BDM MIXED METHODS FOR A NONLINEAR ELLIPTIC PROBLEM

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Abstract. A new mixed formulation is introduced to approximate the solution of a nonlinear elliptic problem based on the Brezzi-Douglas-Marini mixed finite elements. The existence and uniqueness of solutions of the mixed formulation and its discretization are demonstrated. Optimal error estimates in \( L^2(\Omega), L^\infty(\Omega), \) and \( H^{-s}(\Omega) \) are derived. Numerical results are presented to compare the present approach with the standard one and to justify the theoretical results obtained here.

Key words: Mixed method, error estimate, nonlinear elliptic problem, BDM element

AMS(MOS) subject classifications(1985 revision). Primary 65N15, 65N30; Secondary 41A10, 41A25

1. Introduction

In this paper we shall consider the Dirichlet problem

\[
\begin{align*}
(1.1a) & \quad Lu = -\text{div} \ A(x, \nabla u) = f(x), \quad x \in \Omega, \\
(1.1b) & \quad u = 0, \quad x \in \partial \Omega,
\end{align*}
\]

in a bounded domain \( \Omega \subset \mathbb{R}^2 \) with a \( C^2 \) boundary \( \partial \Omega \), where \( \nabla u \) denotes the gradient of a scalar function \( u \), and \( \text{div} \ w \) denotes the divergence of a vector function \( w \). We shall assume that \( A : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}^2 \) are twice continuously differentiable with bounded derivatives through second order; moreover, we shall assume that \( L \) is strictly elliptic at \( u(x) \) in the sense that there is a constant \( \alpha_0 > 0 \) such that

\[
(1.2) \quad \xi^T DA(x, p) \xi \geq \alpha_0 \| \xi \|^2_{L^2}, \quad \forall \xi \in \mathbb{R}^2, \quad \forall (x, p) \in \overline{\Omega} \times \mathbb{R}^2,
\]

where \( DA(x, p) = (\partial A_i / \partial p_j) \) is the \( 2 \times 2 \) Jacobian matrix. The variable \( x \) will normally be omitted in this notation below.

It will be also assumed that for some \( \epsilon, 0 < \epsilon < 1 \), and for each \( f \in H^\epsilon(\Omega) \) there exists a unique solution \( u \in H^{2+\epsilon}(\Omega) \) of (1.1).

Mixed finite element methods have been studied \([9], [2]\) to approximate the solution of a quasilinear second order elliptic problem by means of the Raviart–Thomas method \([11]\). However, it has turned out that attempts at using the Brezzi–Douglas–Marini (BDM) method \([1]\) for the numerical solution of (1.1) by applying the ideas given in \([9], [2]\) are not entirely successful. The reason for this is that error equations couple the scalar variable \( u \)

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and the flux variable (i.e., the one for which mixed methods are designed to approximate well); as a consequence, the errors of the former influence those of the latter. Hence, the error estimates for the flux variable are not optimal since the BDM space uses higher order polynomials for this variable than for the scalar.

The object of this paper is to introduce a mixed formulation to approximate the solution of (1.1). This formulation differs from the traditional mixed formulation [11], [1], [9], [2], [3], [4] due to the introduction of a new variable which acts as a Lagrange multiplier. Based on the formulation and the BDM elements [1], a new family of mixed finite elements is developed. While the new variable is introduced in order to derive optimal error estimates both for the flux variable and for the scalar variable, it will be seen that, since there is no continuity requirement on the approximate space associated with this variable, it can be eliminated element by element by static condensation with virtually no cost and thus the system associated with the formulation considered here reduces to traditional ones. Therefore, the error estimates for the flux are improved without increasing the computational cost.

We shall use the fact in the error analysis below that the restriction of the linear elliptic operator

(1.3) \[ Mw = - \text{div} (D\mathbf{A}(\nabla u)\nabla w), \]

to \( H^2(\Omega) \cap H^1_0(\Omega) \) has a bounded inverse, which follows from [7, Theorem 8.12].

In the next section we shall introduce our mixed formulation of problem (1.1) and show the existence and uniqueness of solution of this formulation. Then, based on the formulation, we shall develop a family of mixed finite elements. In §3, we shall demonstrate the existence and uniqueness of an approximation solution. In §§4–6, we shall carry out an asymptotic error analysis. The results of the analysis are optimal error estimates in \( L^2(\Omega) \), \( L^\infty(\Omega) \), and \( H^{-s}(\Omega) \). In §7, some numerical results are presented to compare the present method with other more standard approaches. Finally, a concluding remark is made in §8. Throughout this paper, vectors will be denoted by bold face.

2. A mixed formulation

To introduce a mixed formulation of (1.1), let

\[ \mathbf{V} = H(\text{div}, \Omega) = \{ \mathbf{\tau} : \mathbf{\tau} \in L^2(\Omega), \text{div} \mathbf{\tau} \in L^2(\Omega) \}, \]

with the usual norm

\[ \| \mathbf{\tau} \|_V^2 = \| \mathbf{\tau} \|_{H(\text{div}, \Omega)}^2 = \sum_{i=1}^2 \| \tau_i \|_0^2 + \| \text{div} \mathbf{\tau} \|_0^2, \]
where \( \tau = (\tau_1, \tau_2) \) and \( \| \cdot \|_0 \) denotes the norm on \( L^2(\Omega) \). Then, we can construct our mixed formulation for (1.1): Find \( (u, \lambda, \sigma) \in L^2(\Omega) \times L^2(\Omega) \times V \) such that

\[
\begin{align*}
(A(\lambda), \mu) + (\sigma, \mu) &= 0, \quad \forall \mu \in L^2(\Omega), \\
(u, \text{div} \, \tau) + (\lambda, \tau) &= 0, \quad \forall \tau \in V,
\end{align*}
\]

(2.1)

\[
(v, \text{div} \, \sigma) = (f, v), \quad \forall v \in L^2(\Omega),
\]

where \((\cdot, \cdot)\) indicates the inner product on \( L^2(\Omega) \) or \( L^2(\Omega) \).

Set now

\[
W = L^2(\Omega) \times L^2(\Omega),
\]

with the usual product norm \( \| u \|_W^2 = \| u \|_0^2 + \| \mu \|_0^2, u = (u, \mu) \in W \), and introduce the form \( A(\cdot, \cdot) : W \times W \to \mathbb{R} \) defined by

\[
A(u, v) = (A(\lambda), \mu), \quad u = (u, \lambda), v = (v, \mu) \in W,
\]

and the continuous bilinear form \( B(\cdot, \cdot) : W \times V \to \mathbb{R} \) by

\[
B(u, \tau) = (u, \text{div} \, \tau) + (\lambda, \tau), \quad u = (u, \lambda) \in W, \quad \tau \in V.
\]

Then, (2.1) may be rewritten in the standard form suitable for its mathematical analysis as follows: Find \( (u, \sigma) \in W \times V \) such that

\[
\begin{align*}
(2.2a) \quad & A(u, v) + B(v, \sigma) = F(v), \quad \forall v \in W, \\
(2.2b) \quad & B(u, \tau) = 0, \quad \forall \tau \in V,
\end{align*}
\]

where the continuous linear form \( F(\cdot) \) on \( W \) is given by

\[
F(v) = f(v), \quad v = (v, \mu) \in W.
\]

In order to analyse (2.2), we define the subspace \( Z \) of \( W \) by

\[
Z = \{ v \in W : B(v, \tau) = 0, \quad \forall \tau \in V \}.
\]

**Lemma 2.1.** Let \( v = (v, \mu) \in W \). Then, \( v \in Z \) if and only if \( v \in H^1_0(\Omega) \) and \( \mu = \nabla v \).

**Proof.** First, let \( v = (v, \mu) \in W \) such that \( v \in H^1_0(\Omega) \) and \( \mu = \nabla v \). Then, for all \( \tau \in V \),

\[
B(v, \tau) = (v, \text{div} \, \tau) + (\mu, \tau)
= -(\nabla v, \tau) + (\mu, \tau)
= 0;
\]
i.e., $v \in Z$.

Next, let $v = (v, \mu) \in Z$. Define $\tau \in V$ with $\tau_1 = \phi \in D(\bar{\Omega})$, the restriction of the functions infinitely differentiable and with compact support in $\mathbb{R}^2$ to $\Omega$, and $\tau_2 = 0$. Then, by the definition of $Z$,

$$(v, \partial \phi / \partial x_1) = - (\mu_1, \phi), \quad \forall \phi \in D(\bar{\Omega}).$$

Since $H^1(\Omega) = \overline{D(\bar{\Omega})}$, the closure of $D(\bar{\Omega})$, this implies that $\mu_1 = \partial v / \partial x_1$. Similarly, $\mu_2 = \partial v / \partial x_2$; consequently, $\mu = \nabla v$. Therefore, by the definition of $Z$ and Green’s formula, we have

$$(v, \tau \cdot n)_{\partial \Omega} = 0, \quad \forall \tau \in V,$$

where $n$ denotes the unit outer normal to $\partial \Omega$. Since $v|_{\partial \Omega} \in H^{1/2}(\partial \Omega)$ and the mapping: $\tau \mapsto \tau \cdot n|_{\partial \Omega}$ defined on $V$ is onto $H^{1/2}(\partial \Omega)$, the equation above implies that $v|_{\partial \Omega} = 0$; i.e., $v \in H^1_0(\Omega)$. This completes the proof.

The following result characterizes the relations between the solutions of (1.1) and (2.2).

**Theorem 2.2.** Suppose that, for some $\alpha \in (0, 1), \partial \Omega \in C^{2,\alpha}$ and $f \in C^\alpha(\bar{\Omega})$. Then, if $u \in H^1_0(\Omega)$ is the solution of (1.1), the pair $(u, \sigma) \in W \times V$ defined by

$$(2.3a) \quad u = (u, \lambda), \quad \lambda = \nabla u,$$

$$(2.3b) \quad \sigma = -A(\nabla u),$$

is a solution of (2.2); conversely, if $(u, \sigma) \in W \times V$ is a solution of (2.2) with $u = (u, \lambda)$, then $u \in H^1_0(\Omega)$ is the solution of (1.1) and $\lambda$ and $\sigma$ satisfy (2.3).

**Proof.** First, let $u \in H^1_0(\Omega)$ be the solution of (1.1) and let $\lambda$ and $\sigma$ be defined by (2.3). Then, it follows from Lemma 2.1 that $u = (u, \lambda) \in Z$, so that (2.2b) is satisfied. Now for each $v \in W$ with $v = (v, \mu)$, by (2.3),

$$A(u, v) + B(v, \sigma) = (v, -\text{div} A(\nabla u)) = (v, f);$$

i.e., (2.2a) holds, and thus $(u, \sigma)$ is a solution of (2.2).

Next, let $(u, \sigma) \in W \times V$ be a solution of (2.2) with $u = (u, \lambda)$. Then, (2.2b) implies that $u \in Z$. Hence, by Lemma 2.1, $u \in H^1_0(\Omega)$ and $\lambda = \nabla u$. Therefore, for all $v \in H^1_0(\Omega)$ and $\mu = \nabla v$, by Lemma 2.1 and (2.2a),

$$A(u, v) = F(v), \quad v = (v, \mu) \in W;$$

i.e.,

$$(A(\nabla u), \nabla v) = (f, v), \quad \forall v \in H^1_0(\Omega).$$
Hence, we see that $u$ is a weak solution in $H^1_0(\Omega)$ of (1.1) and that it thus follows by [8, Theorem 2.1], our assumptions made in the theorem, and (1.2) that $u \in H^{2,q}(\Omega)$ for $1 < q < \infty$. In particular, $u \in H^2(\Omega)$. Then, by the Schauder estimates (see, e.g., [10, Theorem 5.6.3]) we find that $u \in C^{2+\alpha}(\overline{\Omega})$. Thus, the weak solution $u$ of (1.1) is in fact its classical solution and belongs to $C^2(\overline{\Omega})$; therefore, Theorem 1 of [5] implies that $u$ is the unique solution of (1.1) and the proof of the theorem has been finished.

To consider the discrete version of (2.1), we now develop a family of mixed finite elements, which is a straightforward extension of the BDM space on triangles [1]. For $0 < h < 1$, let $T_h = \{T\}$ be a quasi-regular triangulation of $\Omega$ by triangles of characteristic parameter $h$. Boundary triangles are allowed to have one curved edge. For $k \geq 1$, let $P_k(T)$ denote the set of restrictions of polynomials of total degree not greater than $k$ to the set $T$ and let $P_k(T) = (P_k(T))^2$. Then, set

$$L_h = \{v \in L^2(\Omega) : v|_T \in P_{k-1}(T), \forall T \in T_h\},$$

$$L_h = \{\mu \in L^2(\Omega) : \mu|_T \in P_k(T), \forall T \in T_h\},$$

$$V_h = \{v \in V : v|_T \in P_k(T), \forall T \in T_h\}.$$

Recall that $v \in V$ if and only if $\tau \cdot n_e$ is continuous across interior edges $e$, where $n_e$ is the exterior normal to $e$.

We can now state a mixed finite element method for approximating the solution of (1.1): Find $(u_h, \lambda_h, \sigma_h) \in L_h \times L_h \times V_h$ such that

$$\begin{align*}
(A(\lambda_h), \mu) + (\sigma_h, \mu) &= 0, &\forall \mu \in L_h, \\
(u_h, \text{div} \tau) + (\lambda_h, \tau) &= 0, &\forall \tau \in V_h, \\
(v, \text{div} \sigma_h) &= (f, v), &\forall v \in L_h.
\end{align*}$$

(2.4)

As mentioned in the introduction, the purpose of the introduction of the Lagrange multiplier $\lambda_h$ is to improve estimates for the error $\sigma - \sigma_h$. Since there is no continuity requirement on the elements in $L_h$, $\lambda_h$ may be eliminated element by element from this system in a virtually cost-free manner to obtain a traditional system involving only $u_h$ and $\sigma_h$. Therefore, system (2.4) can be solved in a standard way.

3. The existence and uniqueness of discrete solution

We shall in this section demonstrate the existence and uniqueness of solution of system (2.4). The argument below will follow the development described by Milner [9] and the author [2], following Douglas and Roberts [6] for treating mixed methods for linear second order elliptic problems, but will be simpler than that given in [9] and [2].

Let $\Pi_h : H^1(\Omega) \to V_h$ denote the BDM projection [1], which satisfies

$$\begin{align*}
\text{div}(\Pi_h \sigma - \sigma), v) &= 0, &\forall v \in L_h, &\sigma \in H^1(\Omega), \\
\|\Pi_h \sigma - \sigma\|_0 &\leq C\|\sigma\|_{H^r}, &\forall \sigma \in H^r(\Omega), &1 \leq r \leq k + 1,
\end{align*}$$

(3.1)

(3.2)
and let $P_h$ and $P_h$ indicate the $L^2$–projections onto $L_h$ and $L_h$, respectively, so that

\begin{align}
\text{(3.3)} \quad \langle \text{div } \tau, v - P_h v \rangle &= 0, \quad \forall \tau \in V_h, \quad v \in L^2(\Omega), \\
\text{(3.4)} \quad \left\| P_h v - v \right\|_{-s} &\leq C \left\| v \right\|_{H^{r+s}} , \quad \forall v \in H^r(\Omega), \quad 0 \leq r, s \leq k,
\end{align}

and

\begin{align}
\text{(3.5)} \quad \langle \mu - P_h \mu, \tau \rangle &= 0, \quad \forall \tau \in V_h, \quad \mu \in L^2(\Omega), \\
\text{(3.6)} \quad \left\| P_h \mu - \mu \right\|_{-s} &\leq C \left\| \mu \right\|_{H^{r+s}} , \quad \forall \mu \in H^r(\Omega), \quad 0 \leq r, s \leq k + 1.
\end{align}

Subtracting (2.4) from (2.1), we get the error equations

\begin{align}
\text{(3.7)} \quad (A(\lambda) - A(\lambda), \mu) + (\sigma - \sigma_h, \mu) &= 0, \quad \forall \mu \in L_h, \\
(u - u_h, \text{div } \tau) + (\lambda - \lambda_h, \tau) &= 0, \quad \forall \tau \in V_h, \\
(v, \text{div } (\sigma - \sigma_h)) &= 0, \quad \forall v \in L_h.
\end{align}

For any $\nu \in L_h$, we have

\begin{align}
\text{(3.8)} \quad A(\lambda) - A(\nu) &= D\tilde{\lambda}(\nu)(\lambda - \nu),
\end{align}

where $D\tilde{\lambda}(\nu) = (\partial \tilde{A}_i / \partial \lambda_j)(\nu)$ is the $2 \times 2$ Jacobian matrix given by

\begin{align*}
\frac{\partial \tilde{A}_i}{\partial \lambda_1} (\nu) &= \int_0^1 \frac{\partial \tilde{A}_i}{\partial \lambda_1}(\nu_1 + t(\lambda_1 - \nu_1), \nu_2)dt, \quad i = 1, 2, \\
\frac{\partial \tilde{A}_i}{\partial \lambda_2} (\nu) &= \int_0^1 \frac{\partial \tilde{A}_i}{\partial \lambda_2}(\nu_1, \nu_2 + t(\lambda_2 - \nu_2))dt, \quad i = 1, 2,
\end{align*}

where $\lambda = (\lambda_1, \lambda_2)$ and $\nu = (\nu_1, \nu_2)$. Thus, by substituting (3.8) with $\nu = \lambda_h$ into (3.7), we see that

\begin{align}
\text{(3.9)} \quad D\tilde{\lambda}(\lambda_h)(\lambda - \lambda_h), \mu) + (\sigma - \sigma_h, \mu) &= 0, \quad \forall \mu \in L_h, \\
(u - u_h, \text{div } \tau) + (\lambda - \lambda_h, \tau) &= 0, \quad \forall \tau \in V_h, \\
(v, \text{div } (\sigma - \sigma_h)) &= 0, \quad \forall v \in L_h,
\end{align}

so that, by shifting $(u, \lambda, \sigma)$ to $(P_h u, P_h \lambda, P_h \sigma)$ on the left-hand side of (3.9) and using (3.1) and (3.5),

\begin{align}
\text{(3.10)} \quad D\tilde{\lambda}(\lambda_h)(P_h \lambda - \lambda_h), \mu) + (\Pi_h \sigma - \sigma_h, \mu) &= D\tilde{\lambda}(\lambda_h)(P_h \lambda - \lambda) + \Pi_h \sigma - \sigma, \mu), \\
(u - u_h, \text{div } \tau) + (P_h \lambda - \lambda_h, \tau) &= 0, \quad \forall \tau \in V_h, \\
(v, \text{div } (\Pi_h \sigma - \sigma_h)) &= 0, \quad \forall v \in L_h.
\end{align}

Let

\begin{align*}
\Phi : L_h \times L_h \times V_h \rightarrow L_h \times L_h \times V_h,
\end{align*}

be a mapping defined by $\Phi((w, \nu, \rho)) = (x, y, z)$, where $(x, y, z)$ is the solution of the system

6
\begin{align}
(D\tilde{A}(\nu)(P_h\lambda - y),\mu) + (\Pi_h\sigma - z,\mu) &= (D\tilde{A}(\nu)(P_h\lambda - \lambda) + \Pi_h\sigma - \sigma,\mu), \\
\forall \mu &\in L_h, \\
(P_hu - x,\text{div } \tau) + (P_h\lambda - y,\tau) &= 0, \quad \forall \tau \in V_h, \\
(v,\text{div } (\Pi_h\sigma - z)) &= 0, \quad \forall v \in L_h.
\end{align}

Note that (3.11) corresponds to the mixed method for the operator \(N : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)\) given by

\begin{equation}
Nw = -\text{div } (D\tilde{A}(\nu)\nabla w).
\end{equation}

The existence and uniqueness of solution of (3.11) will follow [6], [9], [2] if we can prove that

\begin{equation}
x^T D\tilde{A}(\nu)x \geq \beta \|x\|_{\mathbb{R}^2}^2, \quad \forall x \in \overline{\Omega}, \quad \forall x \in \mathbb{R}^2,
\end{equation}

for some constant \(\beta > 0\). For then, \(N\) will have a bounded inverse mapping \(L^2(\Omega)\) onto \(H^2(\Omega) \cap H^1_0(\Omega)\) (see [7, Theorem 8.12]).

Let

\begin{equation}
\max_{1 \leq i,j,k \leq 2} \left\| \frac{\partial^2 A_i}{\partial p_j \partial p_k} \right\|_{0,\infty} \leq Q,
\end{equation}

for some constant \(Q\). Then, we have

**Lemma 3.1.** Let (3.14) be satisfied and \(\nu\) belong to \(B_1 = \{\mu \in L_h : \|\lambda - \mu\|_{0,\infty} \leq \alpha_0/6Q\}\), where \(\alpha_0\) is defined as in (1.2). Then, (3.13) is satisfied.

**Proof.** For any \(x = (\xi_1,\xi_2) \in \mathbb{R}^2\),

\begin{align}
x^T D\tilde{A}(\nu)x &= \int_0^1 \{x^T D\tilde{A}(\nu_1 + t(\lambda_1 - \nu_1),\nu_2)x \\
&\quad - \left( \frac{\partial A_1}{\partial \lambda_2}(\lambda_1,\nu_2 + t(\lambda_2 - \nu_2)) - \frac{\partial A_1}{\partial \nu_2}(\nu_1 + t(\lambda_1 - \nu_1),\nu_2) \right)\xi_1 \xi_2 \\
&\quad + \left( \frac{\partial A_2}{\partial \lambda_2}(\lambda_1,\nu_2 + t(\lambda_2 - \nu_2)) - \frac{\partial A_2}{\partial \nu_2}(\nu_1 + t(\lambda_1 - \nu_1),\nu_2) \right)\xi_2^2 \} dt.
\end{align}

Thus, by Mean Value Theorem, Cauchy’s inequality, (1.2), and (3.14), we obtain

\begin{equation}
x^T D\tilde{A}(\nu)x \geq \int_0^1 \{\alpha_0 \|x\|_{\mathbb{R}^2}^2 - 3Q\|\lambda - \nu\|_{0,\infty} \|x\|_{\mathbb{R}^2}^2 \} dt \geq \frac{\alpha_0}{2} \|x\|_{\mathbb{R}^2}^2,
\end{equation}

which implies (3.13) with \(\beta = \frac{\alpha_0}{2}\), and the proof of the lemma has been completed.

Note now that the existence of a solution of (2.4) is equivalent to showing that \(\Phi\) has a fixed point. Consequently, the solvability of problem (2.4) will follow from the Brouwer fixed theorem if we can prove that \(\Phi\) maps a ball of \(L_h \times L_h \times V_h\) into itself. For this, let \(L_h = L_h\) with the stronger norm \(\|\mu\|_{L_h} = \|\mu\|_{0,\infty}\).
Theorem 3.2. Let $(3.14)$ be valid and $\delta \leq \alpha_0/12Q$. Then, for sufficiently small $h$, $\Phi$ maps a ball of radius $\delta$ of $L_h \times \mathcal{L}_h \times V_h$ into itself.

To prove this theorem, let us state a duality lemma which can be proved by the argument in [6].

Lemma 3.3. Let $\mathbf{v} \in B_1$, $\xi \in L^2(\Omega), \zeta \in V$ and $g \in L^2(\Omega)$. If $w \in L_h$ satisfies

\[(D\tilde{A}(\mathbf{v})\xi, \mu) + (\zeta, \mu) = (g, \mu), \quad \forall \mu \in L_h,
\]

\[(w, \text{div } \mathbf{r}) + (\xi, \mathbf{r}) = 0, \quad \forall \mathbf{r} \in V_h,
\]

\[(v, \text{div } \zeta) = 0, \quad \forall v \in L_h,
\]

then, for $k \geq 1$,

\[
\|w\|_0 \leq C((\|\xi\|_0 + \|\zeta\|_0 + \|g\|_0)h + \|\text{div } \zeta\|_0 h^{\min(2,k)} + \|g\|_{-1}).
\]

Proof of Theorem 3.2. Let

\[
\|P_h u - w\|_0 \leq \delta, \quad \|\Pi_h \sigma - \rho\|_V \leq \delta, \quad \|P_h \lambda - \nu\|_{0,\infty} \leq \delta.
\]

Note that, since $\delta \leq \alpha_0/12Q$, $\mathbf{v} \in B_1$ for sufficiently small $h$. Now, taking $\mu = P_h \lambda - \mathbf{y}$ in (3.11a) and $\mathbf{r} = \Pi_h \sigma - \mathbf{z}$ in (3.11b) and using (3.11c), we see that

\[
\|P_h \lambda - \mathbf{y}\|_0 \leq C\{\|P_h \lambda - \lambda\|_0 + \|\Pi_h \sigma - \sigma\|_0\}
\]

\[
\leq C h^{1+\epsilon}\{\|\lambda\|_{1+\epsilon} + \|\sigma\|_{1+\epsilon}\},
\]

by (3.2) and (6). Also, taking $\mu = \Pi_h \sigma - \mathbf{z}$ in (3.11a) and using (3.16), (3.2), and (3.6), we get

\[
\|\Pi_h \sigma - \mathbf{z}\|_0 \leq C\{\|P_h \lambda - \mathbf{y}\|_0 + \|P_h \lambda - \lambda\|_0 + \|\Pi_h \sigma - \sigma\|_0\}
\]

\[
\leq C h^{1+\epsilon}\{\|\lambda\|_{1+\epsilon} + \|\sigma\|_{1+\epsilon}\}.
\]

We now apply Lemma 3.3 to (3.11) with

\[
g = D\tilde{A}(\mathbf{v})(P_h \lambda - \lambda) + \Pi_h \sigma - \sigma,
\]

and use (3.11c) which implies that

\[
\text{div } (\Pi_h \sigma - \mathbf{z}) = 0,
\]

to obtain

\[
\|P_h u - x\|_0 \leq C\{\|P_h \lambda - \mathbf{y}\|_0 + \|\Pi_h \sigma - \mathbf{z}\|_0\}h + \|g\|_0
\]

\[
\leq C h^{1+\epsilon}\{\|\lambda\|_{1+\epsilon} + \|\sigma\|_{1+\epsilon}\}.
\]

The quasi-regularity of $T_h$ implies that

\[
\|P_h \lambda - \mathbf{y}\|_{0,\infty} \leq C h^{-1}\|P_h \lambda - \mathbf{y}\|_0
\]

\[
\leq C h^{-1} h^{1+\epsilon}\{\|\lambda\|_{1+\epsilon} + \|\sigma\|_{1+\epsilon}\},
\]

by (3.16). Now, let $h < (\delta/(C(\|u\|_{2+\epsilon} + \|\sigma\|_{1+\epsilon})))^{1/\epsilon}$; the theorem is then proved since (3.17–20) imply that $\|P_h u - x\|_0 \leq \delta, \|\Pi_h \sigma - \mathbf{z}\|_V \leq \delta$, and $\|P_h \lambda - \mathbf{y}\|_{0,\infty} \leq \delta$.

We now state a uniqueness result.
Theorem 3.4. Assume that (3.14) is valid. Then, there is a unique solution of (2.4) in the ball $B_1$.

Proof. Let $(u_h^i, \lambda_h^i, \sigma_h^i) \in L_h \times L_h \times V_h$, $i = 1, 2$, be two solutions of (2.4) in $B_1$. Then, it follows from (2.4) that

$$
(A(\lambda_h^1) - A(\lambda_h^2), \mu) + (\sigma_h^1 - \sigma_h^2, \mu) = 0, \quad \forall \mu \in L_h,
$$

$$
(u_h^1 - u_h^2, \text{div} \, \tau) + (\lambda_h^1 - \lambda_h^2, \tau) = 0, \quad \forall \tau \in V_h,
$$

$$
(v, \text{div} (\sigma_h^1 - \sigma_h^2)) = 0, \quad \forall v \in L_h,
$$

so that

$$
(\Delta A'(c)(\lambda_h^1 - \lambda_h^2), \mu) + (\sigma_h^1 - \sigma_h^2, \mu) = 0, \quad \forall \mu \in L_h,
$$

$$
(u_h^1 - u_h^2, \text{div} \, \tau) + (\lambda_h^1 - \lambda_h^2, \tau) = 0, \quad \forall \tau \in V_h,
$$

$$
(v, \text{div} (\sigma_h^1 - \sigma_h^2)) = 0, \quad \forall v \in L_h,
$$

where

$$
\Delta A'(c) = \begin{pmatrix}
\frac{\partial A_1}{\partial \lambda_1}(c_1) & \frac{\partial A_1}{\partial \lambda_2}(c_1) \\
\frac{\partial A_2}{\partial \lambda_1}(c_2) & \frac{\partial A_2}{\partial \lambda_2}(c_2)
\end{pmatrix},
$$

c_1 = \lambda_h^1 + t(\lambda_h^2 - \lambda_h^1), \text{ and } c_2 = \lambda_h^1 + \theta(\lambda_h^2 - \lambda_h^1), \text{ for some } t \text{ and } \theta : 0 \leq t, \theta \leq 1. \text{ Thus, for any } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,

$$
\xi^T \Delta A'(c) \xi \geq \xi^T \Delta A(c_1) \xi + \left( \frac{\partial A_2}{\partial \lambda_1}(c_2) - \frac{\partial A_2}{\partial \lambda_1}(c_1) \right) \xi_1 \xi_2
$$

$$
+ \left( \frac{\partial A_2}{\partial \lambda_2}(c_2) - \frac{\partial A_2}{\partial \lambda_2}(c_1) \right) \xi_2^2.
$$

This, together with (1.2) and (3.14), implies that

$$
\xi^T \Delta A'(c) \xi \geq \alpha_0 \|\xi\|_{\mathbb{R}^2}^2 - \frac{3\sqrt{2}}{2} Q \|\xi\|_{\mathbb{R}^2}^2 \|\lambda_h^1 - \lambda_h^2\|_{0, \infty}
$$

$$
\geq \left(1 - \frac{\sqrt{2}}{2}\right) \alpha_0 \|\xi\|_{\mathbb{R}^2}^2.
$$

Now, note that, by (3.22c),

$$
\text{div} (\sigma_h^1 - \sigma_h^2) = 0,
$$

and that, taking $\mu = \lambda_h^1 - \lambda_h^2$ in (3.22a) and $\tau = \sigma_h^1 - \sigma_h^2$ in (3.22b) and using (3.24).

$$
(\Delta A'(c)(\lambda_h^1 - \lambda_h^2), \lambda_h^1 - \lambda_h^2) = 0.
$$
Hence, by (3.23) and (3.25), we get that \( \lambda_h^1 = \lambda_h^2 \); so, taking \( \mu = \sigma_h^1 - \sigma_h^2 \) in (3.22a) implies that \( \sigma_h^1 = \sigma_h^2 \). Finally, (3.22b) leads to the equation

\[
(u_h^1 - u_h^2, \text{div } \tau) = 0, \quad \forall \tau \in V_h,
\]

which immediately means that \( u_h^1 = u_h^2 \) since \( \text{div } V_h = L_h \), and the proof of the theorem is complete.

4. \( L^2 \)-error estimates

In this section and the next two sections, we shall carry out an asymptotic error analysis and show that the differences \( u - u_h, \lambda - \lambda_h, \) and \( \sigma - \sigma_h \) are of optimal order in \( L^2(\Omega), L^\infty(\Omega), \) and \( (H^4(\Omega))^\prime \).

Let

\[
\alpha = \lambda - \lambda_h, \quad \beta = P_h \lambda - \lambda_h, \\
\delta = \sigma - \sigma_h, \quad \epsilon = \Pi_h \sigma - \sigma_h, \\
w = P_h u - u_h,
\]

and rewrite (3.9) as

\[
(4.1a) \quad (D \tilde{A}(\lambda_h) \alpha, \mu) + (d, \mu) = 0, \quad \forall \mu \in L_h, \\
(4.1b) \quad (w, \text{div } \tau) + (\alpha, \tau) = 0, \quad \forall \tau \in V_h, \\
(4.1c) \quad (v, \text{div } d) = 0, \quad \forall v \in L_h,
\]

by (3.3).

First, observe that, by (3.1) and (4.1c),

\[
(v, \text{div } e) = (v, \text{div } d) = 0, \quad \forall v \in L_h,
\]

so that \( \text{div } e = 0 \). Hence, by (3.1) and (3.3)–(3.4),

\[
(4.2) \quad \| \text{div } d \|_0 = \| \text{div } (\sigma - \Pi_h \sigma) \|_0 \leq C \| \text{div } \sigma \|_{H^r}^r, \quad 1 \leq r \leq k.
\]

Next, by (4.1) and (3.5), we have

\[
(D \tilde{A}(\lambda_h) \beta, \beta) = (D \tilde{A}(\lambda_h)(P_h \lambda - \lambda), \beta) + (D \tilde{A}(\lambda_h) \alpha, \beta) \\
= (D \tilde{A}(\lambda_h)(P_h \lambda - \lambda), \beta) - (d, \beta) \\
= (D \tilde{A}(\lambda_h)(P_h \lambda - \lambda), \beta) - (\sigma - \Pi_h \sigma, \beta) + (w, \text{div } e) \\
= (D \tilde{A}(\lambda_h)(P_h \lambda - \lambda), \beta) - (\sigma - \Pi_h \sigma, \beta),
\]

which leads to

\[
(4.3) \quad \| \beta \|_0 \leq C(\| P_h \lambda - \lambda \|_0 + \| \sigma - \Pi_h \sigma \|_0),
\]
and thus, by (3.2) and (3.6),

\[
\|\alpha\|_0 \leq \|\beta\|_0 + \|P_h\lambda - \lambda\|_0 \\
\leq C(\|\lambda\|_r + \|\sigma\|_r)h^r, \quad 1 \leq r \leq k + 1.
\]

Also, by (4.1), we see that

\[
(e, e) = (\Pi_h\sigma - \sigma, e) - (D\hat{A}(\lambda_h)\alpha, e);
\]

consequently, by (3.2) and (3.4),

\[
\|d\|_0 \leq \|e\|_0 + \|\Pi_h\sigma - \sigma\|_0 \\
\leq C(\|\Pi_h\sigma - \sigma\|_0 + \|\sigma\|_0) \\
\leq C(\|\lambda\|_r + \|\sigma\|_r)h^r, \quad 1 \leq r \leq k + 1.
\]

Finally, we apply Lemma 3.3 to (4.1) to get

\[
\|w\|_0 \leq C(\|\alpha\|_0 + \|d\|_0)h + \|\text{div } d\|_0 h^{\min(2,k)}.
\]

Our result can be summarized as follows.

**Theorem 4.1.** Let \((u, \lambda, \sigma)\) and \((u_h, \lambda_h, \sigma_h) \in L_h \times L_h \times V_h\) be the solutions of (2.1) and (2.4), respectively. Then, for sufficiently small \(h\),

\[
\|u - u_h\|_0 \leq C\left\{ \begin{array}{ll}
(\|u\|_1 + \|\sigma\|_1)h, & k = 1, \\
(\|u\|_r + \|\sigma\|_{r-1})h^r, & 2 \leq r \leq k, \quad k > 1,
\end{array} \right.
\]

\[
\|\lambda - \lambda_h\|_0 \leq C(\|u\|_{r+1} + \|\sigma\|_r)h^r, \quad 1 \leq r \leq k + 1,
\]

\[
\|\sigma - \sigma_h\|_0 \leq C(\|u\|_{r+1} + \|\sigma\|_r)h^r, \quad 1 \leq r \leq k + 1,
\]

\[
\|\text{div } (\sigma - \sigma_h)\|_0 \leq C\|\text{div } \sigma\|_r h^r = C\|f\|_r h^r, \quad 1 \leq r \leq k,
\]

\[
\|u_h - P_h u\|_0 \leq C\|f\|_k h^{\min(k+2.2k)}.
\]

Note that Theorem 4.1 shows that \(\{(u_h, \lambda_h, \sigma_h)\}\) converges in \(L^2(\Omega) \times L^2(\Omega) \times V\) to \((u, \lambda, \sigma)\) as \(h \rightarrow 0\) both at an optimal rate (for any \(h\)) and with minimal smoothness requirements on the solution of (1.1).

5. **Error estimate in \(L^\infty(\Omega)\)**

Note that, by (4.6) and (4.8)-(4.10),

\[
\|w\|_0 \leq C\left\{ \begin{array}{ll}
(\|u\|_2 + \|\sigma\|_2)h^2, & k = 1, \\
(\|u\|_{r+1} + \|\sigma\|_r)h^{r+1}, & 2 \leq r \leq k + 1, \quad k > 1.
\end{array} \right.
\]
The quasi-regularity of $T_h$ implies that

\[
\|w\|_{0,\infty} \leq C h^{-1} \|w\|_0 \\
\|u\|_2 + \|\sigma\|_2^k, \quad k = 1, \\
(\|u\|_{r+1} + \|\sigma\|_r)^k r, \quad 2 \leq r \leq k + 1, k > 1.
\]

Thus, for $k \geq 1$,

\[
\|u - u_h\|_{0,\infty} \leq \|u - P_h u\|_{0,\infty} + \|w\|_{0,\infty} \\
(\|u\|_{1,\infty} + \|u\|_2 + \|\sigma\|_2)^k, \quad k = 1, \\
(\|u\|_{r,\infty} + \|u\|_{r+1} + \|\sigma\|_r)^k r, \quad 2 \leq r \leq k, k > 1.
\]

6. Error estimates in $(H^s(\Omega))^t$

We shall state a duality lemma for $w$ in a negative norm. Toward that end, let us rewrite (4.1) as

\begin{align*}
(6.1a) & \quad (D(A)(\lambda)\alpha,\mu) + (d,\mu) = ([DA(\lambda) - DA(\lambda_h)]\alpha,\mu), \quad \forall \mu \in L_h, \\
(6.1b) & \quad (w, \text{div } \tau) + (\alpha,\tau) = 0, \quad \forall \tau \in V_h, \\
(6.1c) & \quad (v, \text{div } d) = 0, \quad \forall v \in L_h.
\end{align*}

Observe that (6.1) corresponds to the mixed method for the operator $M : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$ defined by

\[M w = - \text{div } (DA(\lambda)\nabla w).\]

We shall assume that $A(x,p)$ is smooth enough that the Dirichlet problem

\[M \varphi = \psi \quad \text{in } \Omega, \]
\[\varphi = 0 \quad \text{on } \partial \Omega,
\]

has the elliptic regularity result

\[\|\varphi\|_{s+2} \leq C \|\psi\|_s,
\]

if $\psi \in H^s(\Omega)$. Then, the next result can be proved as in [6].

**Lemma 6.1.** Let $\xi \in L^2(\Omega), \zeta \in V$, and $g \in L^2(\Omega)$. If $w \in L_h$ satisfies

\begin{align*}
(DA(\lambda)\xi,\mu) + (\zeta,\mu) = (g,\mu), \quad \forall \mu \in L_h, \\
(w, \text{div } \tau) + (\xi,\tau) = 0, \quad \forall \tau \in V_h, \\
(v, \text{div } \zeta) = 0, \quad \forall v \in L_h.
\end{align*}
then, for $s \geq 0$,

\[
\| w \|_{-s} \leq C(\| \xi \|_0 + \| \zeta \|_0 + \| g \|_0) h^{\min(s+1,k+1)}
+ \| \text{div } \zeta \|_0 h^{\min(s+2,k)} + \| g \|_{-s-1}).
\]

We now apply this lemma to (6.1) with

\[
g = [DA(\lambda) - D\tilde{A}(\lambda_h)]\alpha,
\]

to get

\[
\| w \|_{-s} \leq C(\| \alpha \|_0 + \| d \|_0 + \| DA(\lambda) - D\tilde{A}(\lambda_h) \|_{0,\infty} \times
\]

\[
\| \alpha \|_0 h^{\min(s+1,k+1)} + \| \text{div } d \|_0 h^{\min(s+2,k)}
+ \| DA(\lambda) - D\tilde{A}(\lambda_h) \|_{0,\infty} \| \alpha \|_{-s-1}).
\]

To estimate $\| DA(\lambda) - D\tilde{A}(\lambda_h) \|_{0,\infty}$, note that

\[
\frac{\partial A_1}{\partial \lambda_1}(\lambda) - \frac{\partial \tilde{A}_1}{\partial \lambda_1}(\lambda_h) = \int_0^1 \left[ \frac{\partial^2 A_1}{\partial \lambda_1^2}(\lambda)(1-t)(\lambda_1 - \nu_1)
+ \frac{\partial^2 A_1}{\partial \lambda_2 \partial \lambda_1}(\lambda_2 - \nu_2) \right] dt,
\]

where $\lambda$ is some convex combination of $\lambda$ and $\lambda_h$, and similar expressions hold for $\frac{\partial A_1}{\partial \lambda_2}(\lambda) - \frac{\partial \tilde{A}_1}{\partial \lambda_2}(\lambda_h), \frac{\partial A_2}{\partial \lambda_1}(\lambda) - \frac{\partial \tilde{A}_2}{\partial \lambda_1}(\lambda_h), i = 1, 2$. Consequently, it follows from (3.14) that there is a constant $C$ such that

\[
\| DA(\lambda) - D\tilde{A}(\lambda_h) \|_{0,\infty} \leq C \| \lambda - \lambda_h \|_{0,\infty}.
\]

By (4.3), (3.2), and (3.6), we find that

\[
\| P_h \lambda - \lambda_h \|_0 \leq C(\| P_h \lambda - \lambda \|_0 + \| \sigma - \Pi_h \sigma \|_0)
\leq C h^{1+\epsilon/2}(\| \lambda \|_{1+\epsilon/2} + \| \sigma \|_{1+\epsilon/2}),
\]

so that, by the quasi-regularity of $T_h$,

\[
\| P_h \lambda - \lambda_h \|_{0,\infty} \leq C h^{-1} \| P_h \lambda - \lambda_h \|_0
\leq C h^{\epsilon/2}(\| \lambda \|_{1+\epsilon/2} + \| \sigma \|_{1+\epsilon/2}).
\]

The Sobolev imbedding theorem [7] implies that

\[
H^{1+\epsilon}(\Omega) \subset W^{\epsilon/2,\infty}(\Omega).
\]
Using (6.5)–(6.7) and (3.6), we obtain
\[
\|DA(\lambda) - D\lambda_h\|_{0,\infty} \leq C(\|P_h\lambda - \lambda\|_{0,\infty} + \|P_h\lambda - \lambda_h\|_{0,\infty})
\]
\[
\leq C(h^{\ell/2}\|\lambda\|_{\ell/2,\infty} + h^{\ell/2}(\|\lambda\|_{1+\epsilon/2} + \||\sigma|\|_{1+\epsilon/2}))
\]
\[
\leq Ch^{\ell/2}(\|u\|_{2+\epsilon} + \||\sigma|\|_{1+\epsilon/2}).
\]

Let now $\varphi \in H^s(\Omega)$; then, by (6.1b–c), and (3.1),
\[
(\alpha,\varphi) = (\alpha,\varphi - \Pi_h\varphi) + (\alpha,\Pi_h\varphi)
\]
\[
= (\alpha,\varphi - \Pi_h\varphi) - (w, \text{div} \varphi),
\]
and so, by (3.2),
\[
\|\alpha\|_{-s} \leq C(\|\alpha\|_0 h^{\min(s,k+1)} + \|w\|_{-s+1}).
\]
Hence, it follows from (6.4), (6.8), and (6.9) that there exists a constant, depending on $\|u\|_{2+\epsilon}$ and $\||\sigma|\|_{1+\epsilon/2}$, such that, for sufficiently small $h$,
\[
\|w\|_{-s} \leq C\{(\|\alpha\|_0 + \|d\|_0) h^{\min(s+1,k+1)} + \|\text{div} d\|_0 h^{\min(s+2,k)}\}.
\]

Again, let $\varphi \in H^s(\Omega)$ and use (6.1) to get
\[
(d,\varphi) = (d,\varphi - P_h\varphi) + (d, P_h\varphi)
\]
\[
= (d,\varphi - P_h\varphi) - ([DA(\lambda) - D\lambda_h]\alpha,\varphi - P_h\varphi)
\]
\[
+ ([DA(\lambda) - D\lambda_h]\alpha,\varphi) + (DA(\lambda)\alpha,\varphi - P_h\varphi)
\]
\[
- (\alpha, DA(\lambda)\varphi - \Pi_h(DA(\lambda)\varphi)) + (w, \text{div} (DA(\lambda)\varphi)).
\]
Therefore, we get (with $C$ depending on $\|u\|_{2+\epsilon}$ and $\||\sigma|\|_{1+\epsilon/2}$)
\[
\|d\|_{-s} \leq C(\|\alpha\|_0 + \|d\|_0) h^{\min(s,k+1)} + \|\alpha\|_{-s} + \|w\|_{-s+1}),
\]
by (3.2), (3.6), and (6.8).

We have proved the following theorem.

**Theorem 6.2.** There exists a constant independent of $h$, depending on $\|u\|_{2+\epsilon}$ and $\||\sigma|\|_{1+\epsilon/2}$, such that, for sufficiently small $h$,
\[
\|u - u_h\|_{-s} \leq C(\|u\|_r + \||\sigma|\|_{r-1}) h^{r+s}, \quad 2 \leq r, s \leq k,
\]
\[
\|\lambda - \lambda_h\|_{-s} \leq C(\|u\|_{r+1} + \||\sigma|\|_r) h^{r+s}, \quad 1 \leq r, s \leq k + 1,
\]
\[
\|\sigma - \sigma_h\|_{-s} \leq C(\|u\|_{r+1} + \||\sigma|\|_r) h^{r+s}, \quad 1 \leq r, s \leq k + 1,
\]
\[
\|\text{div} (\sigma - \sigma_h)\|_{-s} \leq C\|\text{div} \sigma\|_r h^{r+s}, \quad 0 \leq r, s \leq k,
\]
\[
\|u_h - P_hu\|_0 \leq C(\|u\|_{k+2} + \||\sigma|\|_{k+1}) h^{\min(k+2,2k)}.
\]
Proof. (6.12)–(6.14) and (6.16) follow immediately from (6.10), (6.9), (3.4), (4.8)–(4.10), and (6.11). In order to derive (6.15), let \( \varphi \in H^s(\Omega) \); then, by (6.1c),
\[
(\text{div } d, \varphi) = (\text{div } d, \varphi - v), \quad \forall \ v \in L_h,
\]
which together with (4.10) implies (6.15), and the proof of the theorem has been completed.

7. Numerical results

In this section two numerical examples are presented to make a comparison of the present approach with the standard one and to justify the error bounds obtained in the last few sections.

Example 1. In the first example we shall apply the above method with \( k = 1 \) to solve the following model problem on a \( 10 \times 10 \) mesh over a uniform triangular decomposition of the unit square:
\[
- \text{div } A(\nabla u) = f(x_1, x_2), \quad (x_1, x_2) \in \Omega = (0, 1)^2,
\]
\[
u = 0, \quad (x_1, x_2) \in \partial \Omega,
\]
where
\[
A(p) = (p_1, 3p_2/2 - \sin(2p_2)/4),
\]
\[
f(x_1, x_2) = 2(x_2 - x_1^2) + (x_1 - x_1^2)(3 - \cos(2(x_1 - x_1^2)(1 - 2x_2))),
\]
with \( p = (p_1, p_2) \). It is easily verified that
\[
D A(p) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + \sin^2 p_2 \end{pmatrix},
\]
for all \( p \in \mathbb{R}^2 \), so that (1.2) holds with \( \alpha_0 = 1 \). Moreover, it can be shown [7] that this problem has a unique solution.

| x     | 0.1000 | 0.2000 | 0.3000 | 0.4000 | 0.5000 | 0.6000 | 0.7000 | 0.8000 | 0.9000 
<table>
<thead>
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<td>0.0525</td>
<td>0.0600</td>
<td>0.0625</td>
<td>0.0600</td>
<td>0.0525</td>
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<tr>
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<td>0.0326</td>
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<td>0.0631</td>
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</tr>
<tr>
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<td>0.0318</td>
<td>0.0492</td>
<td>0.0625</td>
<td>0.0724</td>
<td>0.0625</td>
<td>0.0492</td>
<td>0.0318</td>
<td>0.0190</td>
</tr>
</tbody>
</table>

| x     | 0.1000 | 0.2000 | 0.3000 | 0.4000 | 0.5000 | 0.6000 | 0.7000 | 0.8000 | 0.9000 
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<td>-0.3075</td>
</tr>
<tr>
<td>SA</td>
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<td>-0.3027</td>
<td>-0.3901</td>
</tr>
</tbody>
</table>

15
Tables 1 and 2 show the exact solutions (ES) and numerical solutions of the scalar and the first component of the flux variable at $x_2 = 0.5$ and various points of $x_1$, respectively, to the above problem using the present method (PM) and the standard approach (SA) given in [9], [2], [3]. Evidently, from the tables, we observe that the PM produces the approximate solutions of the flux more accurately than the SA, while both methods have the same accuracy for the scalar.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.1000</th>
<th>0.2000</th>
<th>0.3000</th>
<th>0.4000</th>
<th>0.5000</th>
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<th>0.7000</th>
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<td>PMf</td>
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<td>-0.2592</td>
</tr>
</tbody>
</table>

Table 3 displays the numerical solutions of the scalar PMs and the first component of the flux PMf at the same points as the above to the same problem over a finer mesh $20 \times 20$. It follows from the table that the error estimates given in §5 are observed.

**Example 2.** In this example we shall consider the same problem as in Example 1 with

\[ u = \begin{cases} 
0, & x_1 = 0, \ 0 \leq x_2 \leq 1, \\
0, & x_2 = 0, \ 0 \leq x_1 \leq 1, \\
x_2, & x_1 = 1, \ 0 \leq x_2 \leq 1, \\
x_1^2, & x_2 = 1, \ 0 \leq x_1 \leq 1,
\end{cases} \]

and

\[ A(p) = \left(3p_1/2 - \sin(2p_1)/4, p_2\right), \]

\[ f(x_1, x_2) = -x_2(3 - \cos(4x_1x_2)). \]

In the present case, we have

\[ DA(p) = \begin{pmatrix} 1 + \sin^2 p_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \forall \mathbf{p} \in \mathbb{R}^2, \]

so that (1.2) is again true with $\alpha_0 = 1$.

Again, the mixed formulation (2.4) over the $20 \times 20$ mesh with $k = 1$ is applied. The surface of the solution is plotted in Figure 1; the exact and numerical solutions of the scalar and the second component of the flux at the cut $x_2 = 0.5$ are displayed in Figures 2 and 3. From these pictures, the same conclusions as in the first example can be easily reached.
8. A concluding remark

A new mixed formulation to approximate the solution of a nonlinear elliptic problem is developed; existence and uniqueness of the mixed formulation as well as of its discretization based on the BDM mixed elements are proven. Theoretically and experimentally, it is shown that optimal error estimates are obtained. We end this paper with a remark that the results of sections 3-6 can be obtained in the same way for the BDM spaces [1] over rectangles.
FIG. 3. The ‘-’, ‘- -’, and ‘- × -’ denote ES, PM, and SA.

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<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1011</td>
<td>E.G. Kalnins, Willard Miller, Jr. and Sanchita Mukherjee</td>
<td>Models of q-algebra representations:</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Matrix elements of $U_q(sl_2)$</td>
</tr>
<tr>
<td>1012</td>
<td>Zhangxin Chen and Bernardo Cockburn</td>
<td>Error estimates for a finite element method for the</td>
</tr>
<tr>
<td></td>
<td></td>
<td>drift-diffusion semiconductor device equations</td>
</tr>
<tr>
<td>1013</td>
<td>Chaocheng Huang</td>
<td>Drying of gelatin asymptotically in photographic film</td>
</tr>
<tr>
<td>1014</td>
<td>Richard E. Ewing and Hong Wang</td>
<td>Eulerian-Lagrangian localized adjoint methods for reactive transport</td>
</tr>
<tr>
<td></td>
<td></td>
<td>in groundwater</td>
</tr>
<tr>
<td>1015</td>
<td>Bing-Yu Zhang</td>
<td>Taylor series expansion for solutions of the Korteweg-de Vries equation with respect to their initial values</td>
</tr>
<tr>
<td>1016</td>
<td>Kenneth R. Driesssel</td>
<td>Some remarks on the geometry of some surfaces of matrices associated with Toeplitz eigenproblems</td>
</tr>
<tr>
<td>1017</td>
<td>C.J. Van Duijn and Peter Knabner</td>
<td>Flow and reactive transport in porous media induced by well injection: Similarity solution</td>
</tr>
<tr>
<td>1018</td>
<td>Wasin So, Rank one perturbation and its application to the Laplacian spectrum of a graph</td>
<td></td>
</tr>
<tr>
<td>1019</td>
<td>G. Baccarani, F. Odeh, A. Gnudi and D. Ventura</td>
<td>A critical review of the fundamental semiconductor equations</td>
</tr>
<tr>
<td>1020</td>
<td>T.R. Hoffend Jr.</td>
<td>Magnetostatic interactions for certain types of stacked, cylindrically symmetric magnetic particles</td>
</tr>
<tr>
<td>1021</td>
<td>IMA Summer Program for Graduate Students</td>
<td>Mathematical Modeling</td>
</tr>
<tr>
<td>1022</td>
<td>Wayne Barrett, Charles R. Johnson, and Pablo Tarazaga</td>
<td>The real positive definite completion problem for a simple cycle</td>
</tr>
<tr>
<td>1023</td>
<td>Charles A. McCarthy</td>
<td>Fourth order accuracy for a cubic spline collocation method</td>
</tr>
<tr>
<td>1024</td>
<td>Martin Hanke, James Nagy, and Robert Plemons</td>
<td>Preconditioned iterative regularization for Ill-posed problems</td>
</tr>
<tr>
<td>1025</td>
<td>John R. Gilbert, Esmond G. Ng, and Barry W. Peyton</td>
<td>An efficient algorithm to compute row and column counts for sparse Cholesky factorization</td>
</tr>
<tr>
<td>1026</td>
<td>Xinfu Chen</td>
<td>Existence and regularity of solutions of a nonlinear nonuniformly elliptic system arising from a thermistor problem</td>
</tr>
<tr>
<td>1027</td>
<td>Xinfu Chen and Weiqing Xie</td>
<td>Discontinuous solutions of steady state, viscous compressible Navier-Stokes equations</td>
</tr>
<tr>
<td>1028</td>
<td>E.G. Kalnins, Willard Miller, Jr., and Sanchita Mukherjee</td>
<td>Models of q-algebra representations:</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Matrix elements of the q-oscillator algebra</td>
</tr>
<tr>
<td>1029</td>
<td>W. Miller, Jr. and Lee A. Rubel</td>
<td>Functional separation of variables for Laplace equations in two dimensions</td>
</tr>
<tr>
<td>1030</td>
<td>I. Gohberg and I. Koltracht</td>
<td>Structured condition numbers for linear matrix structures</td>
</tr>
<tr>
<td>1031</td>
<td>Xinfu Chen</td>
<td>Hele-Shaw problem and area preserved curve shortening motion</td>
</tr>
<tr>
<td>1032</td>
<td>Zhangxin Chen and Jim Douglas, Jr.</td>
<td>Modelling of compositional flow in naturally fractured reservoirs</td>
</tr>
<tr>
<td>1033</td>
<td>Harald K. Wimmer</td>
<td>On the existence of a least and negative-semidefinite solution of the discrete-time algebraic Riccati equation</td>
</tr>
<tr>
<td>1034</td>
<td>Harald K. Wimmer</td>
<td>Monotonicity and parametrization results for continuous-time algebraic Riccati equations and Riccati inequalities</td>
</tr>
<tr>
<td>1035</td>
<td>Bart De Moor, Peter Van Overschee, and Geert Schelfhout</td>
<td>$H_2$ model reduction for SISO systems</td>
</tr>
<tr>
<td>1036</td>
<td>Bart De Moor</td>
<td>Structured total least squares and $L_2$ approximation problems</td>
</tr>
<tr>
<td>1037</td>
<td>Chijan Lim</td>
<td>Nonexistence of Lyapunov functions and the instability of the Von Karman vortex streets</td>
</tr>
<tr>
<td>1038</td>
<td>David C. Dobson and Fadil Santosana</td>
<td>Resolution and stability analysis of an inverse problem in electrical impedance tomography – dependence on the input current patterns</td>
</tr>
<tr>
<td>1039</td>
<td>C.N. Dawson, C.J. van Duijn, and M.F. Wheeler</td>
<td>Characteristic-Galerkin methods for contaminant transport with non-equilibrium adsorption kinetics</td>
</tr>
<tr>
<td>1040</td>
<td>Bing-Yu Zhang</td>
<td>Analyticity of solutions of the generalized Korteweg-de Vries equation with respect to their initial values</td>
</tr>
<tr>
<td>1041</td>
<td>Neerchal K. Nagaraj and Wayne A. Fuller</td>
<td>Least squares estimation of the linear model with autoregressive errors</td>
</tr>
<tr>
<td>1042</td>
<td>H.J. Sussmann &amp; W. Liu</td>
<td>A characterization of continuous dependence of trajectories with respect to the input for control-affine systems</td>
</tr>
<tr>
<td>1043</td>
<td>Karen Rudie &amp; W. Murray Wonham</td>
<td>Protocol verification using discrete-event systems</td>
</tr>
<tr>
<td>1044</td>
<td>Rohan Abeyaratne &amp; James K. Knowles</td>
<td>Nucleation, kinetics and admissibility criteria for propagating phase boundaries</td>
</tr>
<tr>
<td>1045</td>
<td>Gang Bao &amp; William W. Symes</td>
<td>Computation of pseudo-differential operators</td>
</tr>
<tr>
<td>1046</td>
<td>Srdjan Stojeanovic</td>
<td>Nonsmooth analysis and shape optimization in flow problem</td>
</tr>
<tr>
<td>1047</td>
<td>Miroslav Tuma</td>
<td>Row ordering in sparse QR decomposition</td>
</tr>
</tbody>
</table>
Onur Toker & Hitay Özbay, On the computation of suboptimal $H^\infty$ controllers for unstable infinite dimensional systems

Hitay Özbay, $H^\infty$ optimal controller design for a class of distributed parameter systems

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H.S. Dumas, F. Golse, and P. Lochak, Multiphase averaging for generalized flows on manifolds

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