UNIQUE SOLVABILITY OF NONLINEAR
VOLterra EQUATIONS IN WEIGHTED SPACES

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Volterra equations in weighted spaces

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Unique solvability of nonlinear Volterra equations in weighted spaces

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Abstract: We investigate integral equations of the form
\begin{equation}
(*) \quad x(t) = g(t) + \int_{-\infty}^{t} F(t,s,x(s))\,ds .
\end{equation}
In general, this equation is history-dependent, so one needs to give an initial condition on 
\(( -\infty,0] \) in order to obtain a unique solution. By introducing a weight function on \( R \), we can
single out a class of admissible solutions, and give conditions for the unique solvability of
\((*) \) in this restricted class. We also study some Fredholm equations on these weighted
spaces. In addition, we also treat a class of equations of the first kind for which similar
conclusions can be drawn. The results of this paper continue the investigation carried out in

1. Introduction. In this paper we consider a Volterra integral equation of the form
\begin{equation}
(1.1) \quad x(t) = g(t) + \int_{-\infty}^{t} F(t,s,x(s))\,ds ,
\end{equation}
which can be regarded as a retarded equation whose delay is infinite. In general this
problem requires that one give an "initial function" on \( ( -\infty,0] \), after which the equation can
be treated with the techniques of standard Volterra equations. Thus the nonuniqueness of
solutions of \((1.1) \) is an intrinsic feature, which occurs even in the linear case (this seems to
have first been noticed by Love [7]). For example, the equation
\begin{equation}
(1.2) \quad x(t) = e^{-t} + \frac{1}{2} \int_{-\infty}^{t} e^{-2(t-s)} x(s) ds
\end{equation}
has the family of solutions \( x(t) = 2e^{-t} + ce^{-3t/2} \), where \( c \) is an arbitrary constant. In this
example, we observe that if we require that \( x(t)/e^{-t} \) be bounded on \( R \), then there exists
only one solution, namely \( x(t) = 2e^{-t} \). Thus we obtain uniqueness in the class of functions satisfying this exponential bound. The same phenomenon has been noticed with regard to the solutions of Abel’s equation (see Linz [6]); in fact, (1.1) can be considered as a type of singular integral equation. The selection mechanism in these examples can be formalized as follows: consider a weight function \( w(t) \), continuous and positive on a subinterval \( I \) of \( \mathbb{R} \) (our main interest is in an unbounded \( I \)) and define the space \( C(I,w) \) consisting of all continuous functions \( x: I \to \mathbb{R} \) such that \( \| x(.) \|_w = \sup_{t \in I} \left[ w(t)^{-1} |x(t)| \right] < \infty \). Then \( C(I,w) \) is a Banach space with the norm \( \| . \|_w \), and one can ask whether the given equation has a unique solution in \( C(I,w) \). In some cases, the appropriate phase space is \( L^p(I,w) \), where \( w \) is a measurable function. This space is defined as the set of all measurable functions on \( I \) such that the weighted norm

\[
\| x(.) \|_{p,w} = \int_I w(t)^{-1} |x(t)|^p \, dt < \infty
\]

It is easy to check that \( L^p(I,w) \) is a Banach space. Problems in weighted spaces arise in the study of spectral properties of Schrödinger operators (cf. Devinatz, Moeckel and Rejto [2]). A similar theory can be constructed for the case of a singular integral equation on a bounded interval (in this case, the "singularity" of the equation comes not from the unboundedness of the domain but from the kernel). The study of this topic in connection with the theory of asymptotic solutions to ODEs seems to have been started by Horn [5], and Love [7], and continued by Erdelyi [3], [4], Willett [11], as well as Olver [8].

The linear case was investigated in Rejto and Taboada [9], where conditions were given for the existence of a unique solution of a linear Volterra equation of the second kind in a weighted space and sharp resolvent estimates were obtained (we refer to this paper for earlier references on the subject). The goal of the present paper is to generalize these results to the nonlinear equation (1.1); these results can also be applied to integro-differential equations with the same type of unbounded delay.

The paper is organized as follows: in Section 2 we prove a general existence and uniqueness result in the space \( C(I,w) \) for a nonlinear Fredholm equation. If the equation is of Volterra type, we show that a much weaker condition guarantees existence and uniqueness. In Section 3 we extend these results to weighted \( L^p \) spaces. In Section 4 we study a class of integral equations of the first kind. Finally, in Section 5, we apply these results to a convolution-type Volterra equation.
2. Existence and uniqueness results in $C(I,w)$

Let $I$ be a (bounded or unbounded) closed subinterval of $\mathbb{R}$ and define $C(I,w)$ as in Section 1. Our first result is the following:

**Theorem 2.1 (Existence)** Consider the equation

(2.1) \[ x(t) = g(t) + \int_I F(t,s,x(s)) \, ds, \quad t \in I, \] where $F$ is continuous and satisfies the Lipschitz condition

(2.2) \[ |F(t,s,x) - F(t,s,y)| \leq L(t,s) |x - y|, \] where $L(t,s)$ is integrable on $I$.

Assume also, for convenience, that $F(t,s,0) = 0$.

Regarding $g$, suppose that

(2.3) \[ g \in C(I,w). \]

Finally, assume that the following inequality holds:

(2.4) \[ B = B(w,F) = \sup_{t \in I} \left| \int_I w(t)^{-1} w(s) \, L(t,s) \, ds \right| \leq \frac{1}{2}. \]

Then (2.1) has at least one solution in $C(I,w)$.

**Proof.** Consider the closed subset of $C(I,w)$ defined by $U = \{ x \in C(I,w) : \| x - g \| \leq b \}$ where $b$ is a number such that $\| g \|_w \leq b$. Define a mapping $T$ on $U$ by

\[ (Tx)(t) = g(t) + \int_I F(t,s,x(s)) \, ds. \]

We will show that (a) $T$ maps $U$ into itself; (b) $T$ is a contraction on $U$.

To prove (a), notice that

\[ \| Tx - g \|_w \leq \left\| \int_I F(t,s,x(s)) \, ds \right\|_w + \left\| \int_I [F(t,s,x(s)) - F(t,s,g(s))] \, ds \right\|_w \]

By using (2.2) and the fact that $F(t,s,0) = 0$, we have
\[ \| \int \left[ F(t, s, g(s)) \right] ds \| \leq \sup_{t \in I} \int w(t)^{-1}w(s) L(t, s) w^{-1}(s)g(s) ds \]
\[ \leq \left\{ \sup_{t \in I} \int w^{-1}(t)w(s) L(t, s) ds \right\} \| g \|_W \leq \frac{1}{2} \| g \|_W \leq \frac{b}{2}. \]

(Here we have used the fact that \( b \geq \| g \|_W \) and assumption (2.4).)

Similarly, we have
\[ \| \int [F(t, s, x(s)) - F(t, s, g(s))] ds \| \leq \| \int \left[ L(t, s) |x(s) - g(s)| \right] ds \| \leq \sup_{t \in I} \int w^{-1}(t)w(s) L(t, s) ds \leq \frac{1}{2} \| x - g \|_W \leq \frac{b}{2}. \]

It follows that \( \| Tx - g \| \leq b \), and \( T \) maps \( U \) into itself.

To prove (b), notice that
\[ \| Tx - Ty \|_W \leq \sup_{t \in I} \int w(t)^{-1}L(t, s) |x(s) - y(s)| ds \leq B \| x - y \|_W \leq \frac{1}{2} \| x - y \|_W. \]

Hence \( T \) is a contraction on \( U \), so there exists a solution, which is unique in \( U \). However, this does not prove uniqueness in \( C(I, W) \). The uniqueness issue is dealt with in the next result, which complements Thm. 2.1.

**Theorem 2.2 (Uniqueness)** Assume (2.2) and (2.3) hold as before, and instead of (2.4), assume that there exists a number \( \theta, 0 \leq \theta < 1 \), such that

\[(2.5) \quad \sup_{t \in I} \left\{ w(t)^{-1}w(s) L(t, s) \right\} ds \leq \theta < 1.\]

Then Eq. (2.1) has at most one solution in \( C(W, R) \).

**Proof.** Let \( x(t) \) and \( y(t) \) be two solutions of (2.1) in \( C(W, R) \). Then by subtraction we obtain

\[(2.6) \quad x(t) - y(t) = \int \left[ F(t, s, x(s)) - F(t, s, y(s)) \right] ds \]

and assumption (2.2) implies

\[(2.7) \quad |x(t) - y(t)| \leq \int L(t, s) |x(s) - y(s)| ds. \]
Let us defined the auxiliary functions \( q(s) = \sup_{t \in I} \{ w(t)^{-1}w(s) | L(t,s) | \} , s \in I \), and \( \phi(t) = w(t)^{-1} | x(t) - y(t) | \). Notice that by assumption (2.5) \( q(.) \) is integrable on \( I \), hence it is finite almost everywhere; notice also that \( \phi(.) \) is a uniformly bounded function. Then, as an immediate consequence of (2.7), we have

\[
(2.8) \quad \phi(t) \leq \int_I q(s)\phi(s) \, ds .
\]

By iteration of (2.8), one easily obtains, for any positive integer \( n \),

\[
(2.9) \quad \phi(t) \leq (\int_I q(\tau) \, d\tau)^n \int_I q(s)\phi(s) \, ds
\]

and assumption (2.5) implies that

\[
(2.10) \quad \phi(t) \leq \theta^n \int_I q(s)\phi(s) \, ds , \text{ therefore }
(2.11) \quad \phi(t) \leq \theta^{n+1} \sup_I \phi.
\]

From (2.11) we conclude, by letting \( n \to \infty \), that \( \phi(t) = 0 \) for all \( t \) in \( I \), which proves the uniqueness of the solution of (2.1).

By putting together the results of Thms. 2.1 and 2.2, and observing that \( B \leq B^* \), we obtain a sufficient condition for existence and uniqueness.

**Theorem 2.3 (Existence and Uniqueness)** Assume that (2.2) and (2.3) hold, and in addition \( B^* \leq \frac{1}{2} \). Then Eq. (2.1) has a unique solution in \( C(I,w) \).

Let us now restrict ourselves to the Volterra equation

\[
(2.12) \quad x(t) = g(t) + \int_a^t F(t,s,x(s)) \, ds
\]

where \( a \) can be finite or \(-\infty\); let \( I = [a,\infty) \) or \( \mathbb{R} \). In this situation, the result of Thm. 2.1 can be considerably strengthened as follows:

**Theorem 2.4** Assume that \( F \) satisfies the same conditions as in Theorem 2.1 on the triangular region \( a \leq s \leq t, t \in I, g \in C(I,w) \), and instead of (2.4) let us assume that the following inequality holds
(2.13) \[ 0 < B^* = \int \sup_{t>s} w(t)^{-1}w(s) L(t,s) \, ds < \infty. \]

Then Eq. (2.5) has a unique solution in \( C(I,w) \).

**Proof.** Let us define a sequence \( \{x_n(t)\} \) of successive approximations by

(2.14) \[ x_0(t) = g(t), \quad x_{n+1}(t) = g(t) + \int_a^t F(t,s,x_n(s)) \, ds. \]

Let us also define the function

(2.15) \[ q(s) = \sup_{t>s} \{ w(t)^{-1}w(s) L(t,s) \}, \ s \in I. \]

Notice that condition (2.6) can be written as \( \int q(s) \, ds < \infty \). By the Lipschitz condition, we see that the successive approximations satisfy

\[ \| x_{n+1} - x_n \|_w \leq \frac{1}{n!} \left( \int q(s) \, ds \right)^n \leq \frac{(B^*)^n}{n!} \| x_0 \|_w. \]

Therefore the sequence \( \{x_n(t)\} \) converges in norm in \( C(I,w) \) to a function \( x(t) \). To see that \( x(t) \) is a solution of (2.12), consider an arbitrary compact subinterval \( J \) of \( I \). Let \( M = \max_J w(t) \). Given \( \varepsilon > 0 \), there exists \( N = N(\varepsilon) \) such that \( \sup_J |x_n(t) - x(t)| \leq M\varepsilon \) for \( n \geq N(\varepsilon) \), which means that \( \{x_n(t)\} \) converges uniformly on \( J \). Therefore, we can pass to the limit in (2.14), and \( x(t) \) is a solution of (2.12) on any compact subinterval of \( I \), hence on all of \( I \).

We next show that this solution is unique in \( C(I,w) \). Let \( x(t) \) and \( y(t) \) be any two solutions of (2.12) in \( C(I,w) \). We have

\[ x(t) - y(t) = \int_a^t [F(t,s,x(s)) - F(t,s,y(s))] \, ds, \] which implies, as in Thm. 2.2,

(2.16) \[ |w(t)^{-1}\{x(t) - y(t)\}| \leq \int w(t)^{-1}w(s) L(t,s)w(s)^{-1} |x(s) - y(s)| \, ds \quad s \in I ; \ s \leq t \]

Let us define \( q(s) = \sup_{t>s} \{ w(t)^{-1}w(s) L(t,s) \}, \ s \in I, \) and \( \phi(t) = w(t)^{-1} |x(t) - y(t)| \). Then, as a consequence of (2.16), we have
(2.17) \[ \phi(t) \leq \int_a^t q(s)\phi(s) \, ds \] .

By induction, we obtain

(2.18) \[ \phi(t) \leq \frac{1}{n!} \left( \int_a^t q(s) \, ds \right)^n \leq \frac{(B*^n)}{n!} . \]

If we let \( n \to \infty \), it follows that \( \phi(t) = 0 \) for all \( t \) in \( I \), hence \( x(t) \) and \( y(t) \) are identical.

**Remark 2.1.** Notice that the conditions for existence and uniqueness are much less restrictive in the case of a Volterra equation. This difference between the behaviors of Fredholm and Volterra equations is typical. To give just one example, in the simplest case of a linear operator, the spectral radius of a Volterra operator is 0, whereas that of a Fredholm operator is usually nonzero (we refer to [9] for a treatment of the linear case in a weighted space).

**Remark 2.2** By taking \( w = 1 \), the results of Thms. 2.1-2.4 give conditions for the existence and/or uniqueness of a bounded solution for an integral equation of Fredholm of Volterra type.

### 3. Existence and Uniqueness in \( L^p(I,w) \)

In this section, we develop an existence and uniqueness theory in the Lebesgue space \( L^p(I,w) \), since these arise as the natural solution spaces in many applications. As before, we shall consider both Fredholm and Volterra equations of the form

(3.1) \[ x(t) = g(t) + \int_1^t F(t,s,x(s)) \, ds \quad , \quad t \in I \]

and

(3.2) \[ x(t) = g(t) + \int_a^t F(t,s,x(s)) \, ds \]

and we shall give conditions for the existence and/or uniqueness of solutions of these in \( L^p(I,w) \). Some of the proofs are very similar to those in Section 2, and therefore they will be omitted.
Theorem 3.1 Consider Eq. (3.1), under the following assumptions

(3.3) There exists a measurable function $L(t,s)$ such that

$$|F(t,s,x) - F(t,s,y)| \leq L(t,s) |x - y| \text{ for all } t,s \in I \text{ and } x,y \in R.$$

(3.4) $F(t,s,0) = 0$.

(3.5) $g \in LP(I,w)$

(3.6) Assume that the number $B$ defined by

$$B = \left[ \int_1 w(t)^{-1} \left( \int_I w(s)^{q/p} L(t,s)^{q} ds \right)^{p/q} dt \right]^{1/p}$$

satisfies $B \leq 1/2$. Then (3.1) has at least one solution in $LP(I,w)$.

The proof is very similar to that of Thm. 2.1, with the Hölder inequality playing the same role as the sup estimates; we omit the details.

Theorem 3.2 Assume that (3.3)-(3.5) hold, and in addition $B < 1$. Then Eq. (3.1) has at most one solution in $LP(I,w)$.

Proof. Let $x(.)$ and $y(.)$ be two solutions of (3.1) in $LP(I,w)$. Then, by subtraction, we obtain

$$|x(t) - y(t)| \leq \int_I L(t,s) |x(s) - y(s)| ds$$

which implies, by (3.3) and Hölder's inequality, that

(3.7) $w(t)^{-1/p} |x(t) - y(t)| \leq q(t) \left[ \int_I w(s)^{-1} |x(s) - y(s)|^p ds \right]^{1/p}$

where

(3.8) $q(t) = w(t)^{-1/p} \left[ \int_I w(s)^{q/p} L(t,s)^q ds \right]^{1/q}.$
By defining $\phi(t) = w(t)^{1/p} |x(t) - y(t)|$, we can rewrite (3.7) as

\[(3.9) \quad \phi(t) \leq q(t) \left[ \int_I \phi(s)^p \, ds \right]^{1/p} .\]

By raising both sides to the $p$-th power and integrating in $t$ over $I$, we obtain

\[\int_I \phi(t)^p \, dt \leq \int_I q(t)^p \, dt \cdot \int_I \phi(s)^p \, ds .\]

Therefore, since $\int_I q(t)^p \, dt = B < 1$, we have

\[\|\phi\|_p \leq B \|\phi\|_p ,\]

which implies that $\|\phi\|_p = \|x(.) - y(.)\|_{w,p} = 0$. Hence the solutions $x$ and $y$ coincide in $L^p(I,w)$.

**Theorem 3.3** Consider now the Volterra equation (3.2), where $F$ satisfies the same conditions as in Thm. 3.1 on the triangular region $a \leq s \leq t$, $t \in I$, and $g \in L^p(I,w)$. Let us also assume, instead of (3.6), that $B < \infty$. Then Eq. (3.2) has a unique solution in $L^p(w,I)$.

**Proof.** The existence of solutions follows by a procedure very similar to that of Thm 2.3, with $q(t)$ replaced by the function defined in (3.8). In order to prove uniqueness, we proceed as in Thm. 3.2 (with the appropriate change in the range of integration), to obtain

\[(3.10) \quad \phi(t) \leq q(t) \left[ \int_a^t \phi(s)^p \, ds \right]^{1/p} .\]

By the $L^p$ version of the Gronwall inequality (cf. [W], Lemma 2.2), this implies that $\|\phi\|_p = 0$, therefore $x = y$ in $L^p(I,w)$, which establishes uniqueness.
4. Equations of the First Kind

We now extend the above results to equations of the first kind. Let us note that, even in very simple examples, there is a lack of uniqueness of solutions. For example the integral equation

\[
(4.1) \quad \int_0^t (2t - 3s) x(s) \, ds = 0
\]

has the infinite family of solutions \( x(t) = ct \), where \( c \) is an arbitrary constant.

As we shall see, one can give (rather stringent) conditions for the unique solvability of Volterra equations of the first kind in a weighted space by using an approach similar to that of Section 2.

**Theorem 4.1** Consider the Volterra equation of the first kind

\[
(4.2) \quad \int_a^t F(t,s,x(s)) \, ds = g(t),
\]

where \( a \geq -\infty \), and let \( w \) be a weight function on \( I = [a,\infty) \) or \( \mathbb{R} \) and assume:

\[
(4.3) \quad F(t,s,x) \text{ and } \frac{\partial}{\partial t} F(t,s,x) \text{ are continuous for } a \leq s \leq t, x \in \mathbb{R}, \text{ and } \frac{\partial}{\partial t} F(t,s,x) \text{ satisfies the following Lipschitz condition in } x:
\]

\[
(4.4) \quad \frac{\partial}{\partial t} F(t,s,x) - \frac{\partial}{\partial t} F(t,s,y) \mid \leq L(t,s) \mid x - y \mid,
\]

where \( L(t,s) \) is an integrable function.

Assume also that

\[
(4.5) \quad \text{The equation } F(t,t,x) = z \text{ has a unique solution } x \text{ for all } z \in \mathbb{R} \text{ and } t \in I = [a,\infty),
\]

\[
(4.6) \quad \text{There exists a } \theta > 0 \text{ such that } \mid K(t,t,x) - K(t,t,y) \mid \geq \theta \mid x - y \mid \text{ for all } x,y \in \mathbb{R} \text{ and all } t \in I,
\]
(4.7) \quad g(a) = 0 \text{ and } g'(t) \in C(w,R).

Finally, assume that

(4.8) \quad B^* = \int \sup_{t>s} t w(t)^{-1} w(s) \ L(t,s) \ ds < \infty.

Then Eq. (3.2) has a unique solution in $C(w,R)$.

**Proof.** Let us differentiate (3.2) to obtain

(4.9) \quad F(t,t,x(t)) + \int_a^t \frac{\partial}{\partial t} F(t,s,x(s)) \ ds = g'(t).

Define now a sequence $\{x_n(.)\}$ in $C(w,R)$ by taking an arbitrary $x_0(.)$ in $C(w,R)$, and

(4.10) \quad F(t,t,x_{n+1}(t)) + \int_a^t \frac{\partial}{\partial t} F(t,s,x_n(s)) \ ds = g'(t).

Notice that assumption (3.5) ensures that $x_{n+1}(.)$ is a well-defined function. In order to prove that $x_{n+1}(.)$ remains in $C(w,R)$, and that the sequence converges in $C(w,R)$, let us estimate

(4.11) \quad | F(t,t,x_{n+1}(t)) - F(t,t,x_n(t)) | \leq \int_a^t (F_t(t,s,x_n(s)) - F_t(t,s,x_{n-1}(s))) \ ds \leq \int_a^t L(t,s) \ |x_{n+1}(s) - x_n(s)| \ ds.

Therefore, by (3.6), we have

(4.12) \quad |w(t)^{-1}[x_{n+1}(t) - x_n(t)]| \leq \theta^{-1} \int_a^t \sup_{s \geq t} t w(t)^{-1} w(s) \ L(t,s) \ |w(s)^{-1}[x_{n+1}(s) - x_n(s)]| \ ds.
Define \( q(s) = \sup_{t>s} \{ w(t)^{-1}w(s) | L(t,s) | \} \), \( s \in I \), and \( \phi_{n+1}(t) = w(t)^{-1} | x_{n+1}(t) - x_n(t) | \). Then

\[
(4.13) \quad | \phi_{n+1}(t) | \leq \int_{\mathbb{A}} q(s) \phi_n(s) \, ds
\]

and, by induction, it follows that

\[
(4.14) \quad | \phi_{n+1}(t) | \leq \frac{1}{n!} \left( \int_{\mathbb{A}} q(s) \, ds \right)^n B^* \| x_0 \|_w \leq \frac{(B^*)^{n+1}}{n!} \| x_0 \|_w.
\]

By taking the sup over \( t \in I \) on the left-hand side, we see that

\[
(4.15) \quad \| x_{n+1}(.) - x_n(.) \|_w \leq \frac{(B^*)^{n+1}}{n!} \| x_0 \|_w.
\]

This shows that \( x_{n+1}(.) \in C(w,R) \) for all \( n \geq 0 \), and that the sequence \( \{ x_{n+1}(.) \} \) is convergent in \( C(w,R) \). Therefore, Eq. (3.2) has a solution in \( C(w,R) \). The uniqueness of this solution can be proved by a method very similar to that used in Thm. 2.4, hence we omit the details.

5. Applications

In this section, we show how the existence and uniqueness theorems proved in the previous sections specialize for special types of equations. For simplicity, we shall consider the convolution equation

\[
(5.1) \quad x(t) = g(t) + \int_{-\infty}^{t} a(t - s) f(s, x(s)) \, ds,
\]

where we assume that \( a, g \) and \( f \) are continuous, and \( f \) satisfies

\[
(5.2) \quad | f(s, x) - f(s, y) | \leq \lambda(s) | x - y |,
\]

where \( \lambda \) is a measurable function.

As an application of Thm.2.4, we obtain:
Theorem 5.1 Assume that $g \in C(R,w)$ and

$$\int_{\mathbb{R}} \lambda(s) \sup_{t>s} \{ w(t)^{-1}w(s) |a(t-s)| \} ds < \infty.$$  

Then Eq. (5.1) has a unique solution in $C(R,w)$.

An interesting special case occurs when the weight is monotone:

Corollary 5.2 Let the same assumptions as in Thm. 5.1 hold, and in addition assume that the weight $w(.)$ is monotone nondecreasing. Then, if

$$\int_{\mathbb{R}} \lambda(s) \sup_{t>s} \{|a(t-s)|\} ds < \infty,$$

Eq. (5.1) has a unique solution in $C(R,w)$.

Proof. The result follows by noticing that $w(t)^{-1}w(s) \leq 1$ for all $t > s$.

Corollary 5.3 If, in addition to the assumptions of Cor. 5.2, $|a(.)|$ is monotone nonincreasing, one obtains a further simplification of (5.4):

$$\int_{\mathbb{R}} \lambda(s) ds < \infty.$$  

Remark The methods developed in this paper can also be applied to Volterra integro-differential equations, since these can be transformed into Volterra integral equations (with a modified kernel). The methods can also be easily adapted to nonlinear operators which are sums of Volterra and Fredholm operators, as considered in [11].

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