A CENTRAL LIMIT THEOREM FOR NON-LINEAR VECTOR FUNCTIONALS OF VECTOR GAUSSIAN PROCESSES

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Abstract

Let \( X_t = (X^1_t, X^2_t) \), \( t = \ldots, -1, 0, 1, \ldots \) be a stationary vector Gaussian random process with covariance function \( r_{ij}(n) = EX^i_tX^j_{t+n}, \ i, j = 1, 2 \). Let \( H(x, y) \) and \( K(x, y) \) be real valued functions with Hermit ranks \( \nu_1 \) and \( \nu_2 \), respectively. Let \( Z^N_H = \frac{1}{\sqrt{N}} \sum_{i=1}^N H(X^1_i, X^2_i) \) and \( Z^N_K = \frac{1}{\sqrt{N}} \sum_{i=1}^N K(X^1_i, X^2_i) \), \( N = 1, 2, \ldots \). We consider the limiting distribution of the vector process \( (Z^N_H, Z^N_K) \).

We established the following result: If \( \sum_n |r_{ij}(n)|^{\nu_0} < \infty, i, j = 1, 2 \), where \( \nu_0 = \min\{\nu_1, \nu_2\} \) and some minor conditions hold for the Hermit expansions of \( H \) and \( K \), then \( (Z^N_H, Z^N_K) \rightarrow (Z_H, Z_K) \) in distribution as \( N \) tends to \( \infty \), where \( Z^*_H \) and \( Z^*_K \) have a joint normal distribution.

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1 Introduction

Many authors have studied the asymptotic distribution of non-linear functions of Gaussian processes or fields [1,2,3,4,5,6,7,8]. They have established Central Limit Theorems (CLT) and Non-Central Limit Theorems (NCLT) depending on the Hermit ranks of the functionals and the rates of the correlation functions. The underlying processes or fields dealt with in those papers are real Gaussian and the functionals are one dimensional real-valued functions. We are interested in the generalization of the results obtained in one dimensional case to the vector case. In this paper we will formulate a CLT of 2-dimensional non-linear functionals of 2-dimensional stationary vector Gaussian process.

Let $X_t = (X^1_t, X^2_t)$, $t = \ldots, -1, 0, 1, \ldots$ be a stationary vector Gaussian process with $EX^1_t = EX^2_t = 0$ and $EX^i_t X^j_{t+n} = r_{ij}(n)$ for $i, j = 1, 2$ and for all $t, n$. Without loss of generality we can assume

$$r_{11}(0) = r_{22}(0) = 1$$
$$r_{12}(0) = r_{21}(0) = 0,$$

(1)

because otherwise we can define

$$\tilde{X}^1_t = X^1_t / \sqrt{E(X^1_t)^2}$$
$$\tilde{X}^2_t = (X^2_t - E(X^2_t \tilde{X}^1_t) \tilde{X}^1_t) / \sqrt{E[X^2_t - E(X^2_t \tilde{X}^1_t) \tilde{X}^1_t]^2}.$$ 

Then $\tilde{X} = (\tilde{X}^1_t, \tilde{X}^2_t)$ has the required properties.

Let $H(x, y)$ and $K(x, y)$ be such that $EH(X^1_t, X^2_t) = EK(X^1_t, X^2_t) = 0$ and $E[H(X^1_t, X^2_t)]^2 < \infty, E[K(X^1_t, X^2_t)]^2 < \infty$ for all $t$. Then $H$ and $K$ have the following expansions

$$H(X^1_t, X^2_t) = \sum_{j=0}^{\infty} \sum_{|m|=j} c_m H_m(X^1_t, X^2_t),$$
$$K(X^1_t, X^2_t) = \sum_{j=0}^{\infty} \sum_{|m|=j} d_m K_m(X^1_t, X^2_t),$$

(2)
with
\[ \sum_{j=0}^{\infty} \sum_{|m|=j} c_m^2 m_1! m_2! < \infty, \quad \text{and} \quad \sum_{j=0}^{\infty} \sum_{|m|=j} d_m^2 m_1! m_2! < \infty, \]
where \( m = (m_1, m_2), |m| = m_1 + m_2 \) and \( H_m(x, y) \) is the Hermite polynomial of two variables defined by
\[ H_m(x, y) = (-1)^{|m|} e^{x^2 + y^2} \frac{\partial^{|m|}}{\partial x^{m_1} \partial y^{m_2}} e^{-\frac{x^2 + y^2}{2}}. \]
We say that \( H \) has Hermite rank \( \nu_1 \), and \( K \) has Hermite rank \( \nu_2 \) if \( c_m = 0 \) for \( |m| < \nu_1 \) and \( c_m \neq 0 \) for some \( |m| = \nu_1 \), and \( d_m = 0 \) for \( |m| < \nu_2 \) and \( d_m \neq 0 \) for some \( |m| = \nu_2 \) respectively. Let \( \nu_0 = \min(\nu_1, \nu_2) \). Define, for \( N = 1, 2, \ldots, \)
\[ Z_H^N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} H(X^1_t, X^2_t), \]
\[ Z_K^N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} K(X^1_t, X^2_t). \]
We are interested in the limiting distribution of the process \( (Z_H^N, Z_K^N) \) as \( N \to \infty \). We shall use the following notations.
\[ Z_H^N(j) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \sum_{|m|=j} c_m H_m(X^1_t, X^2_t), \]
\[ Z_K^N(j) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \sum_{|m|=j} d_m K_m(X^1_t, X^2_t). \]
and
\[ H_m^N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} H_m(X^1_t, X^2_t), \]
\[ K_m^N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} K_m(X^1_t, X^2_t). \]
Using (6), we can write (4) as
\[ Z_H^N = \sum_{j=\nu_1}^{\infty} \sum_{|m|=j} c_m H_m^N, \]
\[ Z_K^N = \sum_{j=\nu_2}^{\infty} \sum_{|m|=j} d_m K_m^N. \]
To state the main theorem below, we need two assumptions; one on the structure of the underlying process $X_t$ and the other on the expansions of $H$ and $K$

(i) \[ \sum_{n=\infty}^\infty |r_{ij}(n)|^2 < \infty, \quad i, j = 1, 2. \]  

(ii) \[ \sum_{j=0}^\infty (\sum_{|m|=j} |c_m|)^2 j! < \infty, \]
\[ \sum_{j=0}^\infty (\sum_{|m|=j} |d_m|)^2 j! < \infty. \] (9)

**Theorem 1.** Suppose that the functions $H$ and $K$ have Hermite expansions (2) with ranks $\nu_1$ and $\nu_2$, respectively, and suppose (8) and (9) are satisfied. Then we have

\[ \lim_{N \to \infty} E(Z_H^N(j)Z_K^N(j)) \leq C j! \left( \sum_{|m|=j} |c_m| \right) \left( \sum_{|d|=j} |d_{\ell}| \right) \] (10)

for all $j \geq \max(\nu_1, \nu_2)$ and

\[ \lim_{N \to \infty} E(Z_H^N Z_K^N) = \rho < \infty, \text{ for some } \rho, \]
\[ \lim_{N \to \infty} E(Z_H^N)^2 = \sigma_1^2 < \infty, \text{ for some } \sigma_1^2, \]
\[ \lim_{N \to \infty} E(Z_K^N)^2 = \sigma_2^2 < \infty, \text{ for some } \sigma_2^2. \] (11)

Moreover,

\[ (Z_H^N, Z_K^N) \overset{d}{\to} (Z_H, Z_K), \] (12)

where $(Z_H, Z_K)$ is a jointly Gaussian random vector with

\[ E(Z_H^2) = \sigma_1^2, \quad E(Z_K^2) = \sigma_2^2 \quad \text{and} \quad E(Z_H^* Z_K^*) = \rho. \]

The condition (8) is a generalization of the condition used in [1] for the one dimensional case. The conditions in (9) are stronger than the $L^2$-conditions (3) on $H$ and $K$. In the one dimensional case, we only need the $L^2$-condition for $H$. Here we need the stronger conditions in (9) because, in the expressions in (2), we have, by (1),

\[ E[H_m(X_1^i, X_2^i)H_\ell(X_1^\ell, X_2^\ell)] = 0 \text{ for } m \neq \ell. \] But in expressions in (7), we only have
\[ E[H_m^N H_\ell^N] = 0 \text{ for } |m| \neq |\ell|, \text{ or } E[H_m^N H_\ell^N] \text{ may not be zero for } m \neq \ell \text{ but } |m| = |\ell|. \]
The non-orthogonality of $H^N_m$ and $H^N_\ell$ for $m \neq \ell$ and $|m| = |\ell|$ makes it necessary for the use of assumption (9). It seems that it is very hard to remove conditions like those in (9).

2 Proof of Theorem

First we will introduce a diagram formula about the expectation of a product of Hermit polynomials of standard Gaussian random variables. In order to set up the formula we need some preliminary. We call an undirected graph $G$ with $\sum_{j=1}^p (\ell^1_j + \ell^2_j)$ vertices a diagram of order $l = (\ell^1, \ell^2, \ldots, \ell^p)$, where $\ell^j = (\ell^1_j, \ell^2_j)$ for $j = 1, \ldots, p$, if it satisfies the following three conditions:

(i) The set of vertices $V$ of the graph $G$ has the form

$$V = \bigcup_{j=1}^p S_j,$$

where

$$S_j = L^1_j \cup L^2_j$$

and

$$L^1_j = \{(2j-1, n)|1 \leq n \leq \ell^1_j\},$$

$$L^2_j = \{(2j, n)|1 \leq n \leq \ell^2_j\},$$

$$j = 1, \ldots, p.$$ (for $\ell^k_j = 0, k = 1, 2$, define $L^k_j = \phi$). We call $S_j$ the $j^{th}$ sector of the graph $G$, $L^1_j$ the $(2j-1)^{th}$ level, and $L^2_j$ the $(2j)^{th}$ level of the graph $G$.

(ii) Every vertex is of degree 1.

(iii) Edges may pass only between different sectors, i.e. no edge passes between levels $L^1_j$ and $L^2_j$, for $j = 1, \ldots, p$.

Let $\Gamma = \Gamma(l) = \Gamma (\ell^1, \ldots, \ell^p)$ denote the set of diagrams with properties (i),(ii), and (iii) above. Given a graph $G \in \Gamma$ let $V(G)$ be the set of all edges of $G$ and let $V_G(L^1_{i_1}, L^1_{i_2}), i_1, i_2 = 1, 2; j_1, j_2 = 1, \ldots, p$ be the set of all edges pass between levels
$L_{i_1}^{j_1}$ and $L_{i_2}^{j_2}$. If $\omega \in V_G(L_{i_1}^{j_1}, L_{i_2}^{j_2})$ then define $d_1(\omega) = j_1$ and $d_2(\omega) = j_2$. Define $k_G(j) = |\{\omega \in G|d_1(\omega) = j\}|$. Indeed $k_G(j)$ is the cardinality of the set of edges $w \in V(G)$ which begin at the $j^{th}$ sector and end at sectors of indices higher than $j$.

**Definition 1.** We call a diagram regular if its sectors can be paired in such a way that no edge passes between sectors in different pairs.

Now let’s state the diagram formula for random vectors.

**Lemma 1.** (Diagram Formula) Let $(X_{t_1}, \ldots, X_{t_p})$, $p \geq 2$ be such that $X_{t_i} = (X_{t_i}^1, X_{t_i}^2)$, $i = 1, \ldots, p$ are jointly normal and for each $t, s \in \{t_1, \ldots, t_p\}$ $EX_t^1 = EX_t^2 = 0$, $E(X_t^1)^2 = E(X_t^2)^2 = 1$, $EX_t^1 X_s^2 = 0$, $EX_t^1 X_s^1 = r_{ij}(s-t), i, j = 1, 2$. Then we have

$$E(\prod_{i=1}^{p} H_{\ell}^{\omega}(X_{t_i}^1, X_{t_i}^2)) = \sum_{G \in \Omega(\ell)} \prod_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}),$$

where $u(t_{d_1(\omega)} - t_{d_2(\omega)}) = r_{i_1i_2}(t_{j_1} - t_{j_2})$ if $\omega \in V_G(L_{i_1}^{j_1}, L_{i_2}^{j_2})$ and $1 = (\ell^1, \ldots, \ell^p)$.

The lemma above is just slightly modified from the diagram formula for real-valued random variables. Note that the $(m_1, m_2)^{th}$ Hermit polynomial can be expressed as a product of $m_1^{th}$ and $m_2^{th}$ Hermit polynomials, that is,

$$H_m(x, y) = H_{(m_1, m_2)}(x, y) = H_{m_1}(x) H_{m_2}(y).$$

As a special case of the diagram formula, $p = 2$, we have

$$E(H_m(X_{t_1}^1, X_{t_1}^2) H_{\ell}(X_{s_1}^1, X_{s_1}^2)) = 0$$

(14)

if $|m| \neq |\ell|$, because there is no diagram between two sectors with different number of vertices. The property (14) is called the orthogonality of Hermit polynomials. Note that $|m| = m_1 + m_2$. It follows from (14) that

$$E(Z^N_H Z^K_N) = \sum_{j=\nu}^{\infty} E[Z^N_H(j) Z^K_N(j)],$$

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where $\nu = \max(\nu_1, \nu_2)$. First we will compute $E[Z_H^N(j)Z_K^N(j)]$ for $j \geq \nu$.

\[
E(Z_H^N(j)Z_K^N(j)) = \frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \sum_{|m|=|\ell|=j} c_m d_{\ell} E[H_{m_1}(X_{t_1}^1)H_{m_2}(X_{t_2}^2)H_{l_1}(X_{l_1}^1)H_{l_2}(X_{l_2}^2)]
\]

\[
= \sum_{|m|=|\ell|=j} c_m d_{\ell} \frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \sum_{G \in \Gamma(m, \ell)} \prod_{\omega \in \mathcal{V}(G)} u(t_{d_1}(\omega) - t_{d_2}(\omega))
\]

\[
= \sum_{|m|=|\ell|=j} c_m d_{\ell} \frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} \sum_{s=\max(0, m_1-l_2)}^{\min(m_1, l_1)} \binom{m_1}{s} \binom{l_1}{s} \binom{m_2}{l_2-s} \binom{l_2-m_1+s}{l_2-s}
\]

\[
\times[r_{t_1}^{s}(t_1-t_2) r_{t_2}^{m_1-s}(t_1-t_2) r_{t_1-t_2}^{s}(t_1-t_2) r_{t_1-t_2}^{l_1-s}(t_1-t_2)]
\]

\[
= \sum_{|m|=|\ell|=j} c_m d_{\ell} \sum_{s=\max(0, m_1-l_2)}^{\min(m_1, l_1)} \frac{m_1! m_2! l_1! l_2!}{s!(l_1-s)!(m_1-s)!(l_2-m_1+s)!(l_1-s)!}
\]

\[
\times \left[ \frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} r_{t_1}^{s}(t_1-t_2) r_{t_2}^{m_1-s}(t_1-t_2) r_{t_1-t_2}^{s}(t_1-t_2) r_{t_1-t_2}^{l_1-s}(t_1-t_2) \right].
\]  

(15)

With the condition (8) of the covariance functions we can show that

\[
\frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} r_{t_1}^{s}(t_1-t_2) r_{t_2}^{m_1-s}(t_1-t_2) r_{t_1-t_2}^{s}(t_1-t_2) r_{t_1-t_2}^{l_1-s}(t_1-t_2) \leq C_1
\]

(16)

for some constant $C_1$ and for all $N$. In fact, by Hölder’s inequality, we have

\[
\frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} r_{t_1}^{s}(t_1-t_2) r_{t_2}^{m_1-s}(t_1-t_2) r_{t_1-t_2}^{s}(t_1-t_2) r_{t_1-t_2}^{l_1-s}(t_1-t_2)
\]

\[
\leq \sum_{u=-N+1}^{N-1} \left(1 - \frac{|u|}{N}\right)|r_{t_1}^{s}(u)||r_{t_2}^{m_1-s}(u)||r_{t_2}^{l_1-s}(u)||r_{t_2}^{l_1-s}(u)]]^s
\]

\[
\leq \left[ \sum_u |r_{t_1}^{s}(u)|^s \right] \cdot \left[ \sum_u |r_{t_2}^{m_1-s}(u)|^s \right] \cdot \left[ \sum_u |r_{t_2}^{l_1-s}(u)|^s \right] \cdot \left[ \sum_u |r_{t_2}^{l_1-s}(u)|^s \right]
\]

\[
\leq \max_{i,j=1,2} \left\{ \sum_u |r_{ij}^{s}(u)|^s \right\} = C_1
\]

(17)

If some of the exponents $s, m_1-s, l_2-m_1+s$ and $l_1-s$ are zero, then the corresponding factor with zero exponent is bounded by 1. Therefore (16) holds true. By (16) and
the following identity

$$\sum_{s=\max(0,m_1-l_2)}^{\min(m_1,l_1)} \frac{m_1!m_2!l_1!l_2!}{s!(l_1-s)!(m_1-s)!(l_2-m_1+s)!} = (m_1 + m_2)!.$$ 

we have

$$|E(Z_H^N(j)Z_K^N(j))| \leq C_1 j! \left( \sum_{|m|=j} |c_m||\sum_{|\ell|=j} |d_\ell| \right). \quad (18)$$

By (18), Schwartz’s inequality and (9)

$$|E(Z_H^N Z_K^N)| = \left| \sum_{j=\nu}^{\infty} E(Z_H^N(j)Z_K^N(j)) \right| \leq \sum_{j=\nu}^{\infty} |E(Z_H^N(j)Z_K^N(j))|$$

$$\leq C_1 \sum_{j=\nu}^{\infty} j! \left( \sum_{|m|=j} |c_m||\sum_{|\ell|=j} |d_\ell| \right)$$

$$\leq C_1 \sqrt{\left[ \sum_{j=\nu}^{\infty} \left( \sum_{|m|=j} |c_m| \right)^2 j! \right] \left[ \sum_{j=\nu}^{\infty} \left( \sum_{|\ell|=j} |d_\ell| \right)^2 j! \right]}.$$

Now we will show the existence of \( \rho \) in (11). To show this it is enough to show that \( \{E(Z_H^N Z_K^N)\}_N \) is a Cauchy sequence in \( N \). For a given \( \varepsilon > 0 \) choose \( K \) sufficiently large so that, if \( K < M \leq N \),

$$|E(Z_H^N Z_K^N) - E(Z_H^M Z_K^M)|$$

$$= \left| \sum_{j=\nu}^{\infty} E(Z_H^N(j)Z_K^N(j)) - \sum_{j=\nu}^{\infty} E(Z_H^M(j)Z_K^M(j)) \right|$$

$$= \left| \sum_{j=\nu}^{\infty} [E(Z_H^N(j)Z_K^N(j)) - E(Z_H^M(j)Z_K^M(j))] \right|$$

$$= \left| \sum_{j=\nu}^{\infty} \left( \sum_{|m|=|\ell|=j} c_m d_{\ell j} \times \left[ \frac{1}{N} \sum_{t_1=1}^{N} \sum_{t_2=1}^{N} r_{11}^s (t_1 - t_2) r_{12}^{m_1-s} (t_1 - t_2) r_{22}^{l_2-m_1+s} (t_1 - t_2) r_{21}^{l_1-s} (t_1 - t_2) \right] \right. \right.$$
Then the absolute value of the expression inside the bracket of the last term is

\[
\sum_{M \leq |n| < N} (1 - \frac{|n|}{N}) r_{11}^{-s}(n) r_{12}^{m_1-s}(n) r_{22}^{l_2-m_1+s}(n) r_{21}^{l_1-s}(n) \]

\[+ \sum_{|n| < M} (1 - \frac{|n|}{M}) r_{11}^{-s}(n) r_{12}^{m_1-s}(n) r_{22}^{l_2-m_1+s}(n) r_{21}^{l_1-s}(n) \]

\[= \sum_{M \leq |n| < N} (1 - \frac{|n|}{N}) r_{11}^{-s}(n) r_{12}^{m_1-s}(n) r_{22}^{l_2-m_1+s}(n) r_{21}^{l_1-s}(n) \]

\[+ \sum_{|n| < M} [(1 - \frac{|n|}{N}) - (1 - \frac{|n|}{M})] r_{11}^{-s}(n) r_{12}^{m_1-s}(n) r_{22}^{l_2-m_1+s}(n) r_{21}^{l_1-s}(n) \]

\[\leq \sum_{M \leq |n| < N} |r_{11}^{-s}(n) r_{12}^{m_1-s}(n) r_{22}^{l_2-m_1+s}(n) r_{21}^{l_1-s}(n)| \]

\[+ \sum_{|n| < M} |n| (\frac{1}{M} - \frac{1}{N}) |r_{11}^{-s}(n) r_{12}^{m_1-s}(n) r_{22}^{l_2-m_1+s}(n) r_{21}^{l_1-s}(n)| \]

\[\leq \sum_{|n| < M} |n| (\frac{1}{M} - \frac{1}{N}) |r_{11}^{-s}(n) r_{12}^{m_1-s}(n) r_{22}^{l_2-m_1+s}(n) r_{21}^{l_1-s}(n)| \]

\[+ \sum_{K < |n|} |r_{11}^{-s}(n) r_{12}^{m_1-s}(n) r_{22}^{l_2-m_1+s}(n) r_{21}^{l_1-s}(n)|. \]

Since the second term above is the tail of a convergent series by (16) we can find a sufficiently large \( K \) which makes the second term less than \( \frac{\varepsilon}{2} \) and choose \( M \) and \( N \), so large that the first term is also less than \( \frac{\varepsilon}{2} \). Hence we have

\[
|E(Z_H^N Z_K^N) - E(Z_H^M Z_K^M)| \leq \varepsilon \sum_{j=\nu}^{\infty} \sum_{|m|=|\ell|=j} |c_m||d_\ell||j!|
\]

\[\leq \varepsilon \sum_{j=\nu}^{\infty} j! \sum_{|m|=|\ell|=j} |c_m||d_\ell|
\]

\[\leq \varepsilon \sum_{j=\nu}^{\infty} j! (\sum_{|m|=j} |c_m|)(\sum_{|\ell|=j} |d_\ell|)
\]

\[\leq \varepsilon \sqrt{\sum_{j=\nu}^{\infty} j!(\sum_{|m|=j} |c_m|)^2} \left( \sum_{j=\nu}^{\infty} j!(\sum_{|\ell|=j} |d_\ell|)^2 \right)^{1/2}.
\]

This implies the existence of \( \rho \) as defined in (11). As a special case, if \( H = K \) in the argument above, we have

\[
\lim_{N \to \infty} E(Z_H^N)^2 = \sigma_1^2 < \infty \quad \text{and} \quad \lim_{N \to \infty} E(Z_K^N)^2 = \sigma_2^2 < \infty.
\]

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We may assume that the expansions in (2) have only finite terms. To see this let \( H_1, K_1 \) be defined by, for a large number \( T \),
\[
H_1(x, y) = \sum_{j=T}^{\infty} \sum_{|m|=j} c_m H_m(x, y)
\]
\[
K_1(x, y) = \sum_{j=T}^{\infty} \sum_{|\ell|=j} d_\ell H_\ell(x, y).
\]

Then, by (18) and Schwartz’s inequality,
\[
|E(Z_{H_1}^N, Z_{K_1}^N)| = |\sum_{j=T}^{\infty} E(Z_H^N(j)Z_K^N(j))| \\
\leq \sum_{j=T}^{\infty} |E(Z_H^N(j)Z_K^N(j))| \\
\leq C_1 \left[ \sum_{j=T}^{\infty} j!( \sum_{|m|=j} |c_m|^2) \right] \left[ \sum_{j=T}^{\infty} j!( \sum_{|\ell|=j} |d_\ell|^2) \right] \\
\leq C_1 \left[ \sum_{j=T}^{\infty} j!( \sum_{|m|=j} |c_m|^2) \right] \left[ \sum_{j=T}^{\infty} j!( \sum_{|\ell|=j} |d_\ell|^2) \right].
\]

Since
\[
\sum_{j=\nu}^{\infty} j!( \sum_{|m|=j} |c_m|^2) < \infty \quad \text{and} \quad \sum_{j=\nu}^{\infty} j!( \sum_{|\ell|=j} |d_\ell|^2) < \infty,
\]
we can find a sufficiently large \( T \) such that
\[
|E(Z_{H_1}^N, Z_{K_1}^N)| < \varepsilon.
\]

Because of relation (19) we can restrict ourselves to the special case when \( H \) and \( K \) are polynomials, in other words, when the sums in (2) are finite.

In order to prove the remaining part of the Theorem we shall apply the method of moments. We will show that the moments of random vector \((Z_H^N, Z_K^N)\) tend to the moments of an appropriate jointly Gaussian random vector. Suppose a set \( \{1, 2, \ldots, n\} \), \( n \) is even, is given. We call the \( n/2 \) ordered pairs \((\tau(1), \tau(2)), (\tau(3), \tau(4)), \ldots, (\tau(n-1), \tau(n))\) a pairing of \( \{1, \ldots, n\} \) if \( \tau \) is a permutation on \( \{1, \ldots, n\} \) and satisfies

\[
(i) \quad \tau(2i - 1) < \tau(2i), \quad \text{for} \quad i = 1, \ldots, \frac{n}{2},
\]
\[
(ii) \quad \tau(2i - 1) < \tau(2j - 1), \quad \text{for} \quad 1 \leq i < j \leq \frac{n}{2},
\]
and this pairing is denoted by $\tau$. Let $P$ be the set of all pairing of $\{1, 2, \ldots, n\}$.

The following lemma is well-known and can be derived from Lemma 1.

**Lemma 2.** Let $X$ and $Y$ be two random variables with $EX = EY = 0, EX^2 = \sigma_1^2, EY^2 = \sigma_2^2$ and $EXY = \rho$. Then $X$ and $Y$ are jointly Gaussian, if and only if, for every positive integers $m$ and $n$,

$$E(X^mY^n) = \begin{cases} \sum_{\tau \in P} \prod_{i=1}^{m+n} h_i(\tau) & \text{if } m + n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

where $P$ is the set of all pairings of $\{1, 2, \ldots, m + n\}$ and

$$h_i(\tau) = \begin{cases} \sigma_1^2 & \text{if } \tau(2i - 1), \ \tau(2i) \leq m \\ \rho & \text{if } \tau(2i - 1) \leq m < \tau(2i) \\ \sigma_2^2 & \text{if } \tau(2i - 1), \ \tau(2i) > m, \end{cases}$$

$i = 1, \ldots, \frac{m+n}{2}$.

By Lemma 2, it is enough to show that, for all positive integers $m$ and $n$,

$$\lim_{N \to \infty} E[(Z_H^N)^m(Z_K^N)^n] = \begin{cases} \sum_{\tau \in P} \prod_{i=1}^{\frac{m+n}{2}} h_i(\tau) & \text{if } m + n \text{ is even} \\ 0 & \text{otherwise}. \end{cases} \quad (20)$$

Since we may assume

$$H(x, y) = \sum_{j=\nu_1}^{K_1} \sum_{|\ell|=j} c_\ell H_{\ell} \text{ and } K(x, y) = \sum_{j=\nu_2}^{K_2} \sum_{|\ell|=j} d_\ell H_{\ell},$$

we have

$$Z_H^N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \sum_{j=\nu_1}^{K_1} \sum_{|\ell|=j} c_\ell H_{\ell}(X_{t1}^1, X_{t2}^2)$$

$$Z_K^N = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} \sum_{j=\nu_2}^{K_2} \sum_{|\ell|=j} d_\ell H_{\ell}(X_{t1}^1, X_{t2}^2)$$

and

$$(Z_H^N)^m = \frac{1}{(\sqrt{N})^m} \sum_{t_1=1}^{N} \cdots \sum_{t_m=1}^{N} \sum_{j \in E(m, K_1)} \sum_{i=1}^{m} c_{\ell}^{i} H_{\ell_1}^{i}(X_{t1}^1) H_{\ell_2}^{i}(X_{t2}^2)$$

$$(Z_K^N)^n = \frac{1}{(\sqrt{N})^n} \sum_{t_1=1}^{N} \cdots \sum_{t_n=1}^{N} \sum_{j \in E(n, K_2)} \sum_{i=1}^{n} d_{\ell} H_{\ell_1}^{i}(X_{ti}^1) H_{\ell_2}^{i}(X_{ti}^2)$$

$$E[(Z_H^N)^m(Z_K^N)^n] = \frac{1}{(\sqrt{N})^{m+n}} \sum_{t \in A} \sum_{j \in E} \sum_{i \in B} \sum_{i=1}^{m+n} h_i(\tau) \sum_{i=1}^{m+n} H_{\ell_1}^{i}(X_{t1}^1) H_{\ell_2}^{i}(X_{t2}^2)],$$

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where

\[ A = A(m+n, N) = \{ t = (t_1, \ldots, t_{m+n}) \mid 1 \leq t_i \leq N, \ i = 1, \ldots, m+n \}, \]

\[ E = E(m+n, (K_1, K_2)) \]

\[ = \{ j = (j_1, \ldots, j_{m+n}) \mid \nu_1 \leq j_i \leq K_1, \text{if } 1 \leq i \leq m; \]

\[ \nu_2 \leq j_i \leq K_2, \text{if } m + 1 \leq i \leq m+n \}, \]

\[ B = B(m+n, j) \]

\[ = \{ l = (\ell^1, \ldots, \ell^{m+n}) \mid 0 \leq l^1_i, l^2_i \leq j_i; \ |\ell^i| = l^1_i + l^2_i = j_i, \ i = 1, \ldots, m+n \}, \]

\[ \overline{c} = \prod_{i=1}^{m} c_{\ell^i}, \]

\[ \overline{d} = \prod_{i=m+1}^{m+n} d_{\ell^i}, \]

and

\[ E(m, K_1) = \{ j = (j_1, \ldots, j_m) \mid \nu_1 \leq j_i \leq K_1, \text{if } 1 \leq i \leq m \}. \]

Note that \( B(m+n, j) \) and \( E(m+n, (K_1, K_2)) \) does not increase with \( N \). Let’s fix \( j \) and \( l \). Then we have by the Diagram Formula

\[ \frac{1}{(\sqrt{N})^{m+n}} \sum_{t \in A} \overline{c} \overline{d} E[ \prod_{i=1}^{m+n} H_{l^1_i} (X^1_{x_i}) H_{l^2_i} (X^2_{x_i}) ] \]

\[ = \frac{1}{(\sqrt{N})^{m+n}} \sum_{t \in A} \overline{c} \overline{d} \sum_{G \in \Gamma(j, l)} \prod_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \]

where \( \Gamma(j, l) \) is defined as the set of all diagrams \( \Gamma(j, l) = \Gamma(\ell^1, \ldots, \ell^{m+n}) \), with \( 0 \leq l^1_i, l^2_i \leq j_i, \ |\ell^i| = l^1_i + l^2_i = j_i, \ i = 1, \ldots, m+n \). Define

\[ T_G(j, l, N) = \frac{1}{(\sqrt{N})^{m+n}} \sum_{t \in A} \prod_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) . \] (21)

For clarity, in the following, we shall denote \( \Gamma(j, l) \) as the set \( \Gamma(l) \) of diagrams such that \( l \in B(m+n, j) \) and denote \( G(j, l) \) as an element of \( \Gamma(j, l) \).

We need the following proposition to prove our theorem and we shall delay the proof of this proposition to the next section.

**Proposition 1.** If \( G = G(j, l) \in \Gamma(j, l) \) is not a regular diagram, then

\[ \lim_{N \to \infty} T_G(j, l, N) = 0. \]
First, we shall prove the relation (20) by applying Proposition 1. By (21), we have

\[ E[(Z^N_H)^m(Z^N_K)^n] = \sum_j \sum_1 \sum_{G \in \Gamma(j,1)} \bar{c} \bar{d} \ T_G(j,1,N). \]  

(22)

Let \( \Gamma^*(j,1) \) denote the set of all regular diagrams in \( \Gamma(j,1) \). If \( m+n \) is an odd number then \( \Gamma^*(j,1) \) is empty. Hence the Proposition 1 together with the relation (22) imply that

\[ \lim_{N \to \infty} E[(Z^N_H)^m(Z^N_K)^n] = 0 \text{ if } m+n \text{ is an odd number}, \]  

(23)

If \( m+n \) is an even number, then write \( m+n = 2q \). Let’s fix a diagram \( G(j,1) \in \Gamma^*(j,1) \). Then there is a pairing \( \tau = ((\tau(1), \tau(2)), \ldots, (\tau(2q-1), \tau(2q))) \) of \( \{1, \ldots, 2q\} \) such that edges go only between sectors \( S_{\tau(2i-1)} \) and \( S_{\tau(2j)} \), \( i = 1, \ldots, q \). Note that \( |\bar{t}_{\tau(2i-1)}| = j_{\tau(2i-1)} = j_{\tau(2i)} = |\bar{t}_{\tau(2i)}| \), or \( S_{\tau(2i-1)} \) and \( S_{\tau(2j)} \) have the same cardinality for \( i = 1, \ldots, q \). Now we can write

\[ T_G(j,1,N) = \frac{1}{(\sqrt{N})^{2q}} \sum_{t \in A} \prod_{i=1}^{q} \left( \prod_{\omega \in V(G_i)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \right), \]

where \( V(G_i) = \{ \omega \in V(G) | d_1(\omega) = \tau(2i-1), d_2(\omega) = \tau(2i) \} \). Denote

\[ E^*(p, (K_1, K_2)) \]

= \{ j \in E \mid \exists \text{ a pairing } \tau \text{ of } \{1, \ldots, p\} \text{ with } j_{\tau(2i-1)} = j_{\tau(2i)}; \forall i = 1, \ldots, \frac{p}{2} \},

where \( p = m+n \). If \( j \) is not in \( E^*(p, (K_1, K_2)) \), then \( \Gamma^*(j,1) = \phi \). By Proposition 1

\[ \lim_{N \to \infty} E[(Z^N_H)^m(Z^N_K)^n] = \sum_j \sum_1 \sum_{G \in \Gamma(j,1)} \bar{c} \bar{d} \lim_{N \to \infty} T_G(j,1,N) \]

= \sum_{J \in E^*} \sum_{\omega \in B} \sum_{G \in \Gamma^*} \bar{c} \bar{d} \lim_{N \to \infty} T_G(j,1,N) \]

= \lim_{N \to \infty} \frac{1}{(\sqrt{N})^{m+n}} \sum_{t \in A} \sum_{\omega \in V(G)} \sum_{\omega \in V(G)} \bar{c} \bar{d} \prod_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \]

By comparing individual terms appeared in each side of the following equation we can show relation below holds true.

\[ \sum_{J \in E^*} \sum_{\omega \in B} \sum_{G \in \Gamma^*} \bar{c} \bar{d} \prod_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \]

= \sum_{\tau \in E^*} \left( \sum_{j \in E^*} \sum_{\omega \in B} \sum_{G \in \Gamma^*} \alpha_{2i-1} \alpha_{2i} \prod_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \right),
where

\[ E^*_i = E^*_i(m+n, (K_1, K_2)) \]
\[ = \{ j_i = (j_{\tau(2i-1)}, j_{\tau(2i)}) | \nu_1 \leq j_{\tau(k)} \leq K_1, \ if \ \tau(k) \leq m; \]
\[ \nu_2 \leq j_{\tau(k)} \leq K_2, \ if \ \tau(k) > m; k = 2i - 1, 2i \}, \]

\[ B_i = B_i(m+n, j_i) \]
\[ = \{ l_i = (\ell^{\tau(2i-1)}, \ell^{\tau(2i)}) | [\ell^{\tau(2i-1)}] = [\ell^{\tau(2i)}] = j_{\tau(2i-1)} = j_{\tau(2i)}; \]
\[ 0 \leq l_i^{\tau(k)}, l_i^{\tau(k)} \leq j_{\tau(k)} \ for \ k = 2i - 1, 2i \}, \]

\[ \Gamma^*_i = \Gamma^*(j_i, l_i), \]

\[ \alpha_k = \begin{cases} 
\sigma^{\tau(k)}_2 & if \ \tau(k) \leq m \\
\sigma^{\tau(k)}_2 & \tau(k) > m \\
\end{cases} 
\]

Therefore

\[ \lim_{N \to \infty} \frac{1}{(\sqrt{N})^{m+n}} \sum_{t \in A} \sum_{j \in E^*} \sum_{l \in B} \sum_{G \in \Gamma^*(j, l)} \sum_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \]
\[ = \lim_{N \to \infty} \frac{1}{(\sqrt{N})^{m+n}} \left[ \sum_{\tau \in P} \prod_{i=1}^{m+n} \left[ \sum_{t_i \in T_i} \sum_{j_i \in E_i^*} \sum_{l_i \in B_i} \sum_{G \in \Gamma^*_i} \sum_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \right] \right] \]
\[ = \sum_{\tau \in P} \prod_{i=1}^{m+n} h_i(\tau), \]

where the sum \[ \sum_{t_i \in T_i} \] means the double sum \[ \sum_{t_{\tau(2i-1)}} \sum_{t_{\tau(2i)}} \]

\[ h_i(\tau) = \begin{cases} 
\sigma^{\tau(2i-1)}_2, \tau(2i) \leq m \\
\rho \ & \tau(2i-1) \leq m < \tau(2i) \\
\sigma^{\tau(2i-1)}_2 \ & m < \tau(2i-1), \tau(2i) \end{cases} \]

Hence the theorem is proved.

3 Proof of the Proposition 1

Let \( G = G(j, l) \) be a non regular diagram. Let \( \sigma \) be a permutation on \( \{1, 2, \ldots, p\} \),

where \( p = m + n \). Define \( \sigma G \) in the following way:
(i) The $\sigma(j)^{th}$ sector of $\sigma G$ is $(l_1^j, l_2^j)$.

(ii) $\omega = \{(m_1, n_1), (m_2, n_2)\} \in V(G)$ if and only if

$$\sigma \omega = \{(\sigma(m_1), n_1), (\sigma(m_2), n_2)\} \in \sigma V(G).$$

Observe that, for $G = G(j, l) \in \Gamma(j, l) = \Gamma((l_1^1, l_2^1), \ldots, (l_1^p, l_2^p))$,

$$\sum_{t_1=1}^{N} \cdots \sum_{t_p=1}^{N} \prod_{i=1}^{p} C_{T^i} \prod_{\omega \in V(G)} u(t_{d_1}(\omega) - t_{d_2}(\omega))$$

$$= \sum_{t_{\sigma(1)}=1}^{N} \cdots \sum_{t_{\sigma(p)}=1}^{N} \prod_{i=1}^{p} C_{T^\sigma(i)} \prod_{\sigma \omega \in \sigma V(G)} u(t_{d_1}(\sigma \omega) - t_{d_2}(\sigma \omega))$$

$$= \sum_{t_1=1}^{N} \cdots \sum_{t_p=1}^{N} \prod_{i=1}^{p} C_{T^\sigma(i)} \prod_{\omega \in \sigma V(G)} u(t_{d_1}(\omega) - t_{d_2}(\omega)) \quad (24)$$

For every diagram $G$ there exists a permutation $\sigma$ such that $G' = \sigma G$, where $G'$ has the following property; $G' \in \Gamma((l_1^1, l_2^1), \ldots, (l_1^p, l_2^p))$ with some pairs of integers $(l_1^1, l_2^1), \ldots, (l_1^p, l_2^p)$ such that

$$l_1^1 + l_2^1 \leq l_1^2 + l_2^2 \leq \cdots \leq l_1^p + l_2^p. \quad (25)$$

Because of relation (24) it is enough to prove the proposition only for the diagram $G \in \Gamma(j, l)$ which have the property (25). Now we can write

$$\frac{1}{(\sqrt{N})^p} \sum_{t \in A(p, N)} \prod_{i=1}^{p} C_{T^i} \prod_{\omega \in V(G)} u(t_{d_1}(\omega) - t_{d_2}(\omega))$$

$$\leq \frac{1}{(\sqrt{N})^p} \sum_{t \in A(p, N)} \prod_{i=1}^{p} \left| C_{T^i} \right| \prod_{\omega \in V(G)} \left| u(t_{d_1}(\omega) - t_{d_2}(\omega)) \right| \right]. \quad (26)$$

But

$$\prod_{\omega \in V(G)} |u(t_i - t_{d_2}(\omega))| \leq \frac{1}{k_G(i)} \sum_{\omega \in V(G)} |u(t_i - t_{d_2}(\omega))| ^{k_G(i)}, \quad (27)$$

by Jensen’s inequality. Let

$$A_{\gamma}(i) = \bigcup_{n \geq i} V_G(L_n^i, L_n^m),$$

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where $\delta, \gamma = 1, 2$, so that

$$\{ \omega \in V(G) \mid d_1(\omega) = i \} = \bigcup_{\delta, \gamma = 1, 2} A_{\delta \gamma}(i).$$

Therefore we have

$$\left| \frac{1}{(\sqrt{N})^p} \sum_{t \in A(p,N)} \prod_{i=1}^{p} c_{t,i} \prod_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \right|$$

$$\leq \frac{1}{(\sqrt{N})^p} \sum_{t \in A(p,N)} \prod_{i=1}^{p} \left[ \left| c_{t,i} \right| \frac{1}{k_G(i)} \sum_{\omega \in V(G)} \left| u(t_i - t_{d_2(\omega)}) \right|^{k_G(i)} \right]$$

$$= \frac{1}{(\sqrt{N})^p} \sum_{t_2=1}^{N} \cdots \sum_{t_p=1}^{N} \prod_{i=1}^{p} \left[ \frac{1}{k_G(i)} \sum_{\omega \in V(G)} \left| u(t_i - t_{d_2(\omega)}) \right|^{k_G(i)} \right]$$

$$\times \left[ \prod_{i=2}^{p} \left[ \frac{1}{k_G(i)} \sum_{\omega \in V(G)} \left| u(t_i - t_{d_2(\omega)}) \right|^{k_G(i)} \right] \right]$$

$$= \frac{1}{(\sqrt{N})^p} \sum_{t_2=1}^{N} \cdots \sum_{t_p=1}^{N} \prod_{i=1}^{p} \left[ \frac{1}{k_G(i)} \sum_{\omega \in V(G)} \left| u(t_i - t_{d_2(\omega)}) \right|^{k_G(i)} \right]$$

$$\times \left[ \sum_{t_1=1}^{N} \left[ \frac{1}{k_G(1)} \sum_{\omega \in V(G)} \left| u(t_1 - t_{d_2(\omega)}) \right|^{k_G(1)} \right] \right]. \quad (28)$$

Observe that

$$\sum_{t_1=1}^{N} \frac{1}{k_G(1)} \sum_{\omega \in V(G)} \left| u(t_1 - t_{d_2(\omega)}) \right|^{k_G(1)}$$

$$= \frac{1}{k_G(1)} \sum_{t_1=1}^{N} \sum_{\delta, \gamma = 1, 2} \sum_{\omega \in A_{\delta \gamma}(1)} \left| r_{\delta \gamma}(t_1 - t_{d_2(\omega)}) \right|^{k_G(1)}$$

$$\leq \frac{1}{k_G(1)} \sum_{\delta, \gamma = 1, 2} \sum_{\omega \in A_{\delta \gamma}(1)} \sup_{1 \leq \nu \leq N} \sum_{t_1=1}^{N} \left| r_{\delta \gamma}(t_1 - \nu) \right|^{k_G(1)}$$

$$\leq \frac{1}{k_G(1)} \sum_{\delta, \gamma = 1, 2} \sum_{\omega \in A_{\delta \gamma}(1)} \sum_{|s| < N} \left| r_{\delta \gamma}(s) \right|^{k_G(1)}$$

$$\leq \sum_{|s| < N} \sum_{\delta, \gamma = 1, 2} \left| r_{\delta \gamma}(s) \right|^{k_G(1)}. \quad (29)$$

By (28) and (29) and the iteration of the above procedure for $t_2, \ldots, t_p$ we have

$$\left| \frac{1}{(\sqrt{N})^p} \sum_{t \in A(p,N)} \prod_{i=1}^{p} c_{t,i} \prod_{\omega \in V(G)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \right|$$

$$\leq \frac{1}{(\sqrt{N})^p} \prod_{i=1}^{p} \left| c_{t,i} \right| \sum_{|s| < N} \sum_{\delta, \gamma = 1, 2} \left| r_{\delta \gamma}(s) \right|^{k_G(i)}. \quad (30)$$
Since \( t_1^i + t_2^i \geq \nu_0 \) for all \( i \), we have

\[
\sum_{|s|<N} \sum_{\delta, \gamma=1,2} |r_{\delta \gamma}(s)|^{k_G(i)} \leq C \cdot N^{1-g(i)} \tag{31}
\]

if \( k_G(i) = 0 \) or \( k_G(i) = t_1^i + t_2^i \), where \( g(i) = \frac{k_G(i)}{t_1^i + t_2^i} \). On the other hand, we shall claim that

\[
\sum_{|s|<N} \sum_{\delta, \gamma=1,2} |r_{\delta \gamma}(s)|^{k_G(i)} = o(N^{1-g(i)}) \tag{32}
\]

if \( 0 < k_G(i) < t_1^i + t_2^i \). Indeed, because of \( \sum |r_{\delta \gamma}(s)|^\ell < \infty \); \( \ell \geq \nu_0 \); \( \delta, \gamma = 1,2 \), for any \( \varepsilon > 0 \), we can find a positive integer \( N_0 \) such that

\[
\sum_{|s|>N_0} |r_{\delta \gamma}(s)|^\ell < \varepsilon \quad \delta, \gamma = 1,2 \quad \text{for all} \quad \ell \geq \nu_0 .
\]

By Hölder’s inequality, the left side of (31) is

\[
\sum_{|s|\leq N_0} \sum_{\delta, \gamma=1,2} |r_{\delta \gamma}(s)|^{k_G(i)} + \sum_{N_0<|s|<N} \sum_{\delta, \gamma=1,2} |r_{\delta \gamma}(s)|^{k_G(i)} 
\leq C(\varepsilon) + \left( \sum_{N_0<|s|<N} 1 \right)^{1-g(i)} \cdot \left\{ \sum_{\delta, \gamma=1,2} \left( \sum_{N_0<|s|<N} |r_{\delta \gamma}(s)|^{|\vec{t}^i|g(i)} \right) \right\} 
\leq C(\varepsilon) + 4\varepsilon^{g(i)}(2N)^{1-g(i)} .
\]

Since \( \varepsilon \) is arbitrary small, relation (32) holds true. Now (30), (31) and (32) imply that

\[
\left| \frac{1}{(\sqrt{N})^p} \sum_{t \in A(p,N)} \prod_{i=1}^p c_{\ell^i} \prod_{\omega \in G(V)} u(t_{d_1(\omega)} - t_{d_2(\omega)}) \right| = O(N^{\frac{p}{2} - \sum_{i=1}^p g(i)}) \tag{33}
\]

and (33) holds with \( o(\cdot) \) if \( 0 < k_G(i) < |\vec{t}^i| \) for some \( i \). Now observe that there are two cases of a non regular diagram. The first case is the one such that at least one of inequality in (25) is strict. The second one is that all terms in (25) are equal. Consider the first case. A non regular diagram of the first case satisfies one of the following properties:

(i) \( 0 < k_G(i) < |\vec{t}^i| \) for some \( i \)
(ii) $G$ contains an edge between sectors of different cardinality. Clearly, if (i) holds, Proposition 1 follows. Also we can show that

$$\sum_{i=1}^{p} g(i) \geq \frac{p}{2},$$

where this inequality is strict if $G$ contains an edge connecting sectors of different cardinality, and hence Proposition 1 follows. For any given edge $\omega \in V(G)$, define the numbers $p_1(\omega)$ and $p_2(\omega)$ as the cardinalities of the $d_1(\omega)^{th}$ and $d_2(\omega)^{th}$ sectors, respectively. Because of property (25) we have $p_1(\omega) \leq p_2(\omega)$ for all $\omega \in V(G)$. Hence

$$2 \sum_{i=1}^{p} g(i) = 2 \sum_{i=1}^{p} \frac{k_G(i)}{|\ell^i|} = 2 \sum_{\omega \in V(G)} \frac{1}{p_1(\omega)} \geq \sum_{\omega \in V(G)} \left( \frac{1}{p_1(\omega)} + \frac{1}{p_2(\omega)} \right) = p.$$  (35)

Note that the inequality is strict if (ii) is true, because it implies $p_1(\omega) < p_2(\omega)$ for some $\omega$. Now consider the second case (i.e. all terms in (25) are equal). For such a diagram, which is non regular, there always exists an integer $j$ such that

$$0 < k_G(j) < |\ell^j|,$$  (36)

because there exists a sector $S_j$ and two edges $\omega_1, \omega_2$ such that

$$d_2(\omega_1) = j \quad \text{and} \quad d_1(\omega_2) = j$$

by the definition of non-regularity of $G$. Therefore (33) holds with $a(\cdot)$ and hence we have proved Proposition 1.
Reference


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