NUMERICAL SCHEMES FOR CONSTRAINED MINIMIZATION PROBLEMS

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Abstract

In this paper we study steepest descent algorithms for solving constrained minimization problems. We solve the resulting Euler-Lagrange equations via introduction of time as an auxiliary variable. We define the notion of weak solutions of the evolution equation and identify the proper numerical schemes. We introduce a new filtering algorithm based on this approach.

1 Necessary Conditions

The problem of constrained optimization is a classical problem that has been studied for a long time. One can easily derive the necessary conditions of Euler and Lagrange by using the first variation.

We quote the following theorem that gives the necessary conditions for general set of constraints. The classical problem of Bolza with inequality constraints is that of finding in the class of arcs \((x, u(x)) = (x, u^1, u^2, \ldots, u^d)\), satisfying the conditions of the form

\[ \phi_\alpha(x, u, u_x) \leq 0 \quad (1 \leq \alpha \leq m'), \quad \phi_\alpha(x, u, u_x) = 0 \quad (m' < \alpha \leq m) \]

\[ x^0 = T^0(b), \quad x^1 = T^1(b), \quad u(x^0) = X^0(b), \quad u(x^1) = X^1(b) \]

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\[ I_\gamma(u) \leq 0 \quad (1 \leq \gamma \leq p'), I_\gamma(u) = 0 \quad (p' < \gamma \leq p) \]

one which minimizes \( I_0(u) \), where

\[ I_\gamma = g(b) + \int_{x_0}^{x_1} L_\gamma(x, u, u_x)dx \quad (\gamma = 0, 1, \ldots, p). \]

Assume that \( L_\gamma \) and \( \phi_\alpha \) are \( C^1 \) in a region \( R \) in \( (x, u, u_x) \) space, and \( (\frac{\partial\phi_\alpha}{\partial u_x}, \delta_{\alpha\beta}\phi_\beta) \) has rank \( m \). Then we have the following result.

**Theorem 1** \([1]\) Suppose that \( u \) is a solution of the classical problem of Bolza described above. Then there exist multipliers

\[ \lambda_0 \geq 0, \quad \lambda_\gamma, \quad \mu_\alpha(x) \quad (\gamma = 1, \ldots, p; \alpha = 1, \ldots, m), \]

not vanishing simultaneously, and functions

\[ F(x, u, u_x, \mu) = \lambda_\gamma L_\gamma + \mu_\gamma \phi_\gamma, \quad G(b) = \lambda_\gamma g_\gamma(\gamma = 0, 1, \ldots, p) \]

such that

- The inequality \( \lambda_\gamma \geq 0 \) \( (1 \leq \gamma \leq p') \) holds with \( \lambda_\gamma = 0 \) if \( I_\gamma(u) < 0 \).
- The multipliers \( \mu_\alpha(x) \) are piecewise continuous and are continuous on every interval of continuity of \( u_x \). Moreover, \( \mu_\alpha(x) \geq 0 \) \( (1 \leq \alpha \leq m') \) with \( \mu_\alpha(x) = 0 \) for each value of \( x \) at which \( \phi_\alpha(x, u, u_x) < 0 \).
- The integral form of the Euler-Lagrange equations

\[ F - u_x F_{u_x} = \int_{x_0}^{x_1} F_x dx + c_1, \quad F_{u_x} = \int_{x_0}^{x_1} F_u dx + c_2 \quad (1) \]

hold along \( u \).
- The transversality condition

\[ dG + [(F - u_x F_{u_x})dT^x + F_{u_x}dX^x]_{t=0}^{t=1} = 0 \quad (2) \]

- At each element \((x, u, u_x)\) on \( u \) the inequality

\[ E(x, u, u_x, v, \mu(x)) \geq \mu_\alpha(x) \phi_\gamma(x, u, u_x) \quad (3) \]

hold whenever \((x, u, u_x)\) is in \( R \), where \( E \) is the Weierstrass \( E \)-function.

\[ E(x, u, u_x, v, \mu) = F(x, u, v, \mu) - F(x, u, u_x, \mu) - (v - u_x) F_{u_x}(x, u, u_x, \mu) \]
2 Introduction of Time

A natural and most common computational procedure for constrained optimization is the gradient projection method, see [3] [4]. Let us consider a simple problem of the form:

$$\text{minimize } f(x) \text{ subject to } Ax = 0.$$  

The procedure starts by finding a point, $x_1$, that satisfies the constraint. An ideal direction is found by taking the variation of the functional and then projecting it onto the manifold of the constraint. Then the next point is:

$$x_2 = x_1 + \alpha_1 g(x_1)$$

where $\alpha_1$ is chosen to minimize $f(x_2)$, and $g(x_1)$ is:

$$g(x_1) = -\delta f(x_1) + A^*(AA^*)^{-1}A\delta f(x_1)$$

Based on this natural idea we propose the following scheme. Let us consider the following problem.

$$\text{minimize } \int_0^1 L(u, u_x)dx$$

Where $x$ is a real variable between 0 and 1, and $L$ is a differentiable function of $u$ and $u_x$. Then we introduce artificial time variable $t$, and assuming that we have an original guess, we wish to approach the solution as time progresses. Then we need to have:

$$\frac{d}{dt} \int L \leq 0$$

$$\frac{d}{dt} \int L(u, u_x) = \int L_u u_t + L_{ux} u_{xt} =$$

by integration by parts we get:

$$\int L_u u_t - \frac{d}{dx} L_{ux} u_t + L_{ux} u_t \bigg|_0^1$$

At this point we can assume either one of the following boundary conditions:
1. Fixed end points:

\[ u_t = 0 \text{ at } x = 0 \text{ and } x = 1 \]

2. Natural boundary conditions (Transversality condition):

\[ L_{u_x} = 0 \text{ at } x = 0 \text{ and } x = 1 \]

3. Periodic conditions:

\[ L_{u_x} u_t|_0^1 = 0 \]

We choose natural boundary conditions and we get:

\[ \frac{d}{dt} \int L = \int \delta F u_t \]

Where \( \delta L \) is the first variation of the functional \( L \) and is:

\[ \delta L = -\frac{d}{dx} L_{u_x} + L_u \]

In order to decrease the functional we define the following evolution equation:

\[ u_t = -\delta L \quad L_{u_x}|_0^1 = 0 \quad (4) \]

This defines a nonlinear partial differential equation for evolution of \( u \). At this point we make the very crucial assumption that this partial differential equation has a solution and its solution is as smooth as we need. In one sense we are transferring the difficulty of proving the existence of the extremum function into the difficult problem of solving the evolution equation. Also our procedure does not guarantee convergence to the absolute minimum, rather it converges to the nearest local minimum. In some sense this is an advantage. In some applications one wishes to stay close to the original guess. In signal processing this is often the case. The processing has to be invariant under translation and rotation. This implies non-uniqueness of the solution. Thus in the class of all possible solutions we wish to find the closest to the original.

**Theorem 2** The functional \( \int L(u, u_x) \) is decreasing as a function of time. If the second variation of \( L \) is positive definite, \( u \) converges to a solution of the equation \( \delta L = 0 \).
Let us define \( g(t) = \int L(u, u_x) \, dx \), then

\[
\frac{d}{dt} g(t) = \frac{d}{dt} \int L = \int \delta Lu_t = -\int (u_t)^2 \leq 0.
\]

Thus as time progresses, the functional decreases till we hit an extremum of the problem.

\[
\frac{d^2}{dt^2} g(t) = \frac{d}{dt} \int L = \int \delta L u_{tt} = \int L_{uu}(u_t)^2 + L_{uu_x} u_t u_{xt} + L_{u_x u_x} (u_t)^2 = \\
\int (L - \frac{d}{dx} L_{u_x}) u_{tt} + L_{u_x} u_{tt} |_{tt} + \int L_{uu}(u_t)^2 + 2 L_{uu_x} u_t u_{xt} + L_{u_x u_x} (u_t)^2 = \\
\int \delta L u_{tt} + \int L_{u_x x}(u_t)^2 + 2 L_{uu_x} u_t u_{xt} + L_{u_x u_x} (u_x)^2
\]

Now note that:

\[
\int \delta L u_{tt} = -\int u_t u_{tt} = \frac{1}{2} \frac{d^2}{dt^2} g(t)
\]

Thus we get:

\[
\frac{1}{2} \frac{d^2}{dt^2} g(t) = \int L_{uu}(u_t)^2 + 2 L_{uu_x} u_t u_{xt} + L_{u_x u_x} (u_x)^2
\]

If the second variation of the functional is positive definite then \( \int (u_t)^2 \) is decreasing in time and \( u \) evolves to an extremal of the functional.

Now we consider the problem of constrained minimization. We consider the following problem:

\[
\text{minimize } \int F(u, u_x)
\]

subject to \( \int G_i(u, u_x) = 0 \)  \( 1 \leq i \leq N \)

The solution must satisfy the following necessary Euler-Lagrange equations with \( \lambda_i \) as Lagrange multipliers.

\[
\delta F + \sum_i \lambda_i \delta G_i = 0 \quad F_{u_x} + \sum_i \lambda_i G_{i u_x} |_{x=0} = 0
\]

(5)
Now using the same procedure as above we let:

$$L = F + \sum_{i} \lambda_i G_i$$

Where $\lambda_i$ are Lagrange multipliers to be determined. Also note that:

$$\int L = \int F$$

Assuming that we start with a function that satisfies the constraints, we write the following evolution equations for $u$.

$$u_t = -\delta L = -\delta F - \sum_i \lambda_i \delta G_i F_{u_x} + \sum_i \lambda_i G_{iu_x}^1|_0 = 0 \quad \frac{d}{dt} \int G_i = 0 \quad (6)$$

Using theorem 3, we have

$$\frac{d}{dt} g(t) = \frac{d}{dt} \int L = \frac{d}{dt} \int F = -\int (u_t)^2$$

$$\frac{d^2}{dt^2} \int F = \frac{d^2}{dt^2} g(t) = \frac{d^2}{dt^2} \int L = \int L_{uu}(u_t)^2 + 2L_{u_x u_x} u_t u_{x_t} + L_{uu u_x} (u_{x_t})^2$$

Note that $\int F$ is decreasing in time for all values of $\lambda_i$, but the second derivative of $g(t)$ depends on the value of the Lagrange multipliers. Also note that although $\lambda_i$ are functions of time, their derivatives do not appear in the rate of change of the functional. That is because time derivatives of $\lambda_i$ are always multiplied by 0.

$$\frac{d}{dt} (\lambda_i \int G_i) = \lambda_i \int G_i + \lambda_i \frac{d}{dt} \int G_i = \lambda_i \frac{d}{dt} \int G_i$$

In order to find the Lagrange multipliers we use the second equation as:

$$0 = \frac{d}{dt} \int G_i = \int G_{iu_x} + G_{iu_x} u_{x_t} = \int \delta G_i u_t + G_{iu_x} u_{x_t}^1 = 0$$

We substitute for $u_t$ and get:

$$\int \delta G_i (\delta F + \sum_j \lambda_j \delta G_j) + G_{iu_x} (\delta F + \sum_j \lambda_j \delta G_j)^1|_0 = 0$$
\[ \int \delta G_i \delta F + \sum_j \lambda_j \int \delta G_i \delta G_j + G_{\text{int}} \delta F \|_0^1 + \sum_j \lambda_j G_{\text{int}} \delta G_j \|_0^1 = 0 \]

\[ \sum_j (\int \delta G_i \delta G_j) \lambda_j + \sum_j G_{\text{int}} \delta G_j \|_0^1 \lambda_j = -\int \delta G_i \delta F - G_{\text{int}} \delta F \|_0^1 \]

Thus we get a \( N \times N \) linear system of equations for \( \lambda \). Let us define matrices \( A, B, \) and \( C \) as:

\[ A_{ij} = \int \delta G_i \delta G_j \]

\[ B_{ij} = G_{\text{int}} \delta G_j \|_0^1 \]

\[ C_i = -\int \delta G_i \delta F - G_{\text{int}} \delta F \|_0^1 \]

Then we have the following system for \( \lambda \):

\[ (A + B)\lambda = C \]

The matrix \( B \) is zero in the case of fixed end points and periodic boundary conditions, in the other case we assume that \( B \) is small compared to \( A \). This procedure is dependent on the fact that we can solve the system for values of \( \lambda \). This can be done if the matrix \( A \) is invertible. The matrix \( A \) is called Gram matrix and we repeat the following well known result.

**Theorem 3 [3] The matrix \( A \) is invertible if and only if the first variations of the constraints are linearly independent.**

If the first variations are linearly dependent then it is easy to see that \( A \) is not invertible. Now assume that \( A \) is not invertible, then there is a linear relation between the rows. Thus assume that there are coefficients \( a_i \) such that:

\[ \sum_i a_i \int \delta G_i \delta G_j = 0 \Rightarrow \sum_j a_j \sum_i a_i \int \delta G_i \delta G_j = 0 \]

\[ \int \sum_j \sum_i a_i \delta G_i a_j \delta G_j = 0 \Rightarrow \int (\sum_i a_i \delta G_i)^2 = 0 \]

Thus the first variations are linearly dependent.

\[ \sum_i a_i \delta G_i = 0 \]
Definition 1 A set of conditions are said to be linearly independent if the matrix $A$, as defined above, is invertible at all points satisfying the constraints.

Actually this condition is too restrictive, one only needs $A$ to be invertible around the extremum point that one wants to calculate. Also if $A$ is not invertible, one still can solve for $\lambda$, but the solution is not unique. As an example let us consider case of one constraint with fixed end points boundary:

$$u_t = -\delta F - \lambda \delta G$$

$$0 = \int \delta G \delta F + \lambda \int \delta G \delta G$$

We can solve for $\lambda$ and get the following Integro-Differential equation for evolution of $u$:

$$u_t = -\delta F + \frac{\int \delta F \delta G}{\int (\delta G)^2} \delta G$$

Note that in this case the first variation of the constraint should not vanish on the manifold.

3 Weak Solutions

In this section we present formulation of the evolution problem as a weak solution. In the formulation of the minimization problem we wish to minimize $\int L(u, u_x)$. In this formulation one needs to specify the class of the functions to be considered. The integral implicitly assumes existence of the first derivative but the first derivative does not have to be continuous. In classical formulation as was considered in the last section one has the necessary Euler equations for the function with the accessory Weierstrass-Erdeman corner conditions. The two can be combined in the integral form of the Euler equation.

$$L_{ux} = \int L_u dx + c$$

This implies continuity of $L_{ux}$ but not $u_x$. Therefore one has to allow for corners in the solution. In this respect one can formulate the wake solution as the following.
Definition 2 Let \( L(u, u_{ux}) \) be continuously differentiable then we define a weak solution of \( L_u - \frac{d}{dx} L_{ux} = 0 \) as the function \( u \) such that

\[
\int L_u \phi + L_{ux} \phi_x = 0
\]

for all smooth \( \phi \) with compact support.

Note that one can derive the corner conditions by integration by parts.

\[
L^+_{ux} = L^-_{ux}
\]

In this relation we can formulate the evolution equation, \( u_t = \frac{d}{dx} L_{ux} - L_u \), as a weak solution.

Definition 3 Let \( L(u, u_{ux}) \) be continuously twice differentiable then we define a weak solution of \( u_t = \frac{d}{dx} L_{ux} - L_u \ u(x, 0) = u_0(x) \) as the function \( u \) such that

\[
\int_{\Omega} u \phi_t - L_{ux} \phi_x - L_u \phi + \int_0^1 u_0(x) \phi dx = 0
\]

for all smooth \( \phi \) with compact support in the \( t - x \) plane.

In this relation one can recover the corner conditions for the evolution equation in the following fashion. Let the curve \( \Gamma = (\alpha(t), t) \) be the path of a moving corner with \( u^I \) and \( u^II \) being two classical solutions separated by the graph of the curve \( \Gamma \). Then let \( \phi \) be a test function with compact support. Using the integration by parts formula we get:

\[
\int_{\Omega} (u \phi_t - L_{ux} \phi_x) = \int_{\partial \Omega} (u \zeta_t - L_{ux} \zeta_x) d\Gamma - \int_{\Omega} \phi (u_t + \frac{d}{dx} L_{ux})
\]

Where \((\zeta_t, \zeta_x)\) is the normal to the boundary and we have,

\[
\int_{\Omega_1 + \Omega_2} u \phi_t - L_{ux} \phi_x - L_u \phi =
\]

\[
\int_{\Omega_1} u \phi_t - L_{ux} \phi_x - L_u \phi + \int_{\Omega_2} u \phi_t - L_{ux} \phi_x - L_u \phi =
\]

\[
\int_{\partial \Omega_1} \phi (u \zeta_t - L_{ux} \zeta_x) dt + \int_{\partial \Omega_2} \phi (u \zeta_t - L_{ux} \zeta_x) dt =
\]
Since we have \((\zeta_t, \zeta_x) = (1, -\alpha'(t))\) in \(\Omega_1\) and \((\zeta_t, \zeta_x) = (-1, \alpha'(t))\) in \(\Omega_2\), we get
\[
\int_\Gamma \phi(\alpha'(t)(u^I - u^I)) + (L_{u^I} - L_{u^I})dt = 0
\]
Let \([u] = u^I - u^I\) then we have,
\[
\alpha'(t)[u] + [L_{u^I}] = 0
\]
since we have \([u] = 0\), we get \([L_{u^I}] = 0\). This calculation does not give us the speed of propagation of the discontinuity, but we can find it in this fashion; let
\[
[u] = u^I - u^I = u^I(\alpha(t), t) - u^I(\alpha(t), t)
\]
Since \(u\) is continuous, then we have: \(0 = \frac{d}{dt}[u] = [u_x]\alpha'(t) + [u_t]\) or equivalently, \([u_x]\alpha'(t) + [\frac{d}{dx} L_{u^I} - L_{u^I}] = 0\).

We conclude this section with an analog of a theorem of Lax and Wendroff about conservation laws [2]. Let us define
\[
v(x, t + \Delta t) = v(x, t) + \frac{\Delta t}{\Delta x}(g_1(x + \frac{\Delta x}{2}) - g_1(x - \frac{\Delta x}{2})) - \Delta t g_2(v, D^+v)
\]
where \(g_1\) is a consistent approximation to \(L_{u^I}\), and \(g_2\) is a consistent approximation to \(L_u\) in the sense that if \(v \rightarrow u\) and \(v' \rightarrow u'\) then \(g_1 \rightarrow L_{u^I}\), and \(g_2 \rightarrow L_u\) when \(\Delta x \rightarrow 0\).

**Theorem 4** Assume that as \(\Delta t\) and \(\Delta x\) converge to zero, \(v(x, t)\) converges boundedly almost everywhere to \(u(x, t)\) and \(D^+v(x, t)\) converges boundedly almost everywhere to \(u'(x, t)\) then \(u\) is a weak solution of \(u_t = \frac{d}{dx} L_{u^I} - L_u\).

\[
\frac{v(x, t + \Delta t) - v(x, t)}{\Delta t} = \frac{g_1(x + \frac{\Delta x}{2}, t) - g_1(x - \frac{\Delta x}{2}, t)}{\Delta x} - g_2(x, t)
\]
multiply the above equation by \(\phi(x, t)\) and integrate over time and space to get
\[
\int_0^1 \int_0^\infty \frac{v(x, t + \Delta t) - v(x, t)}{\Delta t} \phi(x, t) dx dt = \int_0^1 \int_0^\infty \frac{g_1(x + \frac{\Delta x}{2}, t) - g_1(x - \frac{\Delta x}{2}, t)}{\Delta x} \phi(x, t) dx dt - \int_0^1 \int_0^\infty g_2(x, t) \phi(x, t) dx dt
\]
This is equivalent to:

\[
\int_0^1 \int_0^\infty \phi(x, t - \Delta t) - \phi(x, t) \frac{v(x, t)}{\Delta t} dx dt - \int_0^1 \int_0^{\Delta t} \frac{v(x, t)}{\Delta t} \phi(x, t) dx dt
\]

\[
= \int_0^{1-\Delta x} \int_0^\infty \phi(x, t) - \phi(x + \Delta x, t) \frac{g_1(x + \frac{\Delta x}{2}, t)}{\Delta x} dx dt -
\]

\[
\int_0^\infty \int_{-\Delta x}^0 g_1(x + \frac{\Delta x}{2}, t) \phi(x + \Delta x, t) \frac{1}{\Delta x} dx dt +
\]

\[
\int_0^\infty \int_{1-\Delta x}^1 g_1(x + \frac{\Delta x}{2}, t) \phi(x, t) \frac{1}{\Delta x} dx dt
\]

\[
- \int_0^1 \int_0^\infty g_2(x, t) \phi(x, t) dx dt
\]

The boundary integrals drop out by applying the natural boundary conditions and if we pass to the limit we get

\[
\int \int - \phi_t u - \int u \phi = - \int \int \phi_x L_{ux} - \int \int L_u
\]

As an example we consider the functional,

\[
F(u_x) = \frac{1}{4} (u_x - 1)^2 (u_x + 1)^2
\]

Then we have the evolution equation,

\[
u_t = \frac{d}{dx} (u_x^3 - u_x)
\]

With the initial data

\[
u(x, 0) = \sin(2\pi x)
\]

First we use the boundary condition \(u(0, t) = 0\) and \(u(1, t) = 0\). We show the creation of the corner and its evolution in Figure 7.1. If we change to natural boundary conditions we observe that the corners move toward the boundary. In Figure 7.2 we show this situation.
4 Projection on Manifold

In applications one does not have a function that satisfies the constraints. As the first step of the solution one has to find a function that satisfies the constraints. We used the following procedure in our calculations. Let us assume we have \( \int G(u, u_x) = \sigma \) instead of \( \int G(u, u_x) = 0 \). We again introduce time variable \( t \) and let:

\[
\int G = \sigma e^{-t}
\]

then we have:

\[
\int \delta G u_t = -\sigma e^{-t} = -\int G
\]

Now we move in direction of the first variation according to

\[
u_t = \beta \delta G \quad G_{u_t} 1_0^1 = 0
\]

where \( \beta \) is to be found, but

\[
\int \delta G u_t = \beta \int (\delta G)^2 = -\int G
\]

Therefore we find the value of \( \beta \) as

\[
\beta = -\frac{\int G}{\int (\delta G)^2}
\]

\[
u_t = \frac{-\int G}{\int (\delta G)^2} \delta G \quad G_{u_t} 1_0^1 = 0
\]

In order to satisfy more than one constraint, one can march towards one at a time and keep the other ones constant, or try to satisfy all of them at the same time. The appropriate solution depends on the constraints and the desired accuracy and is highly problem dependent.

5 Constrained Heat Operator

In this part let us study the equations in detail for one simple case. Let us consider the following problem:

\[
\text{minimize} \quad \frac{1}{2} \int_0^1 (u_x)^2 dx
\]
subject to \( \int_0^1 (u - u_0) dx = 0 \) \( \int_0^1 \frac{1}{2} (u - u_0)^2 dx - \sigma = 0 \)

This problem can be motivated in this fashion. Let us consider a signal corrupted by additive random noise. Let the true signal be \( u_i \) and the observed signal be \( u_{oi} \), then we have,

\[ u_{oi} = u_i + X_i \]

Assuming \( X_i \) is a random variable with known p.d.f. then we have,

\[ \int (u - u_0) \approx \sum_i (u_i - u_{oi}) \Delta x = - \sum_i X_i \Delta x = - \frac{1}{N} \sum_i X_i. \]

For large values of \( N \) we have

\[ S_N = - \frac{1}{N} \sum_i X_i \rightarrow -E(X) \text{ in pr.} \]

Using the same argument we have,

\[ \int (u - u_0)^n \approx \sum_i (u_i - u_{oi})^n \Delta x = - \sum_i X_i^n \Delta x = - \frac{\sum_i X_i^n}{N}. \]

In light of the above argument we can impose the following constrains on the signal,

\[ \int (u - u_0)^n = (-1)^n E(X^n) \]

we wish to emphasize that these equations are valid in the limit of large number of signal values and also in a probability sense.

One can look at this procedure as an approximation problem. Let us elaborate on that. In calculating the Fourier coefficients of a function we consider the space:

\[ M = \{ u | u(x) = \sum_n a_n e^{2\pi i n x} \} \]

then we minimize the following norm;

\[ \int (\sum_n a_n e^{2\pi i n x} - u_0(x))^2 \]
In our approach we have the function space:

\[ M = \{ u | \int (u - u_0) = 0, \frac{1}{2} \int (u - u_0)^2 = \sigma \} \]

and we minimize the following norm:

\[ \int F(u_x) = \int u_x^2 \]

We have \( F(u, u_x) = (u_x)^2, \ G_1(u) = u - u_0, \) and \( G_2(u) = \frac{1}{2}(u - u_0)^2 - \sigma. \)

The first variations of the constraints are \( \delta G_1 = 1 \) and \( \delta G_2 = u - u_0. \) They are linearly independent because if we have

\[ \alpha + \beta(u - u_0) = 0, \]

then we take the integral over the unit interval and

\[ \alpha + \beta \int (u - u_0) = 0 \rightarrow \alpha = 0 \]

\( \beta \) must be zero otherwise, \( u - u_0 = 0 \) which contradicts \( \int (u - u_0)^2 = 2\sigma. \) One can see that the points where the first variations of the constraints vanish are potential pitfalls in calculations and they should be avoided.

As the first step of the calculations we need to find a point on the constraints. The first constraint is easy to satisfy, but the second one needs some work. We use the following equation:

\[ u_t = \beta(u - u_0) \quad \beta = -\frac{1}{2} \frac{\int (u - u_0)^2 + \sigma}{\int (u - u_0)^2} \]

One can see that if \( g(t) = \frac{1}{2} \int (u - u_0)^2 - \sigma, \) then it satisfies the following differential equation:

\[ \frac{d}{dt} g(t) = -g(t) \]

and its solution is:

\[ g(t) = g(0)e^{-t} \]

After calculating a guess function that satisfies the constraints, we solve the following equations:

\[ u_t = u_{xx} - \lambda_1 - \lambda_2(u - u_0) \]
\[ u_x(0) = 0 \quad u_x(1) = 0 \]

\[
\begin{pmatrix}
1 & \int(u - u_0) \\
\int(u - u_0) & \int(u - u_0)^2
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2
\end{pmatrix} = 
\begin{pmatrix}
\int u_{xx} \\
\int u_{xx}(u - u_0)
\end{pmatrix}
\]

Note that the linear system is solvable as long as we avoid \( u - u_0 \). The second variation of the system is

\[
\begin{pmatrix}
\lambda_2 & 0 \\
0 & 1
\end{pmatrix}
\]

and as long as \( \lambda_2 \) is positive we approach a solution of the variational problem.

Let us verify the relations for the rate of change of the functional and its second derivative directly. First note that \( \lambda_1 = 0 \) and also note that we have:

\[
\int(u - u_0)u_t = 0
\]

\[
\int(u_t)^2 + \int(u - u_0)u_{tt} = 0
\]

\[ u_{xx} = u_t + \lambda_2(u - u_0) \]

Now let us define \( g(t) = \frac{1}{2} \int(u_x)^2 \), then

\[
g'(t) = \int u_x u_{xt} = -\int u_{xx} u_t = -\int(u_t + \lambda_2(u - u_0)u_t =
\]

\[
-\int(u_t)^2 - \lambda_2 \int(u - u_0)u_t = -\int(u_t)^2
\]

\[
g''(t) = -2\int u_t u_{tt} = \int(u_{xt})^2 + \int u_x u_{xtt} =
\]

\[
\int(u_{xt})^2 - \int u_{xx} u_{tt} =
\]

\[
\int(u_{xt})^2 - \int u_{tt} + \lambda_2(u - u_0))u_{tt} =
\]

\[
\int(u_{xt})^2 - \int u_t u_{tt} - \lambda_2 \int(u - u_0))u_{tt} =
\]

\[
\int(u_{xt})^2 + \frac{1}{2}g''(t) + \lambda_2 \int(u_t)^2 =
\]
Therefore we have:

\[
\frac{1}{2}g''(t) = \int (u_{xt})^2 + \lambda_2 \int (u_t)^2
\]

This agrees with the general formula.

Now let us look at the solution of the evolution equation. Let \( u(t, x) \) and \( u_0(x) \) be represented by:

\[
u(t, x) = \sum_{n=1}^{\infty} a_n(t) \cos(n \pi x) \quad u_0(x) = \sum_{n=1}^{\infty} b_n \cos(n \pi x)
\]

then we have the following system of equations for \( a_n(t) \)

\[
a_n'(t) = -(n^2 \pi^2 + \lambda_2) a_n(t) + \lambda_2 b_n.
\]

Note that \( \lambda_2 \) is not constant and is a function of \( a_n(t) \) and is:

\[
\lambda_2 = \frac{1}{\sigma} \sum_{k=1}^{\infty} -\frac{n^2 \pi^2}{2} a_k(t)(a_k(t) - b_k)
\]

We observe that \( \lambda_2 \) is not constant but is varying slowly with respect to other variables. Let us assume it is constant and positive, then we have:

\[
a_n(t) = \frac{\lambda_2}{\lambda_2 + n^2 \pi^2} b_n + \left( a_n(0) - \frac{\lambda_2}{\lambda_2 + n^2 \pi^2} \right) e^{-t(n^2 \pi^2 + \lambda_2)}
\]

Now let us look at the steady state solutions of this equation. We have

\[
a_n = \frac{\lambda_2}{\lambda_2 + n^2 \pi^2} b_n
\]

\[
\sigma = \sum_{n=1}^{\infty} \frac{1}{2} \left( 1 - \frac{\lambda_2}{n^2 \pi^2} \right)^2 b_n^2
\]

\[
\frac{d\sigma}{d\lambda_2} = -\sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2(1 + \frac{\lambda_2}{n^2 \pi^2})^3} b_n^2 < 0
\]

If \( \lambda_2 > -\pi^2 \), then \( \sigma \) is a monotone decreasing function of \( \lambda_2 \) and for each value of \( \sigma \) there is a unique Lagrange multiplier. If we are only interested in positive values of \( \lambda \), then this requires \( 2\sigma \leq \int u_0^2 \) which is reasonable since it
means that the $L_2$ norm of the noise be less than the norm of the corrupted signal.

Let us discuss one specific problem. Let

$$u_{xx} - \lambda(u - u_0) = 0$$

$$u_0(x) = sgn(x) = \frac{x}{|x|} \quad -1 \leq x \leq 1$$

with $u_x(-1) = 0$  $u_x(1) = 0$ We can save some computations if we solve half of the above problem, which is:

$$u_{xx} - \lambda(u - u_0) = 0$$

$$u_0(x) = 1 \quad 0 \leq x \leq 1$$

with $u(0) = 0$  $u_x(1) = 0$ If $\lambda$ is positive, then let $\lambda = k^2$, then the solution is

$$u(x) = 1 - \frac{\cosh k(x - 1)}{\cosh k}$$

$$\sigma(k) = \frac{1}{\cosh^2 k} \left( \frac{1}{2} + \frac{\sinh 2k}{4k} \right)$$

One can see that $\sigma(0) = 1$ and that it is monotone decreasing function of $k$. If $\lambda$ is negative, then let $\lambda = -k^2$. The solution is

$$u(x) = 1 - \frac{\cos k(x - 1)}{\cos k}$$

$$\sigma(k) = \frac{1}{\cos^2 k} \left( \frac{1}{2} + \frac{\sin 2k}{4k} \right)$$

One can see that $\sigma$ becomes infinite at the eigenvalues of the boundary value problem which are $k = (n + \frac{1}{2})\pi$. If $\sigma$ is less than one then we have a smoothed out version of the $sgn(x)$ function. If we choose $\sigma$ too large then we start to deviate from the function by adding oscillation to the solution.
6 Numerical Schemes

In this section we describe some of the numerical techniques that were used. All the numerical techniques in this section are standard. We use the standard notation.

\[ u_{i,j}^n = u(t_n, x_i, y_j) \]

\[ \Delta_x^+ u_{i,j} = u_{i+1,j} - u_{i,j} \quad \Delta_x^- u_{i,j} = u_{i,j} - u_{i-1,j} \]

\[ D_x^+ u_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \quad D_x^- u_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \]

\[ D_x^0 u_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \]

Now using forward Euler and central differencing of the heat operator we have:

\[ u_{j}^{n+1} = u_{j}^{n} + \Delta t D^+ D^- u_{j}^{n} - \Delta t \lambda^n (u_{j}^{n} - u_{0j}) \]

For \( \lambda^n \) we have the equation:

\[ \sum D^+ D^- u_{j}^{n} (u_{j}^{n} - u_{0j}) - \lambda^n \sum (u_{j}^{n} - u_{0j})^2 = 0 \]

Note that this is equivalent to:

\[ \sum (u_{j}^{n+1} - u_{j}^{n}) (u_{j}^{n} - u_{0j}) = 0 \]

The equations satisfy the first constraint since,

\[ \sum u_{j}^{n+1} - u_{0j} = \sum \Delta t D^+ D^- u_{j}^{n} + (1 - \Delta t \lambda^n) \sum (u_{j}^{n} - u_{0j}) \]

The first sum on the right is a telescopic sum and is zero and the second one is zero in the first step, thus it stays zero for all steps.

Let us check the second constraint:

\[ \sum (u_{j}^{n+1} - u_{0j})^2 \Delta x - \sum (u_{j}^{n} - u_{0j})^2 \Delta x = \sum (u_{j}^{n+1} - u_{j}^{n})^2 \Delta x \approx \Delta t^2 \int u_i^2 \]

Which implies that the second constraint is not satisfied exactly but the error is first order in \( \Delta t \).
Let us try to modify $\lambda$ such that the second constraint is satisfied exactly. First drop the subscript $j$ from the equations and we get

$$\sum (u^{n+1} - u_0)^2 \Delta x - \sum (u^n - u_0)^2 \Delta x = \sum (u^{n+1} - u^n)(u^{n+1} + u^n - 2u_0) \Delta x =$$

$$\sum (\Delta t D Du^n - \Delta t \lambda^n(u^n - u_0))(2(u^n - u_0) + \Delta t D Du - \lambda^n \Delta t (u^n - u_0)) \Delta x =$$

Let $\beta = \Delta t \lambda^n$ then we have

$$\beta^2 \sum (u^n - u_0)^2 \Delta x - 2\beta(\sum (u^n - u_0)^2 \Delta x + \sum \Delta t D Du^n(u^n - u_0) \Delta x)$$

$$+ 2\sum \Delta t D Du(u^n - u_0) \Delta x + \Delta t^2 \sum (D Du^n)^2 \Delta x$$

Then let $\alpha = \Delta t \sum D Du^n(u^n - u_0) \Delta x$ and $\gamma = \sum (D Du)^2 \Delta x$ then

$$\sigma \beta^2 - 2(\sigma + \alpha) \beta + 2\alpha + \Delta t^2 \gamma = 0$$

We can solve for $\beta$

$$\beta = 1 + \frac{\alpha}{\sigma} - \sqrt{1 + \frac{\alpha^2}{\sigma^2} - \Delta t^2 \frac{\gamma}{\sigma}}$$

Note that in the previous approximation we had $\beta = \frac{a}{\sigma}$.

For our calculation we used a second order scheme to gain accuracy in time. The discretization in time was based on second order Runge-Kutta which improved accuracy so we could use larger time steps.

The results of the numerical results from the heat operator are shown in 7.3 and 7.4.

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Figure 2: Moving Corner
Figure 3: Heat Operator 1
Figure 4: Heat Operator 2
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