ANALYSIS OF THE FINITE ELEMENT APPROXIMATION OF MICROSTRUCTURE IN MICROMAGNETICS

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ANALYSIS OF THE FINITE ELEMENT APPROXIMATION
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Abstract. The solution to nonconvex variational problems is often characterized by microstructure. The variational problem for the magnetization field in micromagnetics has a nonconvex constraint and energy minimizing sequences of magnetization fields can have oscillations whose scale converges to zero but whose amplitude remains finite. We show that the finite element approximation of the magnetization field for this variational problem does not converge pointwise as the mesh is refined, but that nonlinear functions of the approximate magnetization fields converge weakly (or equivalently, all local spatial averages of nonlinear functions of the approximate magnetization field converge as the mesh is refined, which implies the convergence as the mesh is refined of the probability distribution of the approximate magnetization fields in local spatial domains). We give a norm to measure the convergence of this microstructure, and we prove a rate of convergence in this norm.

Key words. finite element method, microstructure, micromagnetics, magnetization field

AMS(MOS) subject classifications. 65N15, 65N30, 35J20, 35J70, 73C60

1. Introduction. There has been great progress during the past several years in the analysis and computation of nonconvex variational problems which are important mathematical models in many areas of science and technology [4–5, 9–18, 20–29]. The solutions to these variational problems often are characterized by a microstructure [4–5, 11, 23–28]. In [12, 15–16], we introduced new concepts to analyze the finite element approximation of the microstructure found in martensitic materials. In this paper, we show how these concepts can be utilized to analyze the finite element approximation of the magnetization field for some ferromagnetic materials which exhibit microstructure in the form of fine-scale magnetic domains.

The magnetization field of many ferromagnetic materials has a fine scale structure or a microstructure. This microstructure is modeled by the theory of micromagnetics [6–8]. The theory of micromagnetics provides a free energy density for the magnetization field, and magnetization fields with microstructure minimize the bulk energy [25–26].

The micromagnetics variational problem is a nonconvex variational problem, and it has been recently shown that these variational problems can fail to attain a minimum value for any classical function [25–26]. Minimizing sequences of magnetization fields can have

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oscillations whose scale converges to zero but whose amplitude remains finite. Numerical approximations of such fields will clearly not converge in any $L^p$ space as the mesh is refined, even locally. However, we demonstrate in this paper that nonlinear functions of the approximate magnetization fields converge weakly, i.e., all local spatial averages of nonlinear functions of the approximate magnetization field converge as the mesh is refined, which implies the convergence as the mesh is refined of the probability distribution of the approximate magnetization fields in local spatial domains [2]. To give error estimates for the convergence of this microstructure, we utilize the norm that we introduced in [16] to analyze the numerical approximation of martensitic microstructure. This theory validates the computational results for the microstructure in micromagnetics reported in [29].

2. The Theory of Micromagnetics. We denote the magnetization field by $m : \Omega \to \mathbb{R}^3$ where $\Omega \subseteq \mathbb{R}^3$ represents the spatial domain of the ferromagnetic material with boundary $\partial \Omega$. We assume for simplicity that the boundary $\partial \Omega$ is the union of triangular faces $\Gamma_l$ which are defined as the convex hull of noncollinear points $\{a_{l,1}, a_{l,2}, a_{l,3}\} \subset \mathbb{R}^3$ or

$$\Gamma_l = \left\{ x = \sum_{i=1}^{3} \beta_i a_{l,i} : \beta_i \geq 0, \sum_{i=1}^{3} \beta_i = 1 \right\}.$$

Further, we have that

$$\partial \Omega = \bigcup_{l=1}^{L} \Gamma_l \quad \text{and} \quad \text{int} \Gamma_l \cap \text{int} \Gamma_j = \emptyset \text{ if } l \neq j$$

where $\text{int} \Gamma_l$ is the interior of $\Gamma_l$.

We assume that the material is magnetically saturated, so that

$$|m(x)| = f(\theta) \quad x \in \Omega,$$

where $\theta$ is the temperature and $||$ denotes the Euclidean norm. Since we shall not be concerned here with temperature variations, we can assume without loss of generality that

(2.1) $$|m(x)| = 1 \quad x \in \Omega.$$

The free energy for the magnetization field is the sum of the anisotropic energy of the crystal and the magnetostatic energy. The anisotropic energy density $\phi(m) : S^2 \to \mathbb{R}$ where

$S^2 = \{ m \in \mathbb{R}^3 : |m| = 1 \}$

has the material symmetry property

$$\phi(Rm) = \phi(m) \quad R \in G$$
where $\mathcal{G}$ is the symmetry group of the crystal. In this paper, we shall be concerned only with the uniaxial case (for example, cobalt) where $\phi(m)$ is even and where there is a unit vector $\hat{m}$ such that

$$0 = \phi(\pm \hat{m}) < \phi(m)$$

for all $m \neq \pm \hat{m}$.

More precisely, we shall assume that there exists a positive constant $c_0$ such that

$$\phi(m) \geq c_0 \inf(|m - \hat{m}|^2, |m - (-\hat{m})|^2)$$

$$\phi(\hat{m}) = 0.$$  \hspace{1cm} (2.2)

We will also assume without loss of generality that $\hat{m} = e_3 = (0, 0, 1)$.

The magnetostatic energy density is given by $|H|^2/2$ where $H : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic field. Maxwell's equations in dimensionless units are given by

$$\text{div } B = 0 \quad \text{and} \quad \text{curl } H = 0$$

where $B = H + m : \mathbb{R}^3 \to \mathbb{R}^3$ is the magnetic induction. So, the magnetic potential $u : \mathbb{R}^3 \to \mathbb{R}$ where $H = -\nabla u$ must satisfy the partial differential equation

$$\text{div}(-\nabla u + m \chi_\Omega) = 0 \quad \text{in } \mathbb{R}^3$$  \hspace{1cm} (2.3)

where $\chi_\Omega(x)$ is the characteristic function of $\Omega$ defined by

$$\chi_\Omega(x) = \begin{cases} 1 \text{ if } x \in \Omega \\ 0 \text{ if } x \notin \Omega. \end{cases}$$

Thus,

$$m(x) \chi_\Omega(x) = \begin{cases} m(x) \text{ if } x \in \Omega \\ 0 \text{ if } x \notin \Omega. \end{cases}$$

It is easy to verify by using the Fourier transform that since $m(x) \chi_\Omega(x) \in L^2(\mathbb{R}^3; \mathbb{R}^3) \cap L^1(\mathbb{R}^3; \mathbb{R}^3)$, (2.3) has a unique solution $u(x) \in H^1(\mathbb{R}^3)$ [2, 19].

We sum the contributions of the anisotropic energy and the magnetostatic energy to obtain the bulk energy for the magnetization field given by

$$E(m) = \int_\Omega \phi(m(x)) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx$$  \hspace{1cm} (2.4)

where

$$\text{div}(-\nabla u + m \chi_\Omega) = 0 \quad \text{in } \mathbb{R}^3$$

and where

$$\mathcal{A} = \{ m \in L^\infty(\Omega; \mathbb{R}^3) : |m(x)| = 1 \text{ a.e. in } \Omega \}.$$  \hspace{1cm} (2.5)
It is shown in [25–26] that

\begin{equation}
\inf_{m \in \mathcal{A}} E(m) = 0.
\end{equation}

However, it is also shown in [25–26] that there does not exist \( m \in \mathcal{A} \) such that \( E(m) = 0 \). The direct method of the calculus of variations does not give a magnetization field at which the bulk energy functional (2.4) attains the minimum value even though the bulk energy functional is weakly continuous (since the bulk energy functional is convex). The weak limit of a minimizing sequence of magnetization fields is the zero field \( m \equiv 0 \) which is not in the set of admissible fields \( \mathcal{A} \) (The set of admissible functions (2.5) is not weakly closed.)

Sometimes an “exchange energy” is also included in the bulk energy to account for the tendency of the magnetic moments to align. However, the exchange energy has a negligible effect on the macroscopic magnetic properties of the material, and we therefore neglect this term.

3. The Numerical Approximation and the Statement of the Results. In this section, we define our finite element approximation of the magnetization field. For \( h > 0 \), we define

\[ I_{ijk} = (ih,(i+1)h) \times (jh,(j+1)h) \times (kh,(k+1)h), \quad i, j, k \in \mathbb{Z}, \]

and we define the decomposition of \( \Omega \) into disjoint sets

\[ \Omega_{ijk} = I_{ijk} \cap \Omega \quad i, j, k \in \mathbb{Z}. \]

We note that \( \Omega_{ijk} \neq \emptyset \) for a finite set of \( i, j, k \in \mathbb{Z} \).

Our finite element method is to minimize the bulk energy on the piecewise constant space

\[ \mathcal{A}_h = \{ m \in \mathcal{A} : m(x) \text{ is constant on } \Omega_{ijk} \text{ for } i, j, k \in \mathbb{Z} \}. \]

The finite element approximation is given by \( m_h \in \mathcal{A}_h \) such that

\begin{equation}
E(m_h) = \inf_{m \in \mathcal{A}_h} E(m).
\end{equation}

We note that (3.1) attains a minimum value since the energy is non-negative and since the dimension of \( \mathcal{A}_h \) is finite. The minimum energy for the finite element approximation (3.1) can be attained by more than one approximate magnetization field, though.

Our first result gives an estimate of the energy of finite element approximations given by (3.1).

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**Theorem 1.** There exists a constant $c_1$, independent of $h$, such that if the magnetization field $m_h \in \mathcal{A}_h$ is a solution to the finite element variational problem (3.1), then

$$E(m_h) = \inf_{m \in \mathcal{A}_h} E(m) \leq c_1 h.$$ 

Our second main theorem states that the approximate magnetization fields $m_h(x)$ are locally in the state $e_3$ with probability $1/2$ and locally in the state $-e_3$ with probability $1/2$. In the jargon of the calculus of variations, the measure

$$\nu_x \equiv \frac{1}{2} \delta_{e_3} + \frac{1}{2} \delta_{-e_3} \quad \text{for} \ x \in \Omega,$$

where $\delta_{\hat{m}}$ is the Dirac delta function with unit mass concentrated at $\hat{m}$ for $\hat{m} = e_3, -e_3$ is the unique Young measure associated with minimizing (3.1) [3, 5, 32].

We define the Sobolev Space $[1, 2, 19] \mathcal{U}$ to be the closure of the space $C^\infty_c(\Omega)$, the space of $C^\infty$ functions on $\Omega$ with compact support in $\Omega$, with respect to the norm

$$\|\zeta\|_\mathcal{U}^2 = \|\zeta\|_{L^2(\Omega)}^2 + \|\zeta_{x_i}\|_{L^2(\Omega)}^2 + \|\zeta_{x_2}\|_{L^2(\Omega)}^2$$

and we define $Lip(S^2)$ to be space of Lipschitz continuous functions $\psi : S^2 \rightarrow \mathbb{R}$ with semi-norm

$$\|\psi\|_{Lip(S^2)} = \sup_{\substack{p, q \in S^2 \atop p \neq q}} \frac{|\psi(p) - \psi(q)|}{|p - q|}.$$ 

Next, we define the vector space $\mathcal{V}$ to be the space of functions $F(x, m) : \Omega \times S^2 \rightarrow \mathbb{R}$ such that

$$F(x, m) \in L^2(\Omega; Lip(S^2))$$

and

$$F(x, e_3) - F(x, -e_3) \in \mathcal{U}$$

with semi-norm

$$\|F\|_{\mathcal{V}} = \left( \int_\Omega \|F(x, \cdot)\|_{Lip(S^2)}^2 \, dx \right)^{1/2} + \|F(x, e_3) - F(x, -e_3)\|_\mathcal{U}.$$ 

We will prove the following theorem which gives the weak limit of nonlinear functions of $m_h$.

**Theorem 2.** We have that there exists a constant $c_2$, independent of $h$, such that if the magnetization field $m_h \in \mathcal{A}_h$ is a solution to the finite element variational problem (3.1), then

$$\left| \int_\Omega F(x, m_h) \, dx - \frac{1}{2} \int_\Omega [F(x, e_3) + F(x, -e_3)] \, dx \right| \leq c_2 h^{1/2} \|F\|_{\mathcal{V}} \quad \text{for all} \ F \in \mathcal{V}.$$
We note that the mechanical properties of magnetic materials depend nonlinearly on
the magnetization $m$. Theorem 2 shows that the material property described by the
microscopic density $F(x, m)$ has the macroscopic density (weak limit)

$$\frac{1}{2} F(x, e_3) + \frac{1}{2} F(x, -e_3)$$

for the minimizing microstructure for the energy (3.1).

To estimate the rate of convergence of $m_h(x)$ we define the dual of $\mathcal{V}$, $\mathcal{V}^*$, to be the
set of linear functionals $L : \mathcal{V} \to \mathbb{R}$ for which there exists a constant $c$ such that

$$|\langle L, F \rangle| \leq c \|F\|_\mathcal{V} \quad \text{for all } F \in \mathcal{V}$$

with operator norm [30] defined for $L \in \mathcal{V}^*$ by

$$\|L\|_{\mathcal{V}^*} = \sup_{\substack{F \in \mathcal{V} \\
\|F\|_{\mathcal{V}} = 1}} |\langle L, F \rangle|.$$ 

We can identify with $m_h$ the functional $L_{m_h} \in \mathcal{V}^*$ defined by

$$\langle L_{m_h}, F \rangle \equiv \int_\Omega F(x, m_h(x)) \, dx$$

and we identify with $\nu \equiv \frac{1}{2} \delta_{e_3} + \frac{1}{2} \delta_{-e_3}$ the functional $L_\nu \in \mathcal{V}^*$ defined by

$$\langle L_\nu, F \rangle \equiv \frac{1}{2} \int_\Omega [F(x, e_3) + F(x, -e_3)] \, dx.$$ 

We then have the following theorem which is a direct consequence of (3.3).

**Theorem 3.** We have the error estimate

$$\|L_{m_h} - L_\nu\|_{\mathcal{V}^*} \leq c_2 h^{1/2}.$$ 

**4. Proofs of the Main Theorems.** We first give the proof of Theorem 1.

**Proof of Theorem 1.** We define $p_h(x) \in \mathcal{A}_h$ by

$$p_h(x) = \begin{cases} (0, 0, 1) & \text{if } x \in \Omega_{ijk}, \ i \text{ even} \\
(0, 0, -1) & \text{if } x \in \Omega_{ijk}, \ i \text{ odd.} \end{cases}$$

Then $\phi(p_h(x)) = 0$ for $x \in \Omega$ and

$$E(p_h) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx.$$
Now by (2.3)
\[
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx = \frac{1}{2} \int_{\Omega} \nabla u \cdot p_h \, dx.
\]
Further, \( u \in H^1(\mathbb{R}^3) \) satisfies
\[
(4.1) \quad u = \text{div} \, w
\]
where \( w = (w_1, w_2, w_3) \in C^1(\mathbb{R}^3, \mathbb{R}^3) \) with \( \nabla w \in H^1(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3}) \) is uniquely defined by [19]
\[
(4.2) \quad \Delta w = p_h \chi_{\Omega}, \quad w(x) \to 0 \text{ as } |x| \to \infty.
\]
Note that \( w_1(x) = w_2(x) = 0 \) for all \( x \in \mathbb{R}^3 \). Hence,
\[
\nabla u \cdot p_h = w_{3,33} P_h
\]
where \( P_h : \bar{\Omega} \to \mathbb{R} \) satisfies
\[
P_h(x) = \begin{cases} 
1 & \text{if } x \in \Omega_{ijk}, \text{ } i \text{ even} \\
-1 & \text{if } x \in \Omega_{ijk}, \text{ } i \text{ odd}
\end{cases}
\]
and is extended naturally to \( \partial \Omega \). Thus, by the divergence theorem
\[
\frac{1}{2} \int_{\Omega} \nabla u \cdot p_h \, dx = \frac{1}{2} \int_{\Omega} w_{3,33} P_h \, dx = \frac{1}{2} \int_{\partial \Omega} w_{3,3} P_h n_3 \, dS
\]
where \( n = (n_1, n_2, n_3) = n(S) \) is the unit exterior normal to \( \partial \Omega \) at \( S \in \partial \Omega \) and \( dS \) is the surface area measure on \( \partial \Omega \).

By (4.1) and (4.2),
\[
w_{33}(x) = -\frac{1}{4\pi} \int_{\Omega} \frac{P_h(y)}{|x - y|} \, dy.
\]
So,
\[
w_{3,3}(x) = -\frac{1}{4\pi} \int_{\Omega} P_h(y) \frac{\partial}{\partial x_3} \frac{1}{|x - y|} \, dy
\]
\[
= \frac{1}{4\pi} \int_{\partial \Omega} \frac{P_h(S)n_3}{|x - S|} \, dS.
\]
Since \( 1/|x - S| \in L^1(\partial \Omega) \) for \( x \in \partial \Omega \), it follows from (4.3) that there exists a constant \( c_3 \) such that
\[
\|w_{3,3}\|_{L^\infty(\partial \Omega)} \leq c_3 \|P_h\|_{L^\infty(\partial \Omega)} \leq c_3.
\]
Let

\[ w_{3,3}^{\Gamma_i}(x) = \frac{1}{4\pi} \int_{\Gamma_i} \frac{P_h(S)^{n_3}}{|x - S|} dS \quad \text{for} \ x \in \mathbb{R}^3. \]

It follows from the theory of singular integrals [31] that for \( 1 < p < \infty \) there exists a constant \( c_4 = c_4(p) \) such that

\[ \| \nabla_S w_{3,3}^{\Gamma_i} \|_{L^p(\Gamma_i)} \leq c_4 \| P_h \|_{L^p(\Gamma_i)} \]  

where \( \nabla_S \) denotes the projection of the gradient onto the tangent plane to \( \Gamma_i \). Thus, for \( q = p/(p - 1) \) we obtain by (4.5) and Hölder’s inequality that

\[ \| \nabla_S w_{3,3}^{\Gamma_i} \|_{L^1(\Gamma_i)} \leq \text{area}(\Gamma_i)^{1/q} \| \nabla_S w_{3,3}^{\Gamma_i} \|_{L^p(\Gamma_i)} \]
\[ \leq c_4 \text{area}(\Gamma_i)^{1/q} \| P_h \|_{L^p(\Gamma_i)} \]
\[ \leq c_4 \text{area}(\Gamma_i) \| P_h \|_{L^\infty(\Gamma_i)}. \]

Now

\[ \nabla w_{3,3}^{\Gamma_i}(x) = \frac{1}{4\pi} \int_{\Gamma_i} \frac{P_h(S)^{n_3}}{|x - S|^3} (x - S) dS \quad \text{for} \ x \in \mathbb{R}^3 - \Gamma_i, \]

so there exists a constant \( c_5 \) such that for \( x \in \mathbb{R}^3 - \Gamma_i, \)

\[ |\nabla w_{3,3}^{\Gamma_i}(x)| \leq \frac{1}{4\pi} \left[ \int_{\Gamma_i} \frac{1}{|x - S|^2} dS \right] \| P_h \|_{L^\infty(\Gamma_i)} \]
\[ \leq c_5 [\log(|\text{dist}(x, \Gamma_i)| + 1)] \| P_h \|_{L^\infty(\Gamma_i)} \]
\[ \leq c_5 [\log(|\text{dist}(x, \Gamma_i)| + 1)] \]

where

\[ \text{dist}(x, \Gamma_i) = \inf_{S \in \Gamma_i} |x - S|. \]

Thus, we can obtain from integrating (4.7) that

\[ \| \nabla_S w_{3,3}^{\Gamma_i} \|_{L^1(\Gamma')} \leq c_6 \| P_h \|_{L^\infty(\Gamma_i)} \leq c_6 \quad \text{for all} \ l' \neq l. \]

Hence, it follows from (4.6), (4.8), and the triangle inequality that there exists a constant \( c_7 \) such that

\[ \| \nabla_S w_{3,3} \|_{L^1(\partial \Omega)} \leq c_7 \| P_h \|_{L^\infty(\partial \Omega)} \leq c_7. \]

We can order the sets \( \Gamma_i \) so that

\[ \begin{cases} 
|n_3| > 0 & \text{for} \ S \in \Gamma_i, \ l = 1, \ldots, L_1 \\
n_3 = 0 & \text{for} \ S \in \Gamma_i, \ l = L_1 + 1, \ldots, L.
\end{cases} \]
Then
\[ \partial \Omega = \Gamma \cup \Gamma^\perp \text{ and } \text{int } \Gamma \cap \text{int } \Gamma^\perp = \emptyset, \]
where
\[ \Gamma = \left\{ S \in \partial \Omega : |n_3| > 0 \right\} = \bigcup_{l=1}^{L_1} \Gamma_l \]
and
\[ \Gamma^\perp = \left\{ S \in \partial \Omega : |n_3| = 0 \right\} = \bigcup_{l=L_1+1}^{L} \Gamma_l. \]

Thus,
\[
\frac{1}{2} \int_{\partial \Omega} w_{3,3} P_h n_3 \, dS = \frac{1}{2} \int_{\Gamma} w_{3,3} P_h n_3 \, dS + \frac{1}{2} \int_{\Gamma^\perp} w_{3,3} P_h n_3 \, dS \\
= \frac{1}{2} \int_{\Gamma} w_{3,3} P_h n_3 \, dS.
\]

For \( l = 1, \ldots, L_1 \) there exists affine functions \( y_l : \tilde{\Gamma}_l \subseteq \mathbb{R}^2 \to \mathbb{R} \) and isomorphisms \( \phi_l(x) : \tilde{\Gamma}_l \subseteq \mathbb{R}^2 \to \Gamma_l \) such that
\[ \tilde{\Gamma}_l \equiv \left\{ (x_1, x_2) \in \mathbb{R}^2 : (x_1, x_2, x_3) \in \Gamma_l \right\} \]
and
\[ \phi_l(\xi_1, \xi_2) = (\xi_1, \xi_2, y_l(\xi_1, \xi_2)). \]

Now
\[
\int_{\Gamma} w_{3,3} P_h n_3 \, dS = \sum_{l=1}^{L_1} \int_{\tilde{\Gamma}_l} w_{3,3} P_h n_3 \, dS
\]
and
\[
\int_{\Gamma_l} w_{3,3} P_h |n_3| \, dS = \int_{\tilde{\Gamma}_l} (w_{3,3} \circ \phi_l)(P_h \circ \phi_l) \, d\xi.
\]

Next, for \( i \) an even integer and \( j \in \mathbb{Z} \) set
\[ \hat{I}_{i,j} \equiv (ih, (i+2)h) \times (jh, (j+1)h) \]
and define
\[ \hat{\Lambda}_{h,l} \equiv \bigcup_{\hat{I}_{i,j} \subset \tilde{\Gamma}_l} \hat{I}_{i,j} \]
and

$$\hat{\Lambda}_{h,t} = \hat{\Gamma}_{t} - \hat{\Lambda}_{h,t}.$$ 

Now for $\hat{\Gamma}_{i,j} \subset \hat{\Gamma}_{i}$, we have that

$$\left| \int_{\hat{\Gamma}_{i,j}} (w_{3,3} \circ \phi_{l})(P_{h} \circ \phi_{l}) \, d\xi \right|$$

$$= \left| \int_{j_{h}}^{(j+1)_{h}} \int_{i_{h}}^{(i+1)_{h}} [w_{3,3} \circ \phi_{l}(\xi_{1} + h, \xi_{2}) - w_{3,3} \circ \phi_{l}(\xi_{1}, \xi_{2})] \, d\xi_{2} \, d\xi_{1} \right|$$

$$\leq h \int_{j_{h}}^{(j+1)_{h}} \int_{i_{h}}^{(i+1)_{h}} \left| \frac{\partial w_{3,3} \circ \phi_{l}}{\partial \xi_{1}}(\xi_{1}, \xi_{2}) \right| \, d\xi_{2} \, d\xi_{1}.$$ 

Thus,

(4.13)

$$\left| \int_{\hat{\Gamma}_{i}} (w_{3,3} \circ \phi_{l})(P_{h} \circ \phi_{l}) \, d\xi \right|$$

$$\leq \left| \int_{\hat{\Lambda}_{h,i}} (w_{3,3} \circ \phi_{l})(P_{h} \circ \phi_{l}) \, d\xi \right| + \left| \int_{\hat{\Lambda}_{h,i}} (w_{3,3} \circ \phi_{l})(P_{h} \circ \phi_{l}) \, d\xi \right|$$

$$\leq h \int_{\hat{\Lambda}_{h,i}} \left| \frac{\partial w_{3,3} \circ \phi_{l}}{\partial \xi_{1}} \right| \, d\xi + \int_{\hat{\Lambda}_{h,i}} \left| (w_{3,3} \circ \phi_{l})(P_{h} \circ \phi_{l}) \right| \, d\xi.$$ 

Also, since $(1, 0, \frac{\partial y_{l}}{\partial \xi_{1}})$ is in the tangent plane to $\Gamma_{i}$, we have that

$$(\nabla w_{3,3} \circ \phi_{l}) \cdot (1, 0, \frac{\partial y_{l}}{\partial \xi_{1}}) = (\nabla w_{3,3} \circ \phi_{l}) \cdot (1, 0, \frac{\partial y_{l}}{\partial \xi_{1}}).$$

Hence, by the chain rule

(4.14)

$$\left| \frac{\partial w_{3,3} \circ \phi_{l}}{\partial \xi_{1}} \right| = \left| (\nabla w_{3,3} \circ \phi_{l}) \cdot (1, 0, \frac{\partial y_{l}}{\partial \xi_{1}}) \right|$$

$$= \left| (\nabla w_{3,3} \circ \phi_{l}) \cdot (1, 0, \frac{\partial y_{l}}{\partial \xi_{1}}) \right|$$

$$\leq \left| \nabla w_{3,3} \circ \phi_{l} \right| \sqrt{1 + \left( \frac{\partial y_{l}}{\partial \xi_{1}} \right)^{2}}$$

so

(4.15)

$$\int_{\hat{\Lambda}_{h,i}} \left| \frac{\partial w_{3,3} \circ \phi_{l}}{\partial \xi_{1}} \right| \, d\xi \leq \int_{\hat{\Lambda}_{h,i}} \left| \nabla w_{3,3} \circ \phi_{l} \right| \sqrt{1 + \left( \frac{\partial y_{l}}{\partial \xi_{1}} \right)^{2}} \, d\xi$$

$$\leq \int_{\hat{\Lambda}_{h,i}} \left| \nabla w_{3,3} \circ \phi_{l} \right| \sqrt{1 + \left( \frac{\partial y_{l}}{\partial \xi_{1}} \right)^{2} + \left( \frac{\partial y_{l}}{\partial \xi_{2}} \right)^{2}} \, d\xi$$

$$= \int_{\phi(\hat{\Lambda}_{h,i})} \left| \nabla w_{3,3} \right| \, dS$$

$$\leq \int_{\Gamma_{i}} \left| \nabla w_{3,3} \right| \, dS.$$
Now
\[ \text{area}(\tilde{\Lambda}_{h,l}) \leq \sqrt{5}h \text{ perimeter } \tilde{\Gamma}_l \leq \sqrt{5}h \text{ perimeter } \Gamma_l. \]

Thus, by (4.11), (4.12), (4.13), and (4.15)
\[
| \int_{\Gamma} w_{3,3} P_h n_3 \, dS | \leq h \sum_{l=1}^{L_1} \int_{\Gamma_l} | \nabla_S w_{3,3} | \, dS + \sum_{l=1}^{L_1} \int_{\tilde{\Lambda}_{h,l}} | (w_{3,3} \circ \phi_l) (P_h \circ \phi_l) | \, d\xi \\
\leq h \| \nabla_S w_{3,3} \|_{L^1(\partial \Omega)} + \sum_{l=1}^{L_1} \sqrt{5}h \text{ perimeter } \Gamma_l \| w_{3,3} \|_{L^\infty(\partial \Omega)}.
\]

It follows from the above equation and (4.4) and (4.9) that
\[
| \int_{\Gamma} w_{3,3} P_h n_3 \, dS | \leq (c_7 + \sqrt{5}c_3 \sum_{l=1}^{L_1} \text{ perimeter } \Gamma_l) h.
\]

The theorem follows from (4.16) and (4.10). \(\square\)

Define the operator \(\Pi : \mathcal{A} \to \mathcal{A}\) by
\[
(4.17) \quad \Pi m(x) = (0,0,\text{sign} m_3(x)) \quad \text{for } x \in \Omega,
\]
where \(m = (m_1, m_2, m_3)\). Note that
\[
|m - \Pi m| = \inf(|m - e_3|, |m - (-e_3)|).
\]

The following two lemmas will be used in the proof of Theorem 2.

**Lemma 1.** There exists a constant \(c_8\), independent of \(h\), such that
\[
\int_{\Omega} |m_h - \Pi m_h|^2 \, dx \leq c_8 h.
\]

**Proof of Lemma 1.** It follows from (2.2), (2.4), and Theorem 1 that
\[
\int_{\Omega} |m_h - \Pi m_h|^2 \, dx \leq c_0^{-1} \int_{\Omega} \phi(m_h) \, dx \leq c_0^{-1} E(m_h) \leq c_0^{-1} c_1 h. \quad \square
\]

We next give an estimate for \(m_h\) where \(m_h = (m_{h1}, m_{h2}, m_{h3})\).
Lemma 2. There exists a constant $c_9$, independent of $h$, such that

$$\left| \int_{\Omega} m_{h3}(x) \zeta(x) \, dx \right| \leq c_9 h^{1/2} \| \zeta \|_U \quad \text{for } \zeta \in U.$$ 

Proof of Lemma 2. Let $\eta(x) \in C_\infty^\infty(\mathbb{R}^3)$ be such that $\eta(x) = 1$ on $\Omega$ where $C_\infty^\infty(\mathbb{R}^3)$ is the set of $C_\infty^\infty(\mathbb{R}^3)$ functions with compact support. Since $C_\infty^\infty(\Omega)$ is dense in $U$, we may assume that $\zeta \in C_\infty^\infty(\Omega)$. We may further assume that $\zeta$ has been extended by zero on $\mathbb{R}^3 - \Omega$ so that $\zeta \in C_\infty^\infty(\mathbb{R}^3)$ with support in $\Omega$. Next, let $v(x) = \eta(x)\tilde{v}(x)$ where

$$\tilde{v}(x_1, x_2, x_3) = \int_0^{x_3} \zeta(x_1, x_2, s) \, ds.$$ 

Then

$$\frac{\partial v}{\partial x_3} = \zeta \quad \text{in } \Omega$$

and

$$\|\nabla v\|_{L^2(\mathbb{R}^3)} \leq c_{10} \|\zeta\|_U.$$ 

By (2.3)

$$|(m_{h3} \chi_\Omega, \zeta)| = |(m_{h3} \chi_\Omega, \frac{\partial v}{\partial x_3})|$$

$$\leq |(\nabla u, \nabla v)| + |(m_{h1} \chi_\Omega, \frac{\partial v}{\partial x_1})| + |(m_{h2} \chi_\Omega, \frac{\partial v}{\partial x_2})|$$

$$\leq (\|\nabla u\|_{L^2(\mathbb{R}^3)} + \|m_{h1}\|_{L^2(\Omega)} + \|m_{h2}\|_{L^2(\Omega)}) \|\nabla v\|_{L^2(\mathbb{R}^3)}$$

$$\leq (\sqrt{2}c_1^{1/2}h^{1/2} + c_0^{-1/2}c_1^{1/2}h^{1/2} + c_0^{-1/2}c_1^{1/2}h^{1/2})c_{10} \|\zeta\|_U.$$ 

We now can give the proof of Theorem 2.

Proof of Theorem 2. We have that

$$\left| \int_{\Omega} [F(x, m_h) - \frac{1}{2}F(x, e_3) - \frac{1}{2}F(x, -e_3)] \, dx \right|$$

$$\leq \left| \int_{\Omega} [F(x, m_h) - F(x, \Pi m_h)] \, dx \right|$$

$$+ \left| \int_{\Omega} [F(x, \Pi m_h) - \frac{1}{2}F(x, e_3) - \frac{1}{2}F(x, -e_3)] \, dx \right|$$

$$= I_1 + I_2.$$
We will estimate \( I_1 \) and \( I_2 \) separately. We have from Lemma 1 that

\[
I_1 \leq \int_{\Omega} \|F(x, \cdot)\|_{Lip(S^2)} |m_h - \Pi m_h| \, dx \\
\leq \|F\|_{L^2(\Omega; Lip(S^2))} \|m_h - \Pi m_h\|_{L^2(\Omega)} \\
\leq c_8^{1/2} \|F\|_{L^2(\Omega; Lip(S^2))} h^{1/2}.
\]

Denote \( G(x) = F(x, e_3) - F(x, -e_3) \). Since

\[
F(x, \Pi m_h) - \frac{1}{2} F(x, e_3) - \frac{1}{2} F(x, -e_3) = \begin{cases} \\
\frac{1}{2} G(x) & \text{if } \Pi m_h = e_3 \\
-\frac{1}{2} G(x) & \text{if } \Pi m_h = -e_3
\end{cases}
\]

we have from Lemma 1 and Lemma 2 that

\[
I_2 \leq \left| \int_{\Omega} \frac{1}{2} e_3 \cdot \Pi m_h G \, dx \right| \\
\leq \frac{1}{2} \left| \int_{\Omega} e_3 (m_h - \Pi m_h) G \, dx \right| + \frac{1}{2} \left| \int_{\Omega} e_3 m_h G \, dx \right| \\
\leq \frac{1}{2} \|m_h - \Pi m_h\|_{L^2(\Omega)} \|G\|_{L^2(\Omega)} + \frac{1}{2} \left| \int_{\Omega} m_h G \, dx \right| \\
\leq \frac{1}{2} c_8^{1/2} h^{1/2} \|G\|_{L^2(\Omega)} + \frac{1}{2} c_9 h^{1/2} \|G\|_{U}.
\]

The result follows from (4.21) and (4.22). \( \square \)

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