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CONSISTENT MODEL OF PHASE FIELD TYPE

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Abstract. In a recent paper Penrose and Fife proposed a thermodynamically consistent model of phase field type based on the idea that the value of the entropy functional can not decrease along solution paths, in agreement with what one expects from the second law of thermodynamics. It turns out that the corresponding partial differential equations become more difficult than the one studied by Caginalp and the others. In this paper we prove the global existence, uniqueness and asymptotic behavior of smooth solutions.

Key words. thermodynamically consistent model, phase field equations, global existence and uniqueness, asymptotic behavior.

AMS(MOS) subject classifications. 35K60, 35B35, 82A25

1. Introduction. In a recent paper Penrose and Fife [PF] proposed thermodynamically consistent models of phase field type based on the idea that the value of the entropy functional cannot decrease along solutions paths which is in agreement with what one expects from the second law of thermodynamics. It turns out that for a system with non-conserved order parameter the order parameter \( \varphi \) and the absolute temperature \( \theta \) satisfy the following coupled system of partial differential equations

\[
\varphi_t = K_1 \left\{ \kappa_1 \Delta \varphi + s'_0(\varphi) + \frac{\lambda(\varphi)}{\theta} \right\} \tag{1.1}
\]

\[
\theta_t - \lambda(\varphi) \varphi_t = -M_2 \Delta \left( \frac{1}{\theta} \right) \tag{1.2}
\]

with \( K_1, \kappa_1, M_2 \) being positive constants and \( s_0(\varphi) \) a double well function.

If we assume that the temperature remains fairly close to some “average” value \( \theta_0 \), and we linearize the equations with respect to \( u = \theta - \theta_0 \) and ignore the dependence of \( \lambda(\varphi) \) on \( \varphi \), then we are led to the phase field equations:

\[
\begin{align*}
\tau \varphi_t &= \xi^2 \Delta \varphi + \varphi - \varphi^3 + 2u \tag{1.3} \\
L u_t + \frac{l}{2} \varphi_t &= K \Delta u \tag{1.4}
\end{align*}
\]

studied by Caginalp [C] and the others [EZ] (refer to section 6 in [PF]).

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However, there is no reason to guarantee that the temperature is always close to certain value. Besides, as pointed out in [PF], at least in the case of the solid-liquid phase transitions, \( \lambda(\varphi) \) is proportional to \( \varphi \), i.e., \( \lambda(\varphi) = a\varphi \), rather than being a constant.

In this paper we are concerned with the initial boundary value problem for equations (1.1)–(1.2) in one space dimension:

\[
\begin{align*}
\varphi_t &= K_1 \varphi_{xx} + \varphi - \varphi^3 + \frac{a\varphi}{\theta}, \\
\theta_t - a\varphi\varphi_t &= -K_2 \left( \frac{1}{\theta} \right)_{xx},
\end{align*}
\tag{1.5, 1.6}
\]

\((x, t) \in (0, 1) \times (0, \infty)\)

with positive constants \( K_1, K_2, a \) and subject to the Neumann boundary conditions

\[
\varphi_x|_{x=0,1} = \theta_x|_{x=0,1} = 0
\tag{1.7}
\]

and the initial conditions

\[
\varphi|_{t=0} = \varphi_0(x), \quad \theta|_{t=0} = \theta_0(x) > 0
\tag{1.8}
\]

From the mathematical point of view problem (1.5)–(1.8) are more difficult than equations (1.3)–(1.4). The main difficulties come from appearance of the term \( \frac{1}{\theta} \) which may become singular, and come also from the highly nonlinear term \( a\varphi\varphi_t \) in (1.6). To overcome the first main difficulty, we introduce new unknown function \( u \):

\[
u = \frac{1}{\theta}
\tag{1.9}
\]

Then it turns out that \( \varphi \) and \( u \) satisfy

\[
\begin{align*}
\varphi_t &= K_1 \theta_{xx} + \varphi - \varphi^3 + a\varphi u, \\
u_t + au^2 \varphi \varphi_t &= K_2 u^2 u_{xx},
\end{align*}
\tag{1.10, 1.11}
\]

Our strategy is to estimate \( u \) (and \( \varphi, \varphi_t \)) from above from (1.10)–(1.11) instead of estimating \( \theta \) from below. Once we have estimates on \( \varphi \) and \( \varphi_t \), we proceed to estimate \( \theta \) from above from (1.6) which turns out to give the estimate of \( u \) from below. the higher nonlinearity, due to \( \lambda(\varphi) = a\varphi \) permits us only to get the global existence and uniqueness results in one space dimension.

The main result in this paper is the following
Main Theorem. Suppose \( \theta_0(x) \in H^2 \), \( \varphi_0(x) \in H^3 \) satisfying the compatibility conditions: \( (\theta_0)_x|_{x=0,1} = (\varphi_0)_x|_{x=0,1} = 0 \) and suppose \( \theta_0(x) > 0 \) in \([0, 1]\). Then problem (1.5)-(1.8) admits a unique global classical solution \( (\varphi(x,t), \theta(x,t)) \), \( \varphi \in C(R^+; H^3) \cap C^1(R^+, H^1) \), \( \theta \in C(R^+, H^2) \cap C^1(R^+, L^2) \), \( \theta(x,t) > 0 \) in \([0,1] \times R^+\). Moreover, the \( \omega \)-limit set \( \omega(\varphi_0, \theta_0) = \{(\psi(x), \tilde{\theta}(x))| \exists \eta_n, t_n \to +\infty \text{ such that } \lim_{t_n \to +\infty} \varphi(x, t_n) = \psi(x) \text{ in } H^2, \lim_{t_n \to +\infty} \theta(x, t_n) = \tilde{\theta}(x) \text{ in } H^1 \} \) is a connected compact subset in \( H^2 \times H^1 \) and consists of the equilibria.

This paper is organized as follows. In section 2 we give the outline of the proof of the local existence and uniqueness. In section 3 we proceed to get the uniform a priori estimates which yields the global existence and uniqueness in combination with local existence and uniqueness theorem. In last section we study the asymptotic behavior of solution. Throughout this paper we denote by \( \| \cdot \|_{H^s} \) the norm of usual Sobolev space \( H^s \) in \( \Omega = (0,1) \) and by \( \| \cdot \| \) the \( L^2 \) norm. We also denote by \( C_i \) the universal positive constants and by \( \widetilde{C}_n \) the positive constants depending on initial data.

2. Local Existence and Uniqueness. In this section we prove the local existence and uniqueness theorem for equations (1.10)-(1.11) with Neumann boundary conditions

(2.1) \[ \varphi_x|_{x=0,1} = u_x|_{x=0,1} = 0 \]

and initial conditions

(2.2) \[ \varphi|_{t=0} = \varphi_0(x), \quad u|_{t=0} = u_0(x) = \frac{1}{\theta_0(x)} > 0 \]

Although the system (1.10)-(1.11) is not a diagonal parabolic system, we are still able to prove the local existence and uniqueness, combining the theory of linear parabolic equation and the contraction mapping theorem. Let \( \tilde{\varphi}, \tilde{u} \) be such functions that

\[ \tilde{\theta}(x,t) \in C([0,T]; H^3) \cap C^1([0,T], H^1), \quad \tilde{\varphi}_x|_{x=0,1} = 0, \quad \tilde{\varphi}|_{t=0} = \varphi_0(x) \]

\[ \tilde{u}(x,t) \in C([0,T]; H^2) \cap C^1([0,T], L^2), \quad \tilde{u}_t \in L^2([0,T]; H^1), \quad \tilde{u}_x|_{x=0,1} = 0, \quad \tilde{u}|_{t=0} = u_0(x) \]

and

(2.3) \[ \tilde{u}(x,t) \geq \frac{1}{2} \min_{x \in \Omega} u_0(x) > 0 \]

Let

\[ D(x,t) = K_2 \tilde{u}^2, \quad f(x,t) = \tilde{\varphi} - \tilde{\varphi}^3 + a \tilde{u} \tilde{\varphi}, b(x,t) = a \tilde{u} \tilde{\varphi} \tilde{\varphi}_t \]

Consider the following auxiliary problem for \( \varphi \) and \( u \), respectively

(2.5) \[ \varphi_t = K_1 \varphi_{xx} + f(x,t) \]
(2.6) \[ \varphi_x|_{x=0,1} = 0 \]
(2.7) \[ \varphi|_{t=0} = \varphi_0(x) \]
and

\begin{align}
(2.8) & \quad u_t - Du_{xx} = b(x, t)u = 0 \\
(2.9) & \quad u_x \big|_{x=0,1} = 0 \\
(2.10) & \quad u \big|_{t=0} = u_0(x)
\end{align}

It turns out from the definition (2.4) that

\begin{align}
(2.11) & \quad D \in C([0,T]; H^2) \cap C^1([0,T], L^2) \\
(2.12) & \quad D \geq \alpha > 0
\end{align}

with

\begin{align}
(2.13) & \quad \alpha = \frac{K_2}{4} \left( \min \limits_{\bar{u}} u_0(x) \right)^2 \\
(2.14) & \quad f \in C([0,T]; H^2) \cap C^1(0,T], L^2), f_t \in L^2([0,T]; H^1) \\
(2.15) & \quad b \in C([0,T]; H^1)
\end{align}

By the well known results for linear parabolic equations we have

**Lemma 2.1.** Problem (2.5)–(2.7) and problem (2.8)–(2.10) admits a unique solution \( \varphi \) and \( u \) such that

\begin{align}
(2.16) & \quad \varphi \in C([0,T], H^3) \cap C^1([0,T]; H^1) \\
(2.17) & \quad u \in C([0,T]; H^2) \cap C^1([0,T], L^2), u_t \in L^2([0,T]; H^1)
\end{align}

Moreover, the following estimates hold for \( 0 \leq t \leq T \)

\begin{align}
(2.17) & \quad \|\varphi(t)\|^2 \leq e^t(\|\varphi_0\|^2 + \int_0^t \|f\|^2 d\tau) \\
(2.18) & \quad \|\varphi_x(t)\|^2 \leq e^t(\|\varphi_{0x}\|^2 + \int_0^t \|f\|^2 d\tau) \\
(2.19) & \quad \|\varphi_{xx}(t)\|^2 \leq e^t(\|\varphi_{0xx}\|^2 + \int_0^t \|f_x\|^2 d\tau) \\
(2.20) & \quad \|\varphi_t(t)\|_{H^1}^2 \leq e^t(\|\varphi_t\|_{H^3}^2 + \|f\|_{t=0}^2_{H^1} + \int_0^t \|f_t\|^2 d\tau)
\end{align}
\[
\leq e^t(\tilde{C}_0 + \int_0^t \|f_t\|^2 d\tau)
\]

(2.21) \[\|\varphi_{xx}(t)\|^2 \leq 2e^t(\tilde{C}_0 + \int_0^t \|f_t\|^2 d\tau + t \int_0^t \|f_{xt}\|^2 d\tau)\]

(2.22) \[\|u(t)\|^2 \leq e^{C_3Mt}\|u_0\|^2\]

with

(2.23) \[M = 2|b|_{L^\infty(\Omega \times (0,T))} + \frac{1}{\alpha} |D_x^2|_{L^\infty(\Omega \times (0,T))}\]

(2.24) \[\|u_x(t)\|^2 \leq \|u_{0x}\|^2 + \frac{M_0}{\alpha C_3 M} (e^{C_3Mt} - 1) \|u_0\|^2\]

with

(2.25) \[M_0 = |b|^2_{L^\infty(\Omega \times (0,T))}\]

(2.26) \[\|u_{xx}(t)\|^2 + \alpha \int_0^t \|u_{xx}\|^2 d\tau \leq \|u_{0xx}\|^2 + \frac{C_4}{\alpha} (\|u_{0x}\|^2 + e^{C_3Mt}\|u_0\|^2 + \frac{M_0}{\alpha C_3 M} (e^{C_3Mt} - 1) \|u_0\|^2) \cdot \int_0^t \|b\|^2_{H^1} + (D_x^2)|_{L^\infty}) d\tau\]

(2.27) \[\|u_x(t)\|^2 \leq 2(|D|_{L^\infty(\Omega \times (0,T))}^2 \|u_{xx}(t)\|^2 + |b|_{L^\infty(\Omega \times (0,T))}^2 \|u\|^2) \int_0^t \|u_{tx}\|^2 d\tau \leq C_5(|D|_{L^\infty(\Omega \times (0,T))}^2 \int_0^t \|u_{xx}\|^2 d\tau + |D_x^2|^2_{L^\infty(\Omega \times (0,T))} \cdot (\|u_{0x}\|^2 + \frac{M_0}{\alpha C_3 M} (e^{C_3Mt} - 1) \|u_0\|^2) + \sup_{0 \leq t \leq T} \|b(t)\|^2_{H^1} \cdot (e^{C_3Mt} - 1) \|u_0\|^2\]

Proof. The existence and uniqueness results are well known (see e.g. [LM] and [T]). The straight application of energy method and Gronwall’s inequality yields inequalities (2.17)–(2.28). We omit the details here.

Theorem 2.2. Suppose \(\varphi_0 \in H^3\), \((\varphi_0)_x|_{x=0,1} = 0\), \(u_0 \in H^2\), \((u_0)_x|_{x=0,1} = 0\), \(u_0(x) > 0\). Then there exists a positive constant \(t^*\) depending only on \(\min_{\Omega} u_0(x), \|u_0\|_{H^2}, \|\varphi_0\|_{H^3}, \alpha, K_1, K_2\)
such that problem (1.10)–(1.11), (2.1)–(2.2) admits in $\Omega \times [0, t^*]$ a unique solution $(\varphi(x, t), u(x, t)) \in C([0, t^*]; H^3) \cap C^1([0, T], H^1) \times C([0, t^*]; H^2) \cap C^1([0, t^*]; L^2) \cap H^1([0, t^*], H^1)$. Moreover,

(2.29)  \[ \|\varphi(t)\|_{H^3}^2 \leq 4(\|\varphi_0\|_{H^3}^2 + \tilde{C}_0), \quad \|u(t)\|_{H^2}^2 \leq 2\|u_0\|_{H^2}^2, \quad 0 \leq t \leq t^* \]

**Proof.** To prove the theorem, we adopt the usual contraction mapping theorem.

Let

$$\Sigma_T = \{ (\varphi, u) \mid \varphi \in C([0, T]; H^3) \cap C^1([0, T]; H^1), \varphi_x|_{x=0,1} = 0, \quad \varphi|_{t=0} = \varphi_0(x), \quad u \in C([0, T]; H^2) \cap C^1([0, T]; L^2), \quad u_t|_{t=0} = u_0(x), \quad u|_{t=0} = u_0(x) \}
$$

$$u(x, t) \geq \frac{1}{2} \min_\Omega u_0(x) > 0, \quad \|\varphi(t)\|_{H^3}^2 \leq 4(\|\varphi_0\|_{H^3}^2 + \tilde{C}_0), \quad \|\varphi_t(x)\|_{H^1}^2 \leq 2\tilde{C}_0, \quad \|u(t)\|_{H^2}^2 \leq 2\|u_0\|_{H^2}^2, \quad \|u_t(t)\| \leq \tilde{C}_1,
$$

$$\int_0^T \|u_t\|_{H^1}^2 dt \leq \tilde{C}_1 \}
$$

with $\tilde{C}_1$ being a positive constant depending only on initial data and specified later on. For $(\tilde{\varphi}, \tilde{u}) \in \Sigma_T$, by Lemma 2.1 we have $(\varphi, u)$, a pair of solutions to the auxiliary problems (2.5)–(2.7) and (2.8)–(2.10). We only need to prove that there exists $t^*$, as required in the statement of the Theorem, such that (i) For $(\tilde{\varphi}, \tilde{u}) \in \Sigma_{t^*}$, $(\varphi, u) \in \Sigma_{t^*}$. (ii) The mapping $(\tilde{\varphi}, \tilde{u}) \to (\varphi, u)$ is contractive.

**Proof of (i).** By the definition (2.4) we easily have

$$|b|_{L^\infty(\Omega \times (0, T))} \leq C_6 \sup_{0 \leq t \leq T} (\|\tilde{u}\|_{H^1}^3 + \|\tilde{\varphi}\|_{H^1}^3 + \|\tilde{\varphi}_t\|_{H^1}^3) \leq \tilde{C}_2
$$

(2.32)

$$\sup_{0 \leq t \leq t} \|b\|_{H^1}^2 \leq C_7 \sup_{0 \leq t \leq t} (\|\tilde{u}\|_{H^1}^6 + \|\tilde{\varphi}\|_{H^1}^6 + \|\tilde{\varphi}_t\|_{H^1}^6) \leq \tilde{C}_3
$$

(2.33)

$$|D|_{L^\infty(\Omega \times (0, T))} \leq K_2 |\tilde{u}|_{L^\infty(\Omega \times (0, T))} \leq \tilde{C}_4
$$

(2.34)

$$|D_x|_{L^\infty(\Omega \times (0, T))} \leq 2K_2 |\tilde{u}|_{L^\infty} |\tilde{u}_x|_{L^\infty} \leq \tilde{C}_5
$$

Similarly, we have

(2.35)  \[ \|f\| \leq \tilde{C}_6, \quad \|f_x\| \leq \tilde{C}_7, \quad \|f_t\| \leq \tilde{C}_8 \]

\[
(2.36) \quad \int_0^t \|f_{xt}\|^2 d\tau \leq \tilde{C}_9 t + \tilde{C}_{10} \int_0^t \|\tilde{u}_{xt}\|^2 d\tau \leq \tilde{C}_9 t + \tilde{C}_{11}
\]

Therefore, it turns out from (2.17)-(2.22), (2.24), (2.26)-(2.28) that there exists a positive constant \( t_1 \) such that

\[
(2.37) \quad \|\varphi(t)\|_{H^3}^2 \leq 4(\|\varphi_0\|_{H^3}^2 + \tilde{C}_0), \quad 0 \leq t \leq t_1
\]

\[
(2.38) \quad \|\varphi_t(t)\|_{H^1}^2 \leq 2\tilde{C}_0 \quad 0 \leq t \leq t_1
\]

\[
(2.39) \quad \|u(t)\|_{H^2}^2 \leq 2\|u_0\|_{H^2}^2 \quad 0 \leq t \leq t_1
\]

\[
(2.40) \quad \int_0^t \|u_{xxx}\|^2 d\tau \leq \frac{2}{\alpha} \|u_0\|_{H^2}^2 \quad 0 \leq t \leq t_1
\]

\[
(2.41) \quad \|u_t(t)\|^2 \leq 4(\tilde{C}_4^2 + \tilde{C}_5^2)\|u_0\|_{H^2}^2 \triangleq \tilde{C}_{12} \quad 0 \leq t \leq t_1
\]

\[
(2.42) \quad \int_0^t \|u_{xt}\|^2 d\tau \leq C_5(\tilde{C}_4^2 \cdot \frac{2}{\alpha} \|u_0\|_{H^2}^2 + 2\tilde{C}_5^2 \|u_0\|_{H^2}^2 + 2\tilde{C}_5^2 \|u_0\|^2) \triangleq \tilde{C}_{13}
\]

We now choose \( \tilde{C}_1 \) appearing in (2.30) as

\[
(2.43) \quad \tilde{C}_1 = \max(\tilde{C}_{12}, \tilde{C}_{13})
\]

It turns out that

\[
(2.44) \quad \|u_t(t)\|^2 \leq \tilde{C}_1 \quad 0 \leq t \leq t_1
\]

\[
(2.45) \quad \int_0^t \|u_{tx}\|^2 d\tau \leq \tilde{C}_1 \quad 0 \leq t \leq t_1
\]

It remains to prove that there exists a positive constant \( t_2 \) such that

\[
(2.46) \quad u(x, t) \geq \frac{1}{2} \min_{\Omega} u_0(x) \triangleq \frac{m_0}{2}, \quad 0 \leq t \leq \min(t_1, t_2)
\]

Indeed, by maximum principle we have

\[
(2.47) \quad u(x, t) > 0 \quad \text{in} \quad \Omega \times (0,t_1)
\]
Let

\begin{align}
\lambda &= -(|b|_{L^\infty(\Omega \times (0,T))} + 1), \quad v = e^{-\lambda t} u \\
\end{align}

Then \( v \) satisfies

\begin{align}
(2.49) \quad v_t - Dv_{xx} &= -(\lambda + b)v \\
(2.50) \quad v_x|_{x=0,1} &= 0 \\
(2.51) \quad v|_{t=0} &= u_0(x)
\end{align}

Since \( v > 0 \) and \( (\lambda + b) < 0 \), by the maximum principle \( v \) must achieve its minimum at \( t = 0 \). It turns out that we have

\begin{align}
\min_{\Omega \times (0,t)} u \geq e^{\lambda t} \min_{\lambda \times (0,t)} v = e^{-(|b|_{L^\infty} + 1)t} \quad m \geq \frac{1}{2} m
\end{align}

as long as \( t \leq t_2 \triangleq \frac{\ln 2}{(|b|_{L^\infty} + 1)} \).

Choose

\begin{align}
(2.53) \quad t_0 = \min(t_1, t_2)
\end{align}

then the mapping \((\tilde{\varphi}, \tilde{u}) \to (\varphi, u)\) maps \( \Sigma_{t_0} \) into itself.

Proof of (ii). The method of the proof is standard. Let \((\varphi_1, u_1), (\varphi_2, u_2)\) be two solutions corresponding to \((\tilde{\varphi}_1, \tilde{u}_1)\) and \((\tilde{\varphi}_2, \tilde{u}_2)\), respectively.

Let

\begin{align}
(2.54) \quad \varphi = \varphi_1 - \varphi_2, \quad \tilde{u} = \tilde{u}_1 - \tilde{u}_2, \quad \varphi = \varphi_1 - \varphi_2, \quad u = u_1 - u_2
\end{align}

Then \( \varphi \) and \( u \) satisfy

\begin{align}
(2.55) \quad \varphi_t &= K_1 \varphi_{xx} + (\tilde{\varphi}_1 - \tilde{\varphi}_1^3 + a\tilde{u}_1 \tilde{\varphi}_1) - (\tilde{\varphi}_2 - \tilde{\varphi}_2^3 + a\tilde{u}_2 \tilde{\varphi}_2) \\
(2.56) \quad \varphi_x|_{x=0,1} &= 0 \\
(2.57) \quad \varphi|_{t=0} &= 0
\end{align}

and

\begin{align}
(2.58) \quad u_t - D_1 u_{xx} + bu &= (K_2 \tilde{u}_1^2 - K_2 \tilde{u}_2^2) u_{1xx} + (a\tilde{u}_1 \tilde{\varphi}_1 \tilde{\varphi}_1) - a\tilde{u}_2 \tilde{\varphi}_2 \tilde{\varphi}_2 u_2 \\
(2.59) \quad u_x|_{x=0,1} &= 0 \\
(2.60) \quad u|_{t=0} &= 0
\end{align}
Applying the energy method as before, we obtain

\begin{equation}
\|\varphi(t)\|_{H^3}^2 + \|\varphi_t(t)\|_{H^1}^2 \leq \tilde{C}_{14} \int_0^t (\|\tilde{\varphi}\|_{H^1}^2 + \|\tilde{\varphi}_t\|_{H^1}^2 + \|\tilde{u}\|_{H^1}^2 + \|\tilde{u}_t\|_{H^1}^2) \, d\tau
\end{equation}

\begin{equation}
\|u(t)\|_{H^2}^2 \leq \tilde{C}_{15} \int_0^t (\|\tilde{\varphi}_t\|_{H^1}^2 + \|\tilde{\varphi}\|_{H^2}^2 + \|\tilde{u}\|_{H^1}^2) \, d\tau
\end{equation}

\begin{equation}
\|u_t(t)\|^2 \leq \tilde{C}_{16}(\|u_{xx}\|^2 + \|u\|^2 + \|\tilde{u}\|^2_{H^1} + \|\tilde{\varphi}\|^2 + \|\tilde{\varphi}_t\|^2_{H^1})
\leq \tilde{C}_{17}(\|\tilde{u}\|_{H^1}^2 + \|\tilde{\varphi}\|^2_{H^1} + \|\tilde{\varphi}_t\|^2 + \int_0^t (\|\tilde{\varphi}_t\|_{H^1}^2 + \|\tilde{\varphi}\|_{H^2}^2) \, d\tau
\end{equation}

It turns out that

\begin{equation}
\sup_{0 \leq \tau \leq t} (\|\varphi(\tau)\|_{H^3}^2 + \|\varphi_t(\tau)\|_{H^1}^2 + \|u(\tau)\|_{H^2}^2 + \delta \|u_t(\tau)\|^2)
\leq (\tilde{C}_{18} t + \tilde{C}_{19} \delta)(\sup_{0 \leq \tau \leq t} (\|\tilde{u}\|_{H^2}^2 + \|\tilde{\varphi}\|_{H^3}^2 + \|\tilde{\varphi}_t\|_{H^1}^2)
\leq \tilde{C}_{20} t \sup_{0 \leq \tau \leq t} \|\tilde{u}_t\|^2)
\end{equation}

We first choose \( \delta = \frac{1}{4 \tilde{C}_{19}} \). Then we choose \( t \) small enough so that

\begin{equation}
\tilde{C}_{18} t < \frac{1}{4} \quad \frac{\tilde{C}_{20} t}{\delta} < \frac{1}{2}
\end{equation}

Thus

\begin{equation}
\tau^* = \min \left( t_1, t_2, \frac{1}{4 \tilde{C}_{18}}, \frac{\delta}{2 \tilde{C}_{20}} \right)
\end{equation}

is the required length of the time interval. The proof of Theorem 2.1 is completed.

3. Uniform a priori estimates. This section is the essential part of this paper. First we use the fact that the value of the entropy functional, the negative value of which serves as the Lyapunov functional, cannot decrease to prove the following

**Lemma 3.1.** For any \( t > 0 \) the following estimates hold

\begin{equation}
\|\varphi(t)\|_{H^1} \leq \tilde{C}_1, \quad \|\varphi(t)\|_{L^\infty} \leq \tilde{C}_2
\end{equation}

\begin{equation}
\int_0^t \|\varphi_t\|^2 \, d\tau \leq \tilde{C}_3, \quad \int_0^t \|u_x\|^2 \, d\tau \leq \tilde{C}_4
\end{equation}

\begin{equation}
0 < \tilde{C}_5 \leq \int_0^1 \frac{1}{u} \, dx \leq \tilde{C}_6
\end{equation}
Proof. We multiply (1.10) by $\varphi_t$ and add up with (1.11) multiplying by $\frac{1}{u}$, integrate with respect to $x$, we obtain

$$
(3.4) \quad \frac{d}{dt} \int_0^1 \left( \frac{K_1}{2} |\varphi_x|^2 + \frac{1}{4} \varphi^4 - \frac{1}{2} \varphi^2 + \ln u \right) dx + \|\varphi_t\|^2 + K_2\|u_x\|^2 = 0
$$

Integrating with respect to $t$ yields

$$
(3.5) \quad \int_0^1 \left( \frac{K_1}{2} |\varphi_x|^2 + \frac{1}{4} \varphi^4 - \frac{1}{2} \varphi^2 + \ln u \right) dx + \int_0^t (\|\varphi_t\|^2 + K_2\|u_x\|^2) d\tau
\quad = \quad A \triangleq \int_0^1 \left( \frac{K_1}{2} |\varphi_0x|^2 + \frac{1}{4} \varphi_0^4 - \frac{1}{2} \varphi_0^2 + \ln u_0 \right) dx
$$

$$
(3.6) \quad \int_0^1 \left( \frac{K_1}{2} |\varphi|^2 + \frac{1}{4} \varphi^4 - \frac{1}{2} \varphi^2 \right) dx + \int_0^t (\|\varphi_t\|^2 + K_2\|u_x\|^2) dx
\quad = \quad A + \int_0^1 \ln \frac{1}{u} \, dx \leq A + \int_0^1 \frac{1}{u} \, dx
$$

On the other hand, multiplying (1.11) by $u^{-2}$, integrating with $x$ and $t$ yields

$$
(3.7) \quad \frac{a}{2} \|\varphi\|^2 - \int_0^1 \frac{1}{u} \, dx = \frac{a}{2} \|\varphi_0\|^2 - \int_0^1 \frac{1}{u_0} \, dx
$$

then it follows from (3.6), (3.7) that

$$
(3.8) \quad \frac{K_1}{2} \|\varphi_x\|^2 + \frac{1}{4} \int_0^1 \varphi^4 \, dx - \frac{1}{2} \|\varphi\|^2 + \int_0^t (\|\varphi_t\|^2 + K_2\|u_x\|^2) d\tau
\quad \leq \tilde{C}_7 + \frac{a}{2} \|\varphi\|^2
$$

Applying Young’s inequality $\varphi^2 \leq \frac{\varepsilon}{2} \varphi^4 + \frac{1}{2\varepsilon}$ to (3.8) yields (3.1)–(3.2) and also, by (3.5), (3.7)

$$
(3.9) \quad \int_0^1 \frac{1}{u} \, dx \leq \tilde{C}_6 \quad \text{and} \quad \int_0^1 \ln ud\tau \leq \tilde{C}_8
$$
Applying Jensen’s inequality to the convex function \(-\ln y\) yields

\[
(3.10) \quad - \int_0^1 \ln \frac{1}{u} \, dx \geq - \ln \int_0^1 \frac{1}{u} \, dx
\]

Then it turns out from (3.9)–(3.10) that

\[
(3.11) \quad 0 < \tilde{C}_5 = e^{-\tilde{C}_9} < \int_0^1 \frac{1}{u} \, dx
\]

Thus the proof is completed.

From Lemma 3.1 we have good estimates for \(\varphi\) but very weak estimate for \(\frac{1}{u}\) which is a common feature for the coupled system involving the energy equation. In the sequel we intend to get better estimates for \(u\) and also for \(\varphi\). Let

\[
(3.12) \quad m(t) = \min_{\Omega} u(x, t), \quad M(t) = \max_{\Omega} u(x, t)
\]

By (3.3) we have

\[
(3.13) \quad m(t) \leq \frac{1}{C_5}
\]

The crucial steps are to estimate \(M(t)\) from above and \(m(t)\) from below.

**Lemma 3.2.** The following estimates hold

\[
(3.14) \quad \|\varphi_t(t)\| \leq \tilde{C}_9, \quad \|\varphi(t)\|_{L^2} \leq \tilde{C}_{10}
\]

\[
(3.15) \quad \|u_x(t)\| \leq \tilde{C}_{11}, \quad |u(t)|_{L^\infty} = M(t) \leq \tilde{C}_{12}
\]

\[
(3.16) \quad \int_0^t \|\varphi_{tx}\|^2 \, d\tau \leq \tilde{C}_{13}, \quad \int_0^t \|\frac{u_t}{u}\|^2 \, d\tau \leq \tilde{C}_{14}
\]

**Proof.** Differentiating (1.10) with \(t\), then multiplying it by \(\varphi_t\) and (1.11) by \(u^{-2}u_t\), integrating with respect to \(x\) and adding together, we obtain

\[
(3.17) \quad \frac{1}{2} \frac{d}{dt} \left( \|\varphi_t\|^2 + K_2\|u_x\|^2 \right) + K_1\|\varphi_{tx}\|^2 + 3 \int_0^1 \varphi^2 \varphi_t^2 \, dx
\]

\[
+ \|\frac{u_t}{u}\|^2 = \|\varphi_t\|^2 + a \int_0^1 u\varphi_t^2 \, dx
\]

\[
\leq (1 + aM(t))\|\varphi_t(t)\|^2
\]
For any \( t > 0 \), let \( x^*(t) \) and \( x_*(t) \) be the points where \( u(x, t) \) achieves its maximum and minimum. Then we have

\[
(3.18) \quad M(t) - m(t) = \int_{x_*}^{x^*} u_t dx \leq \int_0^1 |u_x| dx \leq \|u_x(t)\|
\]

which results in

\[
(3.19) \quad M(t) \leq m(t) + \|u_x(t)\| \leq \frac{1}{C_5} + \|u_x(t)\|
\]

Integrating with respect to \( t \), using (3.2), (3.9), we obtain

\[
(3.20) \quad \frac{1}{2} \|\varphi_t(t)\|^2 + \frac{K_2}{2} \|u_x(t)\|^2 + K_1 \int_0^t \|\varphi_{tx}\|^2 d\tau + 3 \int_0^t \int_0^1 \varphi^2 \varphi_t^2 dx d\tau
\]

\[
+ \int_0^t \int_0^1 \frac{u_t^2}{u^2} dx d\tau \leq \tilde{C}_{14} + \tilde{C}_{15} \max_{0 \leq \tau \leq t} \|u_x(\tau)\|
\]

\[
\leq \tilde{C}_{16} + \frac{K_2}{4} \max_{0 \leq \tau \leq t} \|u_x(\tau)\|^2
\]

Taking supreme with respect to \( t \) in both sides of (3.20) yields (3.14)–(3.15). Thus the proof is completed.

We now proceed to get the estimates of the higher order derivatives of \( \varphi \).

**Lemma 3.3.** The following estimates hold

\[
(3.21) \quad \|\varphi_{xt}(t)\| \leq \tilde{C}_{17} , \quad \|\varphi(t)\|_{L^3} \leq \tilde{C}_{18} , \quad \int_0^t \|\varphi_{tt}\|^2 d\tau \leq \tilde{C}_{19}
\]

**Proof.** Differentiating (1.10) with \( t \), then multiplying it by \( \varphi_{tt} \), integrating with \( x \) and \( t \) yields

\[
(3.22) \quad \int_0^t \|\varphi_{tt}\|^2 d\tau + \frac{K_1}{2} \|\varphi_{xt}(t)\|^2 = \frac{K_1}{2} \|K_1 \varphi_{0txx} + \varphi_0 - \varphi_0^3 + a u_0 \varphi_0\|^2
\]

\[
+ \int_0^t \int_0^1 \varphi_{tt}(1 - 3\varphi^2) \varphi_t dx d\tau + \int_0^t \int_0^1 a \varphi_{tt} (u_t \varphi + u \varphi_t) dx d\tau
\]

\[
\leq \tilde{C}_{20} + \frac{1}{2} \int_0^t \|\varphi_{tt}\|^2 d\tau + \tilde{C}_{21} \left( \int_0^t \|u_t\|^2 d\tau + \int_0^t \|\varphi_t\|^2 d\tau \right)
\]

\[
\leq \tilde{C}_{22} + \frac{1}{2} \int_0^t \|\varphi_{tt}\|^2 d\tau
\]

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which results in (3.21). □

In order to estimate $m(t)$ from below, we are going to estimate $\theta$ from above. It follows from (1.6) that $\theta > 0$ satisfies

$$
(3.23) \quad \theta^3 \theta_t - K_2 \theta \theta_{xx} + 2\theta^2_x = a \theta^3 \varphi_t
$$

For any $p \in Z$, $p \geq 2$, we multiply (3.23) by $\theta^{p-4}$ and integrate with $x$ to obtain

$$
(3.24) \quad \frac{1}{p} \frac{d}{dt} \int_0^1 \theta^p dx + K_2 (p - 1) \int_0^1 \theta^{p-4} \theta_x^2 dx = a \int_0^1 \theta^{p-1} \varphi_t dx
$$

$$
\leq \frac{a(p-1)}{p} \int_0^1 \theta^p dx + \frac{a}{p} \int_0^1 |\varphi_{\varphi_t}|^p dx
$$

It follows from Lemma 3.3 that

$$
(3.25) \quad |\varphi_{\varphi_t}|_{L^\infty} \leq \tilde{C}_{23}
$$

Applying Gronwall's inequality to (3.24) yields for any $p \geq 2$.

$$
(3.26) \quad \int_0^1 \theta^p dx \leq e^{a(p-1)t} \left( \int_0^1 \theta_0^p dx + \frac{\tilde{C}_{23}^p}{p-1} \right)
$$

$$
(3.27) \quad \|\theta(t)\|_{L^p} \leq e^{at} (\|\theta_0\|_{L^p} + \tilde{C}_{23}) \leq e^{at} (\|\theta_0\|_{L^\infty} + \tilde{C}_{23})
$$

It turns out by letting $p \to \infty$ that

$$
(3.28) \quad \|\theta(t)\|_{L^\infty} \leq e^{at} (\|\theta_0\|_{L^\infty} + \tilde{C}_{23})
$$

$$
(3.29) \quad u = \frac{1}{\theta} \geq e^{-at} \frac{1}{\|\theta_0\|_{L^\infty} + \tilde{C}_{23}}, \quad \forall \ t > 0
$$

For any $T > 0$, $0 \leq t \leq T$, $x \in [0,1]

$$
(3.30) \quad u(x,t) \geq \alpha_T \triangleq e^{-aT} \frac{1}{\|\theta_0\|_{L^\infty} + \tilde{C}_{23}} > 0
$$

Thus we have proved

**Lemma 3.4.** for any $T > 0$, $0 \leq t \leq T$, $x \in [0,1]$ $u(x,t)$ is bounded from below as given by (3.20).

Once we have had the estimates of $u$ from below, we can proceed to get the estimates for higher order derivatives using usual energy method.
**Lemma 3.5.** For any $T > 0$, $0 \leq t \leq T$ there exists a positive constant $C_T$ depending only on $T$ and $\|u_0\|_{H^2}$, $\|\varphi_0\|_{H^3}$ such that the following estimate holds

\[(3.31) \quad \|u(t)\|_{H^2} + \|u(t)\| + \int_0^t \|u_{xxx}\|^2 d\tau + \int_0^t \|u_{xx}\|^2 d\tau \leq C_T\]

**Proof.** Estimate (3.30) implies that (2.11) is a nondegenerate parabolic equation. Differentiating (2.11) with respect to $x$, then multiplying it by $-u_{xxx}$ and integrate with $x$, we get

\[(3.32) \quad \frac{1}{2} \frac{d}{dt} \|u_{xx}(t)\|^2 + K_2 \int_0^1 u^2 u_{xx}^2 dx = \int_0^1 u_{xxx}(2auu_x \varphi \varphi_t

+ au^2 \varphi_x \varphi_t + au^2 \varphi \varphi_{xt} - 2K_2 uu_x u_{xx}) dx

\leq \frac{K_2 \alpha_T^2}{4} \|u_{xxx}\|^2 + C_T'(\|u\|_{L^\infty}^2 \|\varphi \varphi_t\|_{L^\infty}^2 \|u_x\|^2 + |u|^2_{L^\infty} \|\varphi_x\|^2_{H^1} \|\varphi_t\|^2

+ |u|^2_{L^\infty} \|\varphi\|^2_{L^\infty} \|\varphi_{xt}\|^2 + |u|^2_{L^\infty} \|u_x\|^2_{L^\infty} \|u_{xx}\|)

using

\[(3.33) \quad \|u_{xx}\|^2 = -\int_0^1 u_x u_{xxx} dx \leq \|u_x\| \|u_{xxx}\|

\[(3.34) \quad |u_x|_{L^\infty} \leq \|u_{xx}\|

and the estimates in previous Lemmas we get

\[(3.35) \quad \frac{d}{dt} \|u_{xx}\|^2 + K_2 \alpha_T^2 \|u_{xxx}\|^2 \leq C_T'' + \frac{K \alpha_T^2}{2} \|u_{xxx}\|^2\]

Thus (3.31) follows from integrating (3.35) with respect to $t$ and using equation (2.11). []

The local existence and uniqueness of Theorem 2.2 combining with the a priori estimates in above Lemmas yields that for an arbitrary $T > 0$ the solution can be uniquely extended step by step to the whole interval $[0, T]$. Since $T$ is arbitrary, we have proved the part of global existence and uniqueness of Main Theorem.

**4. Asymptotic behavior as $t \to +\infty$.** The orbit $t \to (\varphi(t), u(t))$ is continuous in $H^3 \times H^1$ by Lemmas 3.2–3.3 and is compact in $H^2 \times C$. The system (1.10)–(1.11), (2.1)–(2.2) has a Lyapunov functional as given in (3.4) which is also continuous in $H^2 \times C$. Then the well known results $[D]$ in the dynamical system theory claim that the $\omega$-limit set defined by

\[(4.1) \quad \omega(\varphi_0, u_0) = \{(\psi(t), v(x))| \exists t_n, t_n \to \infty \}

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such that
\[
\varphi(x, t_n) \xrightarrow{H^2} \psi(x), \quad u(x, t_n) \xrightarrow{C} \nu(x)
\]
is connected compact subset in \(H^2 \times C\). Moreover, the \(\omega\)-limit set consists of the equilibria of the system (1.10)–(1.11), (2.1) (2.2). In what follows we consider the corresponding stationary problems. The stationary problem for \(\varphi\) reads
\begin{align}
K_1\varphi_{xx} + \varphi - \varphi^3 + au\varphi &= 0 \\
\varphi_x|_{x=0,1} &= 0
\end{align}
(4.2) (4.3)

The stationary problem for \(u\) simply yields
\[u \equiv \text{const}
\]
(4.4)

On the other hand, it follows from (1.11) and (2.1) that for all \(t \geq 0\)
\[
\int_0^1 \frac{1}{u} \, dx - \frac{a}{2} \|\varphi\|^2 = C_0 \triangleq \int_0^1 \frac{1}{u_0} \, dx - \frac{a}{2} \|\varphi_0\|^2
\]
(4.5)

Thus, the stationary solutions \(u\) and \(\varphi\) have to satisfy (4.5) which results in
\[
u = \frac{1}{C_0 + \frac{a}{2} \|\varphi\|^2}
\]
(4.6)

We are interested in the solution \(\varphi\) to (4.2)–(4.3) with \(u\) given by (4.6). This is a boundary value problem of nonlinear elliptic equation with nonlocal term. We are looking for the solution \(\varphi\) and \(u\) given by (4.6) with \(u \geq 0\). Observe that the problem always has the required trivial solution
\[
u = \frac{-\left(\frac{a}{2} + C_0\right) + \sqrt{\left(\frac{a}{2} + C_0\right)^2 + 2a^2}}{a^2}
\]
(4.7)

\[
\varphi = \pm \sqrt{\left(\frac{a}{2} - C_0\right) + \sqrt{\left(\frac{a}{2} + C_0\right)^2 + 2a^2}}
\]
(4.8)

On the other hand, we observe that \(u \equiv 0\) cannot be a solution which, combining with the fact that \(\omega\)-limit set is compact, implies that \(\theta(x, t) = \frac{1}{u(x, t)}\) is uniformly bounded. Thus Lemma 3.5 can be improved to have uniform bounded of \(H^2\) norm of \(u\) and also \(\theta\). We can also define \(\omega\)-limit set for \(\theta\). Thus the proof of Main Theorem is complete.

**Remark.** We have studied in [LZ] the multiplicity of solution to the above nonlinear boundary value problem for \(\varphi\) and have proven that the number of solutions are finite which results in that the \(\omega\)-limit set consists in only one point and \((\varphi(x, t), u(x, t))\) will converge to this point as \(t \to +\infty\).
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