tag{3.4} \]

for any \( \eta \in H^1_0(\Omega) \) and \( t \geq 0 \), where \( \eta_t = \partial \eta / \partial x_t \). Take \( \eta = u^\varepsilon - \phi \) and substitute in (3.4). Then, taking into account (3.1) and (3.2), and after some straightforward manipulations, we obtain (3.3). The proposition is proved.

Once the inequalities (3.1)-(3.3) are established we can apply Theorem C in [OU] and conclude that for any subdomain \( \Omega' \subset \subset \Omega \) and any \( T > 0 \) there exists a constant \( \overline{C} = \overline{C}(\text{dist}(\Omega', \partial\Omega), T) \) such that
\[
\sup_{\Omega^c} |Du^\varepsilon(x,t)| \leq C_4 \quad \text{for} \quad t \in [0,T].
\] (3.5)

This estimate, (3.2) and standard results on uniformly parabolic equations imply that there is a subsequence of \{u^\varepsilon\} converging in \(C^k(\overline{\Omega}^c \times [0,T])\), for any \(k \geq 0\), to some function \(u \in C^\infty(\Omega \times [0,\infty))\). The function \(u\) satisfies (1.5), (1.7), and it follows from (3.1) - (3.3) that \(u \in L_\infty(\Omega \times [0,\infty))\) and \(u \in W^{1,1}(\Omega)\) for all \(t \geq 0\). It is shown, as in [OU], section 4.13 (cf. [LT], section 3.2), that \(u\) satisfies (1.11).

**Proof of Theorem 2.** It follows from (1.14) and (2.11) by standard arguments that

\[
|D(u^\varepsilon(x,t) - \phi(x))| \leq C_6 e^{-\mu t} \quad \text{in} \quad \partial \Omega \times [T,\infty).
\]

By Theorem C' in [OU], this inequality and (3.1)-(3.3) imply that there exists some \(\bar{t} \geq T\) such that

\[
|Du^\varepsilon(x,t)| \leq C_6 \quad \text{on} \quad \overline{\Omega} \times [\bar{t},\infty),
\] (3.6)

where \(C_6 = C_6(\bar{t},C_3)\). Then the solutions \{u^\varepsilon\} and all their derivatives admit uniform bounds in \(\overline{\Omega} \times [\bar{t},T']\) independent on \(\varepsilon\) for any \(T' > \bar{t}\) (see [LSU], ch. IV). Therefore, the generalized solution \(u(x,t)\) of (1.5)-(1.7) is in \(C^1(\overline{\Omega} \times [\bar{t},T'])\) and by general results on uniformly parabolic PDE's, it is in \(C^\infty(\overline{\Omega} \times [\bar{t},\infty))\). Consequently, (1.16) and (1.17) are satisfied. The inequality (1.18) follows now from (2.1) after passing to the limit in \(\varepsilon\) for \(t > \bar{t}\) and, if necessary, replacing \(C\) by a larger constant. Theorem 2 is proved.

4. **Proof of Theorem 3**

The proof of this theorem follows the same steps as the proofs of Theorems 1 and 2.

For each \(\varepsilon > 0\) the equation (1.30) is uniformly parabolic and, therefore, the problem (1.30)-(1.32) admits a solution \(u^\varepsilon\) of class \(C^\infty(\overline{\Omega} \times [0,\infty))\). It follows from the maximum principle that \(u^\varepsilon\) is a unique solution.

We begin by establishing \(C^0\)-estimates for all time.

(a) We may assume that the origin of the coordinate system is inside \(\Omega\). Let \(B_\rho\) be a ball of radius \(\rho\) centered at \(O\) and containing \(\Omega\) strictly inside.

Let
\[ v(x, t) = m \left( 2R^2 - \frac{1}{2} x^2 \right) e^{-\mu}, \]

where \( m = \sup_{\Omega} |u_0(x)| R^{-2} \), and \( \mu \) a positive constant to be determined. Using (1.21), we obtain

\[
L^\varepsilon v = -\mu v + \sqrt{1 + m^2 |x|^2} e^{-2\mu} \left( \sum_{i=1}^{n} F_{ii} \right) me^{-\mu} + \varepsilon n \sqrt{1 + m^2 |x|^2} e^{-2\mu} me^{-\mu} \\
\geq \left[ -2\mu R^2 + \alpha_3 (1 + m^2 |x|^2)^{-3/2} \right] me^{-\mu} \\
\geq \left[ -2\mu R^2 + \alpha_3 (1 + m^2 R^2)^{-3/2} \right] me^{-\mu}.
\]

Let \( \mu = 3^{-1} R^{-2} \alpha_3 (1 + m^2 R^2)^{-3/2} \). Then \( L^\varepsilon v > 0 \) in \( \overline{\Omega} \times [0, \infty) \) and, clearly, \( \pm L^\varepsilon u^\varepsilon = 0 < L^\varepsilon v \) in \( \overline{\Omega} \times [0, \infty) \). Obviously, \( |u_0(x)| \leq v(x, 0) \) in \( \Omega \) and \( \pm u^\varepsilon = 0 < v \) on \( \partial \Omega \times [0, \infty) \). By the maximum principle

\[ |u^\varepsilon(x, t)| \leq v(x, t) \text{ in } \overline{\Omega} \times [0, \infty). \tag{4.1} \]

(b) Next we improve the estimate (4.1) for large \( t \).

Let \( \Omega_\delta = \{ x \in \Omega \mid d(x) < \delta \} \) where \( d(x) \equiv \text{dist}(x, \partial \Omega) \) and \( \delta > 0 \) is such that \( d \in C^\infty(\overline{\Omega}_\delta) \). Put \( f(d(x)) = (1/k)(1 - e^{-kd(x)}), x \in \overline{\Omega}_\delta \), where \( k = \text{const} > 0 \) is to be chosen later.

Consider the function

\[ w(x, t) = c[f(d(x)) + g(t)]e^{-\mu} \]

where \( g(t) = [(T - t),]^2 \), \( \mu \) as in (4.1), and \( c > 0 \) a constant to be chosen. Put also \( d_i = \partial d/\partial x_i \) and \( d_{ij} = \partial^2/\partial x_i \partial x_j \) and note that \( |Dd| = 1 \).

We have

\[
L^\varepsilon w = -\mu w + cg \varepsilon e^{-\mu} - \sqrt{1 + |Dw|^2} \\
\times [F_{ij}(f''d_{ij} + f''d_{ij}) + \varepsilon(f'' + f'' \Delta d)] ce^{-\mu} \ (g_i : = \partial g/\partial t).
\]

Since \( f'' < 0 \), we have, using (1.21),
\[-f''F_{ij}(Dw)_{d_j} = -f''e^{2w}F_{ij}(Dw)(\partial w/\partial x_i)(\partial w/\partial x_j)\]
\[
\geq -\frac{f''\alpha_3}{(1 + |Dw|^2)^{3/2}}.
\]

Using the inequality on the right hand side of (1.21) we get
\[
|F_{ij}(Dw)_{d_j}| \leq \frac{C_1\alpha_4}{\sqrt{1 + |Dw|^2}}.
\]

where \(C_1\) is a constant depending on dimension \(n\) and \(C^2\)-norm of \(d(x)\) in \(\bar{\Omega}_\delta\). Recall that \(d(x) \in C^2(\bar{\Omega}_\delta)\) for any \(\delta < \delta_0\) where \(\delta_0\) is the same as at the end of the proof of Lemma 2.2.

Noting that \(g_i \geq -2\) and \(f'' > 0\), we obtain
\[
L^w \geq \left[-\mu(f + g) - 2 - \frac{f''\alpha_3}{1 + c^2f^{2\alpha_3}e^{-2\mu}} - f'C_1\alpha_4 - \right.
\]
\[
-\varepsilon\sqrt{1 + c^2e^{-2\mu}f^{2\alpha_3}(f'' + f'\Delta d)}\right]ce^{-\mu}.
\]  
(4.2)

We now want to show that one can choose \(T \geq 1\), \(c\), \(k\), \(\delta\), and \(\varepsilon_0\) so that
\[
L^w \geq 0 \text{ for all } \varepsilon \leq \varepsilon_0 \text{ in } \Omega_\delta \times (T - 1, \infty), \quad (4.3)
\]
\[
w(x, T - 1) \geq 2me^{-\mu(T - 1)}, \quad \text{(4.4)}
\]
\[
w(x, t) \geq 2me^{-\mu} \text{ in } \{x \in \Omega \mid d(x) = \delta\} \times [T - 1, \infty). \quad \text{(4.5)}
\]

In order to establish (4.3)-(4.5), we proceed as follows. First, restrict \(c\) and \(T \geq 1\) by the requirement
\[
 ce^{-\mu(T - 1)} \leq 1. \quad \text{(4.6)}
\]

Obviously, \(ce^{-\mu} \leq 1\) for all \(t \geq T - 1\). Also, \(g(t) \leq 1\) for all \(t \geq T - 1\). Next, we note that \(f(d(x)) \leq \delta_0\) when \(x \in \bar{\Omega}_\delta\), \(\delta < \delta_0\). Now we obtain from (4.2)
\[
L^w \geq [(2^{-1}k\alpha_3 - C_1\alpha_4)e^{-kd} - 2 - \mu(1 + \delta_0) - \]
\[
-\varepsilon\sqrt{2\left(k + \sup_{\bar{\Omega}_\delta} |\Delta d| \right)}]ce^{-\mu} \text{ in } \Omega_\delta \times [T - 1, \infty).
In this inequality the constant $C_1$ depends on the $C^2$-norm of $w$ in $\Omega_8$. Assuming that $\delta \leq \delta_y/2$, we may suppose that $C_1$ is determined by the bound on the $C^2$-norm of $w$ in $\Omega_{\delta_y/2}$.

Choose $k$ and $\delta$ so that
\[
(2^{-1}k\alpha_3 - C_1\alpha_4)e^{-1} - 2 - \mu(1 + \delta_y) > 0 \text{ and } \delta = \min\{1/k, \delta_y/2\}.
\]

Next let
\[
c = \max\{2m, 2me^\delta\},
\]
and define $T$ as the smallest $T \geq 1$ satisfying (4.6). Then
\[
w(x, T - 1) = c[f + g]e^{-\mu(T - 1)} \geq ce^{-\mu(T - 1)} \geq 2me^{-\mu(T - 1)},
\]
and (4.4) is satisfied. Further,
\[
w(x, t) \mid_{d(x) = \delta} = c(f + g)e^{-\mu} \mid_{d(x) = \delta} \geq c\delta e^{-k\delta}e^{-\mu}
\]
\[
\geq 2me^{-\mu} \text{ for } t \geq T - 1,
\]
and (4.5) is also satisfied. On the other hand, $w(x, t) \mid_{\partial\Omega} \geq 0 = u^\varepsilon(x, t)$. This, (4.3)-(4.5), and the maximum principle imply that for $\varepsilon \leq \varepsilon_0$
\[
|u^\varepsilon(x, t)| \leq w(x, t) \text{ in } \Omega_{\delta} \times [T - 1, \infty).
\]

For the rest of this section it is assumed without further reminding that $\varepsilon \leq \varepsilon_0$.

Since $g(t) = 0$ for $t \geq T$ we conclude that
\[
|u^\varepsilon(x, t)| \leq C_2d(x)e^{-\mu} \text{ in } \Omega_{\delta} \times [T, \infty),
\]
where $C_2 = ce^{-1}$. Combining the last inequality with (4.1) and adjusting the constant $C_2$ if necessary we conclude that
\[
|u^\varepsilon(x, t)| \leq C_2d(x)e^{-\mu} \text{ in } \Omega \times [T, \infty). \tag{4.7}
\]
(c) A standard argument shows now that

$$\left| \frac{\partial u^\varepsilon}{\partial \nu} \right| \leq C_2 e^{-\omega} \text{ in } \partial \Omega \times (T, \infty),$$

where $\nu$ is the exterior unit normal vector field on $\partial \Omega$. Since $u^\varepsilon(x, t) = 0$ on $\partial \Omega$, we conclude that

$$|Du^\varepsilon(x, t)| \leq C_2 e^{-\omega} \text{ in } \partial \Omega \times (T, \infty). \quad (4.8)$$

(d) We have the following analogue of Proposition 3.1. Let $M_0 = \sup_{\Omega} |u_0|$. Then

$$\begin{align*}
\sup_{\Omega \times [0, \infty)} |u^\varepsilon(x, t)| &\leq M_0, \quad (4.9) \\
\sup_{\Omega \times [0, \infty)} |u_t^\varepsilon(x, t)| &\leq M, \quad (4.10)
\end{align*}$$

where $M$ is a constant depending on the $C^2$-norm of $u_0$. Also,

$$\int_{\Omega} \left( |Du^\varepsilon| + \varepsilon |Du^\varepsilon|^2 \right) dx \leq c_0 \quad \text{for all } t \geq 0, \quad (4.11)$$

where $c_0 = c_0(M_0, M, \alpha_0)$.

The inequalities (4.9)-(4.11) are established similarly to (3.1)-(3.3) except that when proving (4.11) one needs to take into account (1.19).

As in [LU] and [OU], section 4.4, we will use the tangential operator $\delta$ defined on $C^1(\Omega)$ as

$$\delta g = \nabla g - \langle \nabla g, N \rangle N,$$

where $g \in C^1(\Omega)$, $\nabla g = (g_1, g_2, \ldots, g_n, 0)$, $g_i = \partial g / \partial x_i$, $N = (-Du^\varepsilon, 1)/\nu$, and $\nu = \sqrt{1 + |Du^\varepsilon|^2}$. Evidently,

$$|\delta g|^2 \geq |Dg|^2 \left( 1 - \frac{|Du^\varepsilon|^2}{\nu^2} \right) = |Dg|^2 / \nu^2, \quad (4.12)$$

and using (1.21) we get

$$\alpha_\nu \nu^{-1} |\delta g|^2 \leq F_{ij} g_i g_j \leq \alpha_\nu \nu^{-1} |Dg|^2. \quad (4.13)$$

Rewrite the equation (1.30) as

$$u_t^\varepsilon - \nu \frac{dF_i^\varepsilon}{dx_i} = 0 \text{ in } \Omega \times [0, \infty), \quad (4.14)$$

where $F_i^\varepsilon = F_i + \varepsilon p_i$ and $p = Du^\varepsilon$. Here and below in this section we omit the argument $Du^\varepsilon$ at $F$.
and its derivatives.

Applying the operator \((p_\nu/v)d/dx_k\) to (4.14) we obtain

\[
v_i - v \frac{d}{d x_i} \left(F^\varepsilon_{ij} v_j\right) + v \Lambda = \left(\frac{p_k v_k}{v}\right) \frac{dF^\varepsilon_i}{d x_i},
\]

(4.15)

where \(v_i = \partial v/\partial t, v_i = \partial v/\partial x_i\), and

\[
\Lambda = \left[ \frac{d}{d x_j} \left(\frac{p_k}{v}\right) \right] \frac{dF^\varepsilon_i}{d x_k} = \frac{\partial}{\partial p_k} \left(\frac{p_k}{v}\right) F^\varepsilon_{ij} u^\varepsilon_{ij}\kappa_i.
\]

Since

\[
\frac{\partial}{\partial p_k} \left(\frac{p_k}{v}\right) \xi_i \xi_k \geq v^{-3} |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n,
\]

and, because of (1.21), \(\Lambda \geq 0\).

Multiply (4.15) by \(\eta v^{-1}, \eta \in H^1_0(\Omega)\), and integrate over \(\Omega\). After integration by parts on the left, and estimating the right hand side, using (4.14), (4.12), and the inequality \(p^2/v^2 < 1\), we obtain

\[
\int_{\Omega} \left(\frac{v_i}{v} + F^\varepsilon_{ij} v_j + \Lambda \eta\right) dx \leq M \int_{\Omega} \frac{|\delta v|}{v} |\eta| dx \quad \text{for all} \quad t \geq 0,
\]

(4.16)

where \(M\) is the constant from the inequality (4.10).

This inequality replaces the inequality (4.22) in [OU]. Formally both look the same, but the function \(F\) here replaces the function \(\sqrt{1 + p^2}\) in [OU] (also denoted by \(F\) in [OU]). Using (4.16) and (3.1)-(3.3) one derives exact analogues of Theorems C and C' in [OU] for the function \(F\) satisfying conditions (1.19)-(1.21). The arguments used in [OU] to prove Theorems C and C' carry over to our case with only minor modifications; one only needs to use the structure conditions (1.19)-(1.21), inequalities (4.9)-(4.11), and (4.13) instead of the corresponding properties of the function \(\sqrt{1 + p^2}\) and its derivatives. The details are lengthy but straightforward and we will not repeat them here.

Once analogues of Theorems C and C' in [OU] are established we have an estimate of \(|Du^\varepsilon(x, t)|\) in \(\Omega' \times [0, T]\) for any \(\Omega' \subset \subset \Omega\) and \(T > 0\), and, on the account of (4.8), we have an estimate of \(|Du^\varepsilon(x, t)|\) in \(\overline{\Omega} \times [\bar{t}, \infty)\) for some \(\bar{t} \geq 0\). These estimates are similar to (3.5) and (3.6) in section 3 and the proof of Theorem 3 is completed in the same way as the proofs of Theorems 2 and 3.
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