LARGE TIME STUDY OF FINITE ELEMENT METHODS
FOR 2D NAVIER-STOKES EQUATIONS

By

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IMA Preprint Series # 739
December 1990
LARGE TIME STUDY OF FINITE ELEMENT METHODS
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Abstract. The finite element approximations offer a class of widely used numerical schemes for large time simulations of fluids flow. This paper concerns the large time behavior of the finite dimensional dynamical system generated by such approximations to the two dimensional Navier-Stokes equations for the Newtonian incompressible viscous flow with Dirichlet boundary conditions. It is shown that such a finite dimensional approximation admits a global attractor, whose Hausdorff and fractal dimensions are bounded above by a constant independent of the approximation parameter. Two estimates about the approximation solution are given as well, which describe the behavior of approximated solution near t=0 and the attracting property in $H^1$ respectively.

Key words. finite element approximation, Navier-Stokes equations, large time behavior

AMS(MOS) subject classifications. 65N30, 76D05

1. Introduction.

Developments of dynamical system theory in fluid mechanics in the last two decades have contributed greatly to our theoretical understanding of the complicated large time behavior exhibited very often by fluids flow. Among others the theory of global attractor, estimates of its dimensions [3, 5, 21] and the recent concept of inertial manifolds [7] offer a rigorous mathematical approach to the finite dimensional behavior in turbulence.

A global attractor is the maximal compact invariant set in phase space, it attracts every trajectory of the dynamical system [3-5, 21]. It has been known that certain dissipative partial differential equations possess global attractors, including two dimensional Navier-Stokes (2D NSE) equations for the Newtonian incompressible viscous flow [3-5, 15, 21], for which optimal bounds for the dimension of attractors are known as well, compatible to the classical estimates of degrees of freedom of a turbulent flow due to Kolmogorov-Landau-Lifschitz. These estimates are therefore useful for the numerical simulation of fluids flow.

The large time simulation of fluids flow is concentrated on the understanding of turbulent behavior, which is characterized by various instabilities. Yet due to apparent stability

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*This research was supported in part by a grant from the Applied Mathematics and Computational Mathematics Program/DARPA.
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concern, one can only expect convergence to stable structures, as is proved in [11] that under the condition of exponential stability of a particular trajectory a uniform in time error estimate holds for 2D NSE. In [18] it is proved that hyperbolicity and certain qualitative behavior is preserved in the long time by appropriate perturbations. It is known on the other hand that global attractors possess some kind of stability. More precisely, for 2D NSE it is known that the global attractor is upper semicontinuous with respect to spectral Galerkin approximations, as in [9, 21], without any assumption about the long time dynamics. This in a sense validates the numerical simulation of Navier-Stokes equation by the spectral Galerkin methods. We mention here that the attractors of certain partial differential equations are lower semicontinuous [10], there all equilibrium points are supposed to be hyperbolic.

For finite difference methods it is shown [22] that the semidiscrete approximations admit a global attractor as well, with upper bounds of its dimension independent of mesh size. The boundary conditions there are periodic. It is the objective of this paper to obtain similar results in the case of finite element approximations. For this method, error estimates on a finite time interval are known [12-13].

We will study the long time behavior of the finite element approximation of the two dimensional Navier-Stokes equations with Dirichlet boundary conditions. We will concern ourselves with the semidiscrete approximation only, that is, the time variable $t$ is kept continuous and the discretization is only on spatial variables. For the periodic boundary conditions and completely discrete approximations similar techniques are possible. Our analysis remains valid for a three dimensional variant of Navier-Stokes equations due to Ladyzhenskaya [14].

This paper is organized as follows: The standard finite element methods for the equations is recalled in section 2. Due to the generality of the approach of the global attractors a more detailed description of the approximation is postponed until section 5, where some apriori estimates are proved. They demonstrate that the finite dimensional dynamical system generated by the finite element procedure does share the similar estimates to that of the continuous case near $t = 0$ and as $t \to \infty$ respectively. A discrete Sobolev embedding theorem is proved to that end. In section 3 the functional setting of the finite element
supplemented by the conditions:

\[ u(x, t) = 0 \text{ for } x \in \partial \Omega \]
\[ u(x, 0) = u_0(x) \text{ in } \Omega. \]

We require that

\[ \int_{\Omega} p \, dx = 0 \]
for the uniqueness of \( p \), and define

\[ L^2_0(\Omega) = \{ p \in L^2(\Omega) : \int_{\Omega} p \, ds = 0 \}. \]

We will use the following approximation spaces:

\( X_h \): a finite dimensional subspace of \( X \),
\( Q_h \): some subspace of \( L^2_0(\Omega) \).

The finite element approximation to the 2D Navier-Stokes equations is formulated as:

Find \( u_h(t) \in X_h \) and \( p_h(t) \in Q_h \) such that

\[ \left( \frac{d u_h}{d t}, v_h \right) + \nu (\nabla u_h, \nabla v_h) + b(u_h, u_h, v_h) \]
\[ - (p_h, \nabla \cdot v_h) = (f, v_h), \quad \forall v_h \in X_h \]
\[ (\nabla \cdot u_h, q_h) = 0, \quad \forall q_h \in Q_h \]

supplemented by the condition: \( u_h(0) = u^0_h \), where \( u^0_h \in X_h \) is an approximation to \( u_0 \) and

\[ b(u, v, w) = \frac{1}{2}((u \cdot \nabla) v, w) - \frac{1}{2}((u \cdot \nabla) w, v). \]

We remark here that

\[ b(u, v, v) = 0, \quad \forall u, v \in V_h \]
\[ b(u, v, w) = ((u \cdot \nabla) v, w), \quad \text{if } \nabla \cdot u = 0 \]

(2.4), (2.5) can be proved by integration by parts.
Let \( V_h = \{ v_h \in X_h : (q_h, \nabla \cdot v_h) = 0 \ \forall q_h \in Q_h \} \). We assume that \( V_h \) is not empty. Observe here that for \( Q_h \neq L^2_0(\Omega) \), \( V_h \) may not be a subset of \( V \). Methods of this kind are known as mixed finite element methods, referring to the fact that there are two different approximate spaces involved. Finite element approximations where \( Q_h = L^2_0(\Omega) \) are the conforming finite element methods. More detailed discussion and two examples about the finite element methods will be given in section 5. We mention here that under standard assumptions about the finite elements it can be shown [12-13] that \( u_h(t) \to u(t) \) on finite time intervals and for smooth initial condition e.g., \( u_0 \in V \cap H^2(\Omega)^2 \).

We have from the definition of \( V_h \) another formulation from (2.1)-(2.2):

\[
(2.6) \quad (\frac{du_h}{dt}, v_h) + \nu(\nabla u_h, \nabla v_h) + b(u_h, u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h
\]

First we have the following

**PROPOSITION 1.** For any \( u^0_h \in V_h \) there exists a unique \( u_h \in C^1([0, \infty[, V_h) \) satisfying (2.6).

**Proof.** The uniqueness and local existence in time can be obtained similarly as in the proof of well-posedness of the 2D NSE by spectral Galerkin approximations [4, 21]. To show the global existence it suffices to prove the following auxiliary

**LEMMA 1.** Let \( u_h(t) \) satisfies (2.6). Then

\[
(2.7) \quad \| u_h(t) \|^2 \leq \| u^0_h \|^2 \exp(-\nu \mu_1 t) + \frac{\| f \|^2}{\nu \mu_1^2}(1 - \exp(-\nu \mu_1 t))
\]

where \( \mu_1 \) is the smallest eigenvalue of \( -\Delta \) on \( \Omega \) for the Dirichlet boundary condition.

**Proof of Lemma 1.**

Observe \( b(u_h, u_h, v_h) = 0 \) it follows, as in the continuous case, that

\[
\frac{1}{2} \frac{d\| u_h \|^2}{dt} + \nu(\nabla u_h, \nabla u_h) = (f, u_h)
\]

\[
\leq \| f \| \| u_h \|\
\leq \frac{1}{2} \nu \mu_1 \| u_h \|^2 + \frac{1}{2} \frac{\| f \|^2}{\nu \mu_1}
\leq \frac{1}{2} \nu(\nabla u_h, \nabla u_h) + \frac{1}{2} \frac{\| f \|^2}{\nu \mu_1}
\]
\begin{align}
\frac{d\|u_h\|^2}{dt} + \nu (\nabla u_h, \nabla u_h) & \leq \frac{\|f\|^2}{\nu \mu_1} \\
\frac{d\|u_h\|^2}{dt} + \nu \mu_1 \|u_h\|^2 & \leq \frac{\|f\|^2}{\nu \mu_1}
\end{align}

Gronwall’s argument then gives us (2.7). \Box

It follows that \(\|u_h(t)\|\) will not blow up in any finite time, which concludes the proof of Proposition 1. \Box


We seek now to have a dynamical system formulation for the finite element approximations considered in section 2. To this end, we first define the discrete Laplace operator:

\[ \Delta_h : X_h \to X_h, \]

for \( u \in X_h \), \( \Delta_h u \) is defined as the unique element in \( X_h \) such that

\[ -\langle \Delta_h u, v \rangle = (\nabla u, \nabla v), \quad \forall v \in X_h. \]

Denote \( P_h \) as the orthogonal projection operator from \( L^2 \) into \( V_h \), i.e., for \( u \in L^2(\Omega) \) \( P_h u \) is defined to be the unique element in \( V_h \) such that

\[ \langle P_h u, v \rangle = (u, v), \quad \forall v \in V_h \]

Define similarly a bilinear operator \( B_h : V_h \times V_h \to V_h \) such that

\[ (B(u_h, v_h), w_h) = b(u_h, v_h, w_h) \]

where \( b(.,.,.) \) is defined as in section 2. Let \( A_h \) be the operator \(-P_h \Delta_h\) with its domain restricted to \( V_h \). Then (2.6) is equivalent to

\[ \frac{d u_h}{dt} + \nu A_h u_h + B(u_h, u_h) = P_h f \] (3.1)

The operator \( A_h \) will be referred to as the discrete Stokes operator. We refer to [4, 21] for definition and properties of the Stokes operator. \( A_h \) enjoys the following
Theorem 1.

a. The discrete Stokes operator is self-adjoint and positive definite on $V_h$.

b. Let $(0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be the eigenvalues of $A_h$ according to their multiplicity, $N$ is the dimension of $V_h$. Then

\begin{equation}
\bar{\lambda}_j \geq c_0 j \lambda_1
\end{equation}

where $\lambda_1$ is the smallest eigenvalue of the Stokes operator and $c_0$ is a scale invariant positive constant.

In what follows $c_0$ will always be some scale invariant constant, by that we mean a domain depending constant which is independent of the rigid motion and dilation of the domain, unless otherwise specified. $c_0$ may vary.

Proof. a. For any $u, v \in V_h$

$$(A_h u, v) = -(\Delta_h u, v) = (\nabla u, \nabla v)$$

Therefore the assertion a follows.

b. We will prove this part using minimax principles for eigenvalues. Consider the unbounded linear operator $L$ on $L^2(\Omega)^2$:

$$Lu = -\Delta u$$

with domain $H^2(\Omega)^2 \cap H^1_0(\Omega)^2$. It is well known that $L$ is positive definite, densely defined with $L^{-1}$ compact. Let $(0 < )\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \ldots$ be the eigenvalues of $L$ according to its multiplicity. The following classical estimate is valid:

Lemma 2. Let $\Omega \subset R^2$ be open, bounded set of class $C^2$. There exists a scale invariant constant $c_0$ such that

$$\mu_j \geq c_0 j \mu_1, \forall j = 1, 2, \ldots$$

The proof is the same as that of the Stokes operator, see [4, 21].
The classical minimax principle [6] asserts that

\[ \mu_k = \min_{S_k} \max_{u \in S_k} \frac{(\nabla u, \nabla u)}{(u, u)} \]

where \( S_k \) ranges over \( k \)-dimensional subspace of \( H^1_0(\Omega)^2 \). We have, on the other hand, for \( k = 1, 2, \ldots, N \)

\[ \bar{\lambda}_k = \min_{S_k} \max_{u \in S_k} \frac{(\nabla u, \nabla u)}{(u, u)} \]

where \( S_k \) ranges over \( k \)-dimensional subspace of \( V_h \), which is a proper subspace of \( H^1_0(\Omega)^2 \).

It follows therefore

\[ \bar{\lambda}_k \geq \mu_k \geq c_0 j \mu_1. \]

It is simple to observe that \( \mu_1 |\Omega| \) is scale invariant, where \( |\Omega| \) is the area of \( \Omega \); \( \lambda_1 |\Omega| \) is also scale invariant, see [4]; therefore \( \mu_1 = c_0 \lambda_1 \) and the assertion follows. \( \square \)


**Proposition 2.** There exists a bounded absorbing set

\[ B_\rho = \{ u_h \in V_h | \| u_h \| \leq \rho \} \]

for (3.1), i.e., there exists a fixed \( \rho > 0 \), such that for any \( u_h^0 \in V_h \) there exists a \( t_0(\| u_h^0 \|) \) such that for \( t \geq t_0(\| u_h^0 \|) \), the solution \( u_h(t) \) to (3.1) satisfies: \( u_h(t) \in B_\rho. \)

**Proof.** Recall (2.7)

\[ (2.7) \quad \| u_h(t) \|^2 \leq \| u_h^0 \|^2 \exp(-\nu \mu_1 t) + \frac{\| f \|^2}{\nu^2 \mu_1^2} (1 - \exp(-\nu \mu_1 t)) \]

the result follows immediately. \( \square \)

Therefore

**Theorem 2.** Equation (3.1) has a global attractor \( A_h \) which attracts bounded sets in \( V_h \).

We proceed now to estimate the Hausdorff and fractal dimension of the attractor.
Consider the linearized equation of (3.1):
\[
\frac{dv_h}{dt} + A_h v_h + B(u_h, v_h) + B(v_h, u_h) = 0
\]

Denote \( L_h(t) = A_h + B(u_h(t), .) + B(., u_h(t)) \). Let \( v_1(t), \ldots, v_n(t) \) be solutions of
\[
\frac{dv_i}{dt} + L_h(t)v_i = 0
\]
\[
v_i(0) = v_i^0 \in V_h
\]

Then the Wronskian \( ||v_1(t) \wedge \cdots \wedge v_n(t)|| \) satisfies
\[
(4.1) \quad \frac{1}{2} \frac{d}{dt} ||v_1(t) \wedge \cdots \wedge v_n(t)||^2 + ||v_1(t) \wedge \cdots \wedge v_n(t)||^2 \text{Tr}(L_h Q(v_1, \ldots, v_n)) = 0
\]

where \( Q(v_1, \ldots, v_n) \) denotes the orthogonal projection from \( V_h \) onto the linear span of \( v_1, \ldots, v_n \) and \( \text{Tr}(L_h Q(v_1, \ldots, v_n)) \) is the trace of the operator \( L_h Q(v_1, \ldots, v_n) \).

It follows that
\[
(4.2) \quad ||v_1(t) \wedge \cdots \wedge v_n(t)||^2 = ||v_1(0) \wedge \cdots \wedge v_n(0)||^2 \exp\left(-2 \int_0^t \text{Tr}(L_h(s)Q(s)) ds\right)
\]

Without loss of generality we may then assume that \( v_1(s), \ldots, v_n(s) \) are linearly independent for all \( s \geq 0 \).

We will make use of the following Lieb-Thirring inequality [16]:

**Lemma 3.** Let \( \varphi_1, \ldots, \varphi_n \in H^1_0(\Omega), \Omega \subset R^2 \). Assume that the \( \varphi_i \)'s are orthonormal in \( L^2 \). then there exists a constant \( c_L \), independent of \( n \), such that
\[
\int \left( \sum_{i=1}^n |\varphi_i(x)|^2 \right)^2 dx \leq c_L \sum_{i=1}^n \int |\nabla \varphi_i|^2 dx.
\]

**Proposition 3.** Let \( u_h(t) \in A_h \) for \( t \geq 0 \), and \( n \leq N \), the dimension of \( V_h \). Then there exists a constant \( d \geq 0 \) independent of \( h \) such that:
\[
(4.3) \quad ||v_1(t) \wedge \cdots \wedge v_n(t)||^2 \leq ||v_1(0) \wedge \cdots \wedge v_n(0)||^2 \exp\left(-c_0 \lambda_1 \nu \frac{(n(n + 1) - d^2)t}{2}\right)
\]
for $t \geq t_0(\|u_h^0\|) = \frac{\|u_h^0\|_2^2 \nu \mu_1}{\|f\|^2}$.

Proof.

From (4.2) we need only to find the upper bound for

$$-\frac{2}{t} \int_0^t \text{Tr}(L_h(s)Q(s)) \, ds.$$

Since

$$\text{Tr}(L_h(s)Q(s)) = \sum_{k=1}^n (L_h(s)\varphi_k(s), \varphi_k(s))$$

where $\{\varphi_k\}_{k=1}^n$ is an orthonormal basis for the subspace of $V_h$ generated by $v_1, \ldots, v_n$, and since $(B(u_h, \varphi_k), \varphi_k) = 0$, we have, suppressing the argument $s$,

$$\sum_{k=1}^n (L_h\varphi_k, \varphi_k) = \nu \sum_{k=1}^n (A_h\varphi_k, \varphi_k) + \sum_{k=1}^n (B(\varphi_k, u_h), \varphi_k).$$

Now

$$|2 \sum_{k=1}^n (B(\varphi_k, u_h), \varphi_k)|$$

$$\leq \sum_{k=1}^n \int_\Omega |\varphi_k|^2 |\nabla u_h| \, dx + \sum_{k=1}^n \int_\Omega |\varphi_k||\nabla \varphi_k||u_h| \, dx$$

$$\leq \left( \int_\Omega \left( \sum_{k=1}^n |\varphi_k|^2 \right)^2 \, dx \right)^{\frac{1}{2}} \|\nabla u_h\|_{L^2(\Omega)^2} + \int_\Omega \left( \sum_{k=1}^n |\nabla \varphi_k|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^n |\varphi_k|^2 \right)^{\frac{1}{2}} |u_h| \, dx$$

$$\leq \left( \int_\Omega \left( \sum_{k=1}^n |\varphi_k|^2 \right)^2 \, dx \right)^{\frac{1}{2}} \|\nabla u_h\|_{L^2(\Omega)^2}$$

$$+ \left( \int_\Omega \left( \sum_{k=1}^n |\nabla \varphi_k|^2 \right) \, dx \right)^{\frac{1}{2}} \left( \int_\Omega \left( \sum_{k=1}^n |\varphi_k|^2 \right)^2 \, dx \right)^{\frac{1}{4}} \left( \int_\Omega |u_h|^4 \, dx \right)^{\frac{1}{4}}$$

By Lieb-Thirring inequality we have

$$2 | \sum_{k=1}^n (B_h(\varphi_k, u_h), \varphi_k) | \leq c_\nu \left( \sum_{k=1}^n \|\nabla \varphi_k\|^2 \right)^{\frac{1}{2}} \|\nabla u_h\|_{L^2} + c_\nu \left( \sum_{k=1}^n \|\nabla \varphi_k\|^2 \right)^{\frac{3}{4}} \|u_h\|_{L^4(\Omega)^2}$$

Observe

$$\sum_{k=1}^n \|\nabla \varphi_k\|^2 = \sum_{k=1}^n (A_h\varphi_k, \varphi_k) = \text{Tr}(A_hQ(v_1, \ldots, v_n)).$$
using the elementary inequalities for nonnegative $a, b$,

\[
\begin{align*}
    a^{\frac{1}{3}} b^{\frac{2}{3}} &\leq \frac{\nu}{2} a + \frac{b}{2\nu} \\
    a^{\frac{3}{4}} b^{\frac{1}{4}} &\leq \frac{\nu}{2} a + \frac{27}{32\nu^3} b
\end{align*}
\]

it follows that,

\[
\begin{align*}
-\frac{2}{t} \int_0^t Tr(L_h(s)Q(s)) \, ds & \\
&\leq -\frac{2\nu}{t} \int_0^t Tr(A_hQ) \, ds + \frac{2}{t} \int_0^t \sum_{k=1}^n (B(\varphi_k, u_h), \varphi_k) \, ds \\
&\leq -\frac{2\nu}{t} \int_0^t Tr(A_hQ) \, ds + \frac{1}{t} \int_0^t \frac{1}{\nu} Tr(A_hQ)^{\frac{1}{2}} \|\nabla u_h\|_{L^2} \, ds \\
&\quad + \frac{1}{t} \int_0^t c_L^2 (TrA_hQ)^{\frac{3}{2}} \|u_h\|_{L^4(\Omega)^2} \, ds \\
&\leq -\frac{2\nu}{t} \int_0^t Tr(A_hQ) \, ds + \left(\frac{1}{t} \int_0^t (TrA_hQ) \, ds\right)^{\frac{3}{2}} \left(\frac{1}{t} \int_0^t c_L \|\nabla u_h\|_{L^2}^2 \, ds\right)^{\frac{1}{2}} \\
&\quad + \left(\frac{1}{t} \int_0^t (TrA_hQ) \, ds\right)^{\frac{3}{4}} \left(\frac{1}{t} \int_0^t c_L \|u_h\|_{L^4(\Omega)^2}^4 \, ds\right)^{\frac{1}{4}} \\
&\leq -\nu \left(\frac{1}{t} \int_0^t Tr(A_hQ) \, ds\right) + \frac{c_L}{\nu} \left(\frac{1}{t} \int_0^t \|\nabla u_h\|^2 \, ds\right) \\
&\quad + \frac{c_L}{\nu^3} \left(\frac{1}{t} \int_0^t \|u_h\|_{L^4(\Omega)^2}^4 \, ds\right)
\end{align*}
\]

Now since $Tr(A_hQ)$ is the trace of operator which is the restriction of $A_h$ to the linear space spanned by $v_1, v_2, \ldots, v_n$, Theorem 1 gives

\[
Tr(A_hQ) \geq \sum_{k=1}^n \bar{\lambda}_k \geq \frac{n(n+1)}{2} c_0 \lambda_1.
\]

By (2.7) we have for $t \geq t_0(\|u_h^0\|) = \frac{\|u_h^0\|^2 \nu \mu_1}{\|f\|^2}$,

\[
\frac{1}{t} \int_0^t \|\nabla u_h\|^2 \, ds \leq 2 \frac{\|f\|^2}{\nu^2 \mu_1}
\]

(4.4)

Since $u_h(t) \in A_h \in B_\rho$, we have from Proposition 2 that

\[
\|u_h(t)\|^2 \leq 2 \frac{\|f\|^2}{\nu^2 \mu_1^2}.
\]

(4.5)
Recall also that $\mu_1 = \bar{c}_0 \lambda_1$ where $\bar{c}_0$ is positive and scale invariant. Using Ladyzhenskaya’s inequality [14]:

\[(4.6) \quad \|u\|_{L^4} \leq 2\|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2}, \quad \forall u \in H_0^1(\Omega)\]

We have by (4.4)-(4.6) that

\[-\frac{2}{t} \int_0^t \text{Tr}(L_h(s)Q(s)) \, ds \leq -\frac{\nu}{2} n(n + 1) c_0 \lambda_1 + \frac{2c_L\|f\|^2}{\bar{c}_0 \nu^3 \lambda_1} + \frac{4c_L\|f\|^4}{\bar{c}_0^3 \nu^7 \lambda^3}\]

Define $d \geq 0$ independent of $h$ such that

\[d^2 = \frac{4c_L}{\bar{c}_0 c_0} G^2 + \frac{8c_L}{\bar{c}_0^3 c_0} G^4\]

where $G = \frac{\|f\|}{\nu^2 \lambda_1}$ is the Grashof number. Then we have

\[-\frac{2}{t} \int_0^t \text{Tr}(L_h(s)Q(s)) \, ds \leq -\frac{c_0 \lambda_1 \nu}{2} (n(n + 1) - d^2)\]

which completes the proof of the proposition. □

The standard procedures concerning the global Lyapunov exponents yield then the following estimates [3, 20]:

**Theorem 3.** The Hausdorff dimension of the attractor $A_h$ is bounded above by a constant independent of the approximation parameter $h$, i.e., $d_H(A_h) \leq d$

**Theorem 4.** The fractal dimension of the attractor $A_h$ is bounded above by a constant independent of the approximation parameter $h$, i.e., $d_H(A_h) \leq c_0 d$

**Remark 1:** In the case of conforming finite elements, i.e., $V_h \subset V$, we have

\[(B_h(\varphi_k, u_h), \varphi_k) = (\varphi_k \cdot \nabla u_h, \varphi_k).\]

In this case the estimates of

\[-\frac{2}{t} \int_0^t \text{Tr}(L_h(s)Q(s)) \, ds\]
will be the same as for the continuous case. We have then the same optimal bounds for the dimensions as in the continuous case.

Remark 2: Similar consideration applies to some other finite element approximation of dynamical system as well, among them a 3-D variant of of Navier-Stokes equations first proposed by Ladyzhenskaya [14]:

\[
\frac{du}{dt} + (\nu_0 + \nu_1 \|
abla u\|^2)Au + B(u, u) = f
\]

\[
u(0) = u_0
\]

where \(\nu_0, \nu_1\) are positive constants, \(A\) is the Stokes operator.

5. Further Estimates. We will prove in this section some estimates for the semidiscrete dynamical system (3.1) concerning its behavior near \(t = 0\) and as \(t \to \infty\). For simplicity we restrict to the case where \(\Omega\) is a convex polygonal domain in \(R^2\). Let \(T_h\) be a family of regular triangulations of \(\Omega\). Let

\[X_h = \{u_h \in C(\Omega), u_h|_{\partial \Omega} = 0, \text{and } u_h|_K \in P(K)^2, \forall K \in T_h\};\]

\[Q_h = L_0^2(\Omega);\]

or

\[Q_h = \{q_h \in L_0^2(\Omega) : q_h|_K \in Q(K), \forall K \in T_h\}\]

where \(P(K)^2, Q(K)\) are spaces of polynomials whose maximum degree is fixed for all \(h\) and \(K\). Recall that \(V_h = \{v_h \in X_h : (q_h, \nabla \cdot v_h) = 0 \quad \forall q_h \in Q_h\}\). We give two example here:

Example 1:

Let \(W_h\) be a finite dimensional subspace of \(H_0^2(\Omega)\). Define

\[X_h = \{v : v = (\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}), \text{for } \phi \in W_h\},\]

\[Q_h = L_0^2(\Omega).\]

Then \(V_h = X_h \subset V\). This is the conforming finite element case.
Example 2:

Given a regular quadrilateral triangulation of $\Omega$. Define

$$X_h = \{ v \in V : v|_K \in Q^2(K)^2, \forall K \in T_h \},$$

$$Q_h = \{ q \in L^2_0 : q|_K \in P_1(K), \forall K \in T_h \}.$$

where $Q_2$ are the biquadratic polynomials.

To guarantee the stability of the finite element methods, we assume in addition the following standard conditions [12] on the finite element spaces:

**Hypothesis 1:** For each $v \in H^1_0 \cap H^2$ and $q \in H^1$, there exist $v_h \in X_h$ and $q_h \in Q_h$ such that

(H.1) \[ \| \nabla (v - v_h) \| \leq c h \| \Delta v \|, \quad \| q - q_h \|_{L^2/2} \leq c h \| q \|_{H^1/2} \]

**Hypothesis 2:** For every $v \in V \cap H^2$ there exists a $v_h \in V_h$ such that

(H.2) \[ \| \nabla (v - v_h) \| \leq c h \| v \|_{H^2}. \]

(H.2) is related to (H.1) and the standard inf-sup condition [8, 12].

The following result is proved in [12]:

**Lemma 4.** Let H.1 and H.2 be satisfied. Then there exists a constant $c > 0$ independent of $h$ such that

$$\| \Delta_h u_h \| \leq c \| A_h u_h \| \quad \forall u_h \in V_h.$$

We will prove a discrete Sobolev embedding result:

**Lemma 5.** Let H.1 be satisfied. Then there exists a constant $c > 0$ independent of $h$

$$\| u_h \|_{L^\infty} \leq c \| u_h \|^{\frac{1}{2}} \| \Delta_h u_h \|^{\frac{1}{2}}, \quad \forall u_h \in X_h.$$

**Proof.** Consider $u \in H^1_0 \cap H^2$ such that $\Delta u = \Delta_h u_h$. Then $u_h$ is the approximate solution in $X_h$ for $\Delta u = \Delta_h u_h$. It follows by H.1 and the standard finite element estimates that:

(5.1) \[ \| u - u_h \| + h \| \nabla (u - u_h) \| \leq c h^2 \| \Delta_h u_h \|. \]

(5.2) \[ \| u - u_h \|_{L^\infty} \leq c h \| \Delta u \| \]
From the inverse inequality $\|\Delta_h v_h\| \leq c\|v_h\|$ for all $v_h \in X_h$ and the definition of $\Delta_h$,

$$\|\Delta_h u_h\|^2 = (\Delta_h u_h, \Delta_h u_h) = \|\nabla u_h, \nabla \Delta_h u_h\| \leq ch^2\|u_h\|\|\Delta_h u_h\|$$

i.e.,

$$\|\Delta_h u_h\| \leq ch^{-2}\|u_h\|, \quad \|\Delta_h u_h\| \leq ch^{-1}\|\nabla u_h\|,$$

(5.1), (5.2) combined with above lead to

$$\|u\| \leq c\|u_h\|, \quad \|\nabla u\| \leq c\|\nabla u_h\|.$$  

Therefore

$$\|u_h\|_{L^\infty} \leq \|u - u_h\|_{L^\infty} + \|u\|_{L^\infty}$$

$$\leq ch\|\Delta u\| + c\|u\|^{\frac{1}{2}}\|\Delta u\|^{\frac{1}{2}}$$

$$\leq ch\|\Delta_h u_h\| + c\|u_h\|^{\frac{1}{2}}\|\Delta_h u_h\|^{\frac{1}{2}}$$

$$\leq c(h\|\Delta_h u_h\|^{\frac{1}{2}})\|\Delta_h u_h\|^{\frac{1}{2}} + c\|u_h\|^{\frac{1}{2}}\|\Delta_h u_h\|^{\frac{1}{2}}$$

$$\leq c\|u_h\|^{\frac{1}{2}}\|\Delta_h u_h\|^{\frac{1}{2}}$$

We are now ready to handle the trilinear form $b(u, v, w)$ as defined in section 2. Using integration by parts, we have for $u, v, w \in H_0^1$ that,

$$b(u, v, w) = (u \cdot \nabla v, w) + \frac{1}{2}(\nabla \cdot u, v \cdot w)$$

therefore for $u_h \in V_h$,

(5.3)  

$$|b(u_h, u_h, A_h u_h)| \leq c\|u\|_{L^\infty}\|\nabla u_h\|\|A_h u_h\| \leq c\|u_h\|^{\frac{1}{2}}\|\nabla u_h\|\|A_h u_h\|^{\frac{1}{2}}$$

by Lemma 4 and Lemma 5.

The following estimate is then in order:
Let $T > 0$ be fixed. There exists a constant $\rho_0$ depending only on $T, \nu, \mu_1, \|u_0\|$ and $\|f\|$ such that

$$\|\nabla u_h(t)\| \leq t^{-1} \rho_0 \quad \forall t \in (0, T).$$

**Proof.** Take $L^2$ inner product of (5.4) with $A_h u_h$, we obtain by virtue of (5.2) that

$$\frac{1}{2} \frac{d}{dt} \|\nabla u_h\|^2 + \nu(A_h u_h, A_h u_h) \leq \|f\| A_h u_h \| + c \|u_h\|^{\frac{1}{2}} \|\nabla u_h\| A_h u_h \|^{\frac{3}{2}}$$

Apply Young’s inequality to the right hand side terms to obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla u_h\|^2 + \nu(A_h u_h, A_h u_h) \leq \frac{2\|f\|^2}{\nu} + \frac{c}{\nu^3} \|u_h\|^2 \|\nabla u_h\|^4.$$ (5.6)

Denote $y(t) = \|\nabla u(t)\|^2$, $g = \frac{2\|f\|^2}{\nu}$, $h(t) = \frac{c}{\nu^3} \|u_h\|^2 \|\nabla u_h\|^2$. (5.5) then leads to

$$\frac{dy}{dt} \leq g + h(t)y(t)$$

for $0 < s < t$, we have

$$y(t) \leq \exp \left( -\int_s^t h(r) \, dr \right) y(s) + (t - s) \exp \left( -\int_0^t h(r) \, dr \right) g$$

Now integrating over $s \in (0, t)$, we obtain

$$ty(t) \leq \int_0^t \exp \left( -\int_s^t h(r) \, dr \right) y(s) \, ds + g \int_0^t (t - s) \exp \left( -\int_0^t h(r) \, dr \right) \, ds$$

Recall that

$$\int_0^T \|\nabla u_h(s)\|^2 \, ds, \quad \|u_h(t)\|^2$$

are all bounded above by constants depending only on $T, \nu, \mu_1, \|u_0\|$ and $\|f\|$, Proposition 4 then follows. \(\Box\)

We will make use of the uniform Gronwall’s inequality [21]:
Lemma 6. Let \( g, h, y \) be three nonnegative locally integrable functions on \((t_0, +\infty)\) such that \( y \) is locally absolutely continuous on \((t_0, +\infty)\), and satisfy
\[
\frac{dy}{dt} \leq gy + h \quad \forall t \geq t_0,
\]
and furthermore,
\[
\int_t^{t+r} g(s) \, ds \leq a_1, \int_t^{t+r} h(s) \, ds \leq a_2, \int_t^{t+r} y(s) \, ds \leq a_3, \forall t \geq t_0,
\]
where \( r, a_1, a_2, a_3 \) are positive constants. Then
\[
y(t + r) \leq \left( \frac{a_3}{r} + a_2 \right) \exp(a_1), \quad \forall t \geq t_0.
\]

We are now ready to prove the \( H^1 \) attracting property for (3.1), (5.4):

Proposition 5. There exists a fixed \( \rho \leq 0 \), independent of \( h \), such that for any \( u_0 \in H \), there exists a \( t_0 = t_0(\|u_0\|) \) such that for \( t \geq t_0 + 1 \), the solution \( u_h \) to (5.3)-(5.4) satisfies
\[
\|\nabla u_h(t)\| \leq \rho.
\]

Proof. Take as in (2.6) \( t_0 = t_0(\|u_0\|) \) be such that for \( t \geq t_0 \), \( \|u_h\|^2 \leq \frac{2\|f\|^2}{\nu^2 \mu_1^2} \), Recalling that \( \|P_h u_0\| \leq \|u_0\| \). From (2.8) we have:
\[
\int_t^{t+1} \|\nabla u_h\|^2 \leq \frac{\|f\|^2}{\nu^2 \mu_1} + \frac{1}{\nu} \|u_h(t)\|^2.
\]

Thus for \( t \geq t_0 \),
\[
\int_t^{t+1} \|\nabla u_h(s)\|^2 \, ds \leq \frac{\|f\|^2}{\nu^2 \mu_1^2} (\nu \mu_1 + 2)
\]
\[
\int_t^{t+1} \frac{c^3}{\nu} \|u_h\|^2 \|\nabla u_h(s)\|^2 \, ds \leq \frac{2c\|f\|^4}{\nu^8 \mu_1^4} (\nu \mu_1 + 2)
\]
\[
\int_t^{t+1} \frac{2\|f\|^2}{\nu} \, ds = \frac{2\|f\|^2}{\nu}
\]

The desired result then follows from the uniform Gronwall’s inequality and (5.6). \( \square \)
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