NUMERICAL STUDY OF
A SINGULAR SYSTEM OF CONSERVATION LAWS
ARISING IN ENHANCED OIL RESERVOIRS

By

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NUMERICAL STUDY OF
A SINGULAR SYSTEM OF CONSERVATION LAWS
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Abstract. We present a numerical study of the structure of elementary nonlinear waves and their qualitative behavior, for degenerate hyperbolic system of equations that describe the process of polymer flooding of an oil reservoir.

1. Introduction. We consider the physico-chemical flooding model of a horizontal porous medium describing two phase flow of oil and water, where the water contains additive chemicals (polymers) to increase the oil recovery, which has been considered by many authors [1],[2],[5]. The mass conservation equations modeling the one-dimensional displacement of oil by polymer are:

\[ \phi \partial_t s + U \partial_x F(s,c) + \epsilon \partial_x (\psi \partial_x P_c(s,c)) = 0 \]
\[ \partial_t (\phi sc + A(c)) + U \partial_x (cF(s,c)) + \epsilon \partial_x (c\psi \partial_x P_c(s,c)) = \nu \partial_x (D \partial_x c) \]

where \( s = \) saturation of water, \( (1 - s) = \) saturation of oil, \( c = \) concentration of polymer,

\[ F(s,c) = \frac{f_1(s)}{f_1(s) + \frac{\mu_1(c)}{\mu_2(c)} f_2(s)} \]

fractional flow rate of water, \( f_1, f_2 \) relative phase permeabilities, \( \mu_1, \mu_2 \) phase viscosities, \( \phi(x) \) permeability of the medium, \( \psi = \frac{\mu_2}{\mu_2} F, P_c(s,c) = P_2(s,c) - P_1(s,c) \) capillary pressure, \( A(c) = \) amount of polymer sorbed by the porous medium, \( U = u_1 + u_2 \) characteristic velocity satisfying Darcy's law

\[ u_i = -\left( \frac{\phi f_1(s)}{\mu_1(c)} \right) \partial_x P_c(s,c). \]

The diffusion coefficient \( D \) comes from Fick's law of molecular diffusion, \( \epsilon \) is the parameter describing the rate of capillary pressure \( P_c \) to the change in the total hydrodynamic pressure across the reservoir, \( \nu \) is the parameter of convection of the diffusive flow through the reservoir. However, here we study an equivalent but simpler system of equations. We

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assume that $\phi = U = 1$, $\nu = A(c) = 0$, and the capillary effect is restricted to a constant positive value. This will lead to the following viscous system of equations

$$\begin{align*}
\partial_t s + \partial_x F &= \epsilon s_{xx} \\
\partial_t (sc) + \partial_x (cF) &= \epsilon (sc)_{xx}
\end{align*}$$

(1.2)

We consider this system of equations over half plane $-\infty < x < \infty$ and $t > 0$, and the functions $s(x,t), c(x,t)$ are restricted to the square $\Omega = (0,1) \times (0,1)$. The flux function $F(s,c)$ satisfies the following properties:

1. $F(0,c) = 0, F(1,c) = 1, \forall c \in [0,1]$.
2. $\frac{\partial F}{\partial c} < 0, (s,c) \in \Omega$.
3. $\frac{\partial F}{\partial s} > 0, (s,c) \in \Omega$.

In Fig. [1] we plot the surface $F$ over $\Omega$.

![Figure 1 The function F(s,c)](image)

The capillarity parameter $\epsilon$ is usually very small. Thus we can set $\epsilon = 0$ to obtain the corresponding inviscid system of equations

$$\begin{align*}
\partial_t s + \partial_x F &= 0 \\
\partial_t (sc) + \partial_x (cF) &= 0
\end{align*}$$

(1.3)

Following Isaacson and Temple [4], we set $G = F/s$ to be the velocity of the aqueous phase, denote $(s,sc)$ by $U$. Consider the Riemann problem for the quasilinear system of
equations (1.3), that is the initial value with two end states

\[ U_0(x) = \begin{cases} 
    U_l, & x \leq 0, \\
    U_r, & x > 0.
\end{cases} \]

Based on the Lax entropy condition, Isaacson [5] has classified the Riemann type solutions for the system (1.3) with (1.4). The characteristic values for the system (1.3) are \( \lambda_c = G \) and \( \lambda_s = F_s \). These wave speeds are positive for any value of \( s \) and \( c \) in the unit square \( \Omega \). We denote the corresponding elementary waves by \( C \)-waves and \( S \)-waves. The corresponding integral curves are \( G = \text{constant} \), and \( c = \text{constant} \). Thus the \( S \)-waves will be the solution of the Buckley-Leverett equation consisting of shocks and rarefaction waves for which \( c \) is constant. The \( C \)-waves are the rarefaction waves along the curve \( G = \text{constant} \). These waves will be called contact discontinuities, because the characteristic values are equal at the right and left states of the wave. For a flux function \( F \) satisfying (1-3), the eigenvalues \( \lambda_c, \lambda_s \) are equal along a simple curve \( T \) dividing the unit square \( \Omega \) into two regions: the left region \( L \), and the right region \( R \). Thus, the system (1.3) is linearly degenerate and is not diagonalizable along \( T \). The structure of the Riemann solution connecting two end states will be a sequence of \( S \)-waves and \( C \)-waves according to the increase of their corresponding characteristic speeds. In this paper, we will investigate the problem numerically. Instead of studying the inviscid system (1.3), we will consider the viscous system of equations (1.3), computing asymptotic behaviour of the solutions for various values of \( \epsilon \), and studying some questions that are difficult to answer in a rigorous way, such as the stability and the \( \epsilon \) zero limit.

2. Numerical Scheme. We will use finite difference approximations to the following simplified system of equations

\[ \partial_t s + (sG)_x = \epsilon s_{xx} \]
\[ \partial_t c + Gc_x = \epsilon c_{xx} + \epsilon \left( \frac{2cxs_x}{s} \right) \]

Let \( \Delta t, \Delta x \) denote the time and space steps. Define \( s^n_j = s(n\Delta t, j\Delta x) \), and \( c^n_j = c(n\Delta t, j\Delta x) \) for positive integers \( n, j \). We denote \( D_+, D_- \) to be the standard forward, backward and centered finite difference operators. We restrict our computations to a finite interval by introducing artificial boundary conditions so the computations will be stopped before reaching the boundaries, or by introducing moving system of coordinates with a fixed speed as we have done in this analysis. For the system (2.1) we introduce the following implicit, unconditionally stable scheme of second order in time and space \( 0(\Delta t^2 + \Delta x^2) \). The scheme is

\[ \frac{3}{2} \frac{s^{n+1}_j - s^n_j}{\Delta t} - \frac{1}{2} \frac{s^n_j - s^{n-1}_j}{\Delta t} = +\epsilon \frac{D_+ D_- s^{n+1}}{2} + R_s(s^n, c^n, s^{n-1}) \]
\[ \frac{3}{2} \frac{c^{n+1}_j - c^n_j}{\Delta t} - \frac{1}{2} \frac{c^n_j - c^{n-1}_j}{\Delta t} = +\epsilon \frac{D_+ D_- c^{n+1}}{2} + R_c(s^n, c^n, c^{n-1}) \]
where the functions $R_s, R_c$ are defined as follows

$$R_s = -D_0(s^nG(s^n, c^n)) + \epsilon \frac{D_+D_-s^{n-1}}{2}$$
$$R_c = -G(s^n, c^n)D_0c^n + \epsilon \frac{D_+D_-c^{n-1}}{2} + \epsilon \frac{2D_0c^nD_0s^n}{s^n}.$$

Since this finite difference scheme approximates a system of equations of hyperbolic character for small $\epsilon$, the discretization should satisfy the additional stability condition of Courant-Friedrichs-Lewy (CFL), that is

$$\Delta t \leq \frac{\Delta x}{M}$$

where $M$ is a positive constant defined as follows

$$M = \max_{\Omega}(\lambda_s, \lambda_c)$$

In fact in our computations, we take $M = 2$.

The consistency and linearized stability of this scheme is a standard analysis. Tveito [10] has shown similar analysis of an explicit scheme of first order in time for a similar system of equations. However, such an explicit scheme is not efficient for the study of asymptotic (large time) behavior of the solutions and their stability due to restrictions on time steps and round off errors. Therefore, it is essential to work with high order schemes such as the proposed scheme (2.2).

3. Numerical results. We begin this section by specifying the flux function $F$ to be of the form

$$F(s, c) = \frac{s^2}{s^2 + (.5 + c)(1 - s)^2}.$$

Then the characteristic field for fixed value of $\overline{G}$ will be given by the following explicit formula

$$c = -.5 + \frac{s - Gs^2}{\overline{G}(1 - s)^2}$$

and the transition curve $T$ will be given by

$$c = -.5 + \frac{s^2}{1 - s^2}.$$

Figure [2] depicts the curve $T$ and characteristic fields for various values $\overline{G} = 1.08, 1.12, 1.16, 1.2, 1.24$. We choose $U_l = (s_l, c_l) = (.55, .2695473) = U_1$; then we have $\overline{G}(s_l, c_l) = 1.2.$ The characteristic curve defined by $\overline{G}_l$ will intersect the transition curve $T$ at the point $U_4 = (.7142857, .5416664)$. The intersection points of the horizontal line passing through $U_l$
with the transition curve and the curve $\bar{G}$ are $U_2 = (.6594571, c_1)$ and $U_3 = (.7906975, c_1)$. These four points are shown in Fig [2].

3.1 Riemann structure. We demonstrate that the structure of the Riemann type solutions of the inviscid system (1.3) as done in [5] and [6] is the limit of the corresponding profiles of the viscous system (1.2) as $\epsilon$ approaches 0, and for finite time level. We provide the following example to illustrate the validity of this assertion. Let $U_t = U_1$ and take $U_r = (.2, .7) = U_5$ which means that $U_t \in L$ and $U_r \in L$ with $c_r$ greater than the maximum of the characteristic curve corresponding to $U_t$, that is $c_5 > c_4$. See Fig [2]. Let $\Delta x = 1/300, \Delta t = \Delta x/2, \epsilon = .0006$. Isaacson has shown that the connection $U_t \rightarrow U_r$ will be the sequence $S-$wave, $C-$wave, $S-$wave. Figure [3] shows that the solution of (1.3) with this initial data will also evolve into a profile connecting the left state to the right state by $S-C-S$wave profiles. The variable $s$ is plotted at time $T = 0, 3, 9, 15$ on Figure [3], and in Fig [4] we plot the profile of the variable $c$ at the same successive time. As we decrease the value of $\epsilon$, the width of all profiles decreases to zero for a finite time.

3.2 Width of the profiles. In this section we explore the effect of capillarity parameter $\epsilon$ (viscosity) on the thickness of the profiles of the elementary waves. The $S-$profiles are the solution of the Buckley-Leverett equation. We substitute solutions of the form $U = U(\frac{s - Vt}{\epsilon})$ in (1.2), where $V$ is the speed of the shock profile. We can see that the width of the shock profile is of order $\epsilon$ and inverse to the strength of the shock which
leads to the following formula

$$W = \frac{\epsilon}{|U_l - U_r|}.$$

Here $W$ denotes the width. The rarefaction waves of the Buckley-Leverret equation expand according to the following formula

\[(3.2)\]

$$W = (\lambda_r - \lambda_l)t.$$

Next we consider the profile of the $C-$waves which are contact discontinuities with $\lambda_r = \lambda_l = G(s_l, c_l)$. One might expect that the width of the profile of contact discontinuities is of finite thickness. It is not clear how to derive an approximate formula for the width of these waves from theoretical considerations. We obtain the following formula based on computational data

\[(3.3)\]

$$W \approx \frac{1}{\min(|s_r - s_l|, |c_r - c_l|)} \epsilon^\alpha t^\beta$$

where $\alpha \approx \frac{1}{3}, \beta \approx \frac{1}{2}$. Thus we conclude that the width of the profiles of $C-$waves increases with time.
THE S-C-S STRUCTURE At Time T=0,3,9,15

Figure 3 The function $s(x,t)$

Figure 4 The function $c(x,t)$
3.3 Uniqueness. The solution of the Riemann problem of the inviscid system (1.3), corresponding to two end states is unique. However, different connections may correspond to the same end states. This fact is consistent with the entropy admissibility criteria. For example, let \( U_l = U_1 \) and \( U_r = U_4 \), then we may have the following stable connections: \( C^- \)-Wave, \( S - C^- \)-Wave, or \( C \ldots C \ldots C \) waves. However, the one-to-one correspondence between the two end states and the connections will be restored in the viscous system (1.2). This will occur due to the expanding width of the \( C^- \)-waves.

3.4 Stability. In this section, we first show the validity of the following important claim: the maximum value of the initial data \( c(0, x) \) is constant in time. The importance of this stems from the fact that the final configuration of the initial value problem is defined not only by the end states but also by the maximum value of \( c \). In order to demonstrate the validity of this claim we choose the following example. Take \( s(0, x) = s_1 \) and \( c(0, x) = c_1 + \chi \) where \( \chi \) is a characteristic function such that

\[
\chi = \begin{cases} 
\overline{c}, & x \in [a, b], \\
0, & x \notin [a, b]. 
\end{cases}
\]

This initial data is a constant state with a simple perturbation see Fig [5]. We find the numerical solution of the viscous system (1.2) with these initial data, where \( \varepsilon \) is taken to be small enough. In general the solution will consist of two waves moving along the characteristic fields corresponding to \( C^- \) and \( S^- \)-waves. This is precisely what we observe in Fig [6], where this nonlinear wave breaks into a \( C^- \)-wave along the integral curve \( \overline{C} \), and \( S^- \)-waves along \( c = const = c_l \). The \( S^- \)-wave along its characteristic is the solution of the Buckley-Leverett equation, which is a scalar conservation law, so it will decay as an \( N^- \)-wave to zero due to nonlinearity and viscosity. However the \( C^- \)-wave moves as a stable wave with constant upper value equal to \( \overline{c} \). This value \( \overline{c} \) remains constant, because the nonlinear interaction along the \( \overline{C} \) curve has the same speed. However the profile of the \( C^- \)-wave increases in time following the width formula (3.3), and this diffusive process leads to the decay of these waves to a constant state. vanish. This also can be seen from the linearized dispersion relation around a point \((c_1, s_1)\) in the \( C^- \)-direction

\[
(1.9) \\
\lambda = -\varepsilon k^2 - i\overline{G}k
\]

where \( k \) is the wave number and \( \lambda \) is the growth factor. The negative real part \(-\varepsilon k^2\) of \( \lambda \) leads to the decaying of any perturbation to zero; that is we have a stability in the case \( \varepsilon \neq 0 \). We also may conclude the important claim that there is no stability of any waves of the inviscid system (1.3). This observation can be seen in Fig [7]. It is interesting to note that if we perturb a constant state on the transition curve then the perturbations will move along the \( C^- \)-characteristics only. This fact is demonstrated in Fig [8]. We conclude that the viscosity matrix is a stabilizing factor in the solutions with respect to perturbations, and the absence of capillarity from the equations results in the instability of all elementary solutions. This fact is different from the stability results for the strictly hyperbolic system of equations.
THE STATE U AT T=0, 4, 6

THE DEGENERATE STATE U AT T=0
THE DEGENERATE STATE $U$ AT $T=0.4$

Figure 7

THE STATE $U$ AT $T=0.4, 6$

Figure 8
Acknowledgments. I would like to thank Tai Ping Liu for helpful discussions.

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<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>779</td>
<td>Jack Carr and Robert Pego</td>
<td>Self-similarity in a coarsening model in one dimension</td>
</tr>
<tr>
<td>780</td>
<td>J.M. Greenberg</td>
<td>The shock generation problem for a discrete gas with short range repulsive forces</td>
</tr>
<tr>
<td>781</td>
<td>George R. Sell and Mario Taboada</td>
<td>Local dissipativity and attractors for the Kuramoto–Sivashinsky equation in thin 2D domains</td>
</tr>
<tr>
<td>782</td>
<td>T. Subba Rao</td>
<td>Analysis of nonlinear time series (and chaos) by bispectral methods</td>
</tr>
<tr>
<td>783</td>
<td>Nicholas Baumann, Daniel D. Joseph, Paul Mohr and Yuriko Renardy</td>
<td>Vortex rings of one fluid in another free fall</td>
</tr>
<tr>
<td>784</td>
<td>Oscar Bruno, Avner Friedman and Fernando Reitich</td>
<td>Asymptotic behavior for a coalescence problem</td>
</tr>
<tr>
<td>785</td>
<td>Johannes C.C. Nitsche</td>
<td>Periodic surfaces which are extremal for energy functionals containing curvature functions</td>
</tr>
<tr>
<td>786</td>
<td>F. Abergel and J.L. Bona</td>
<td>A mathematical theory for viscous, free-surface flows over a perturbed plane</td>
</tr>
<tr>
<td>787</td>
<td>Gunduz Caginalp and Xinfu Chen</td>
<td>Phase field equations in the singular limit of sharp interface problems</td>
</tr>
<tr>
<td>788</td>
<td>Robert P. Gilbert and Yongzhi Xu</td>
<td>An inverse problem for harmonic acoustics in stratified oceans</td>
</tr>
<tr>
<td>789</td>
<td>Roger Fosdick and Eric Volkmann</td>
<td>Normality and convexity of the yield surface in nonlinear plasticity</td>
</tr>
<tr>
<td>790</td>
<td>H.S. Brown, I.G. Kevrekidis and M.S. Jolly</td>
<td>A minimal model for spatio–temporal patterns in thin film flow</td>
</tr>
<tr>
<td>791</td>
<td>Chao–Nien Chen</td>
<td>On the uniqueness of solutions of some second order differential equations</td>
</tr>
<tr>
<td>792</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>The thermistor problem for conductivity which vanishes at low temperature</td>
</tr>
<tr>
<td>793</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>The thermistor problem with one-zero conductivity</td>
</tr>
<tr>
<td>794</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>Separation of variables for the Dirac equation in Kerr Newman space time</td>
</tr>
<tr>
<td>795</td>
<td>E. Knobloch, M.R.E. Proctor and N.O. Weiss</td>
<td>Finite-dimensional description of doubly diffusive convection</td>
</tr>
<tr>
<td>796</td>
<td>V.V. Pukhnachov</td>
<td>Mathematical model of natural convection under low gravity</td>
</tr>
<tr>
<td>797</td>
<td>M.C. Knaap</td>
<td>Existence and non-existence for quasi-linear elliptic equations with the $p$-laplacian involving critical Sobolev exponents</td>
</tr>
<tr>
<td>798</td>
<td>Stathis Filippas and Wenxiong Liu</td>
<td>On the blowup of multidimensional semilinear heat equations</td>
</tr>
<tr>
<td>799</td>
<td>A.M. Meirmanov</td>
<td>The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution</td>
</tr>
<tr>
<td>800</td>
<td>Bo Guan and Joel Spruck</td>
<td>Interior gradient estimates for solutions of prescribed curvature equations of parabolic type</td>
</tr>
<tr>
<td>801</td>
<td>Hi Jun Choe</td>
<td>Regularity for solutions of nonlinear variational inequalities with gradient constraints</td>
</tr>
<tr>
<td>802</td>
<td>Peter Shi and Yongzhi Xu</td>
<td>Quasistatic linear thermoelasticity on the unit disk</td>
</tr>
<tr>
<td>803</td>
<td>Satyanad Kichenassamy and Peter J. Olver</td>
<td>Existence and non-existence of solitary wave solutions to higher order model evolution equations</td>
</tr>
<tr>
<td>804</td>
<td>Dening Li</td>
<td>Regularity of solutions for a two-phase degenerate Stefan Problem</td>
</tr>
<tr>
<td>805</td>
<td>Marek Fila, Bernhard Kawohl and Howard A. Levine</td>
<td>Quenching for quasilinear equations</td>
</tr>
<tr>
<td>806</td>
<td>Yoshihazu Giga, Shun'ichi Goto and Hitoshi Ishii</td>
<td>Global existence of weak solutions for interface equations coupled with diffusion equations</td>
</tr>
<tr>
<td>807</td>
<td>Mark J. Friedman and Eusebius J. Doedel</td>
<td>Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study</td>
</tr>
<tr>
<td>808</td>
<td>Mark J. Friedman</td>
<td>Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds</td>
</tr>
<tr>
<td>809</td>
<td>Peter W. Bates and Songmu Zheng</td>
<td>Inertial manifolds and inertial sets for the phase-field equations</td>
</tr>
<tr>
<td>810</td>
<td>J. López Gómez, V. Márcquez and N. Wolanski</td>
<td>Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition</td>
</tr>
<tr>
<td>811</td>
<td>Xinfu Chen and Fahuai Yi</td>
<td>Regularity of the free boundary of a continuous casting problem</td>
</tr>
<tr>
<td>812</td>
<td>Eden, A., Foias, C., Nicolaenko, B. and Temam, R.</td>
<td>Inertial sets for dissipative evolution equations Part I: Construction and applications</td>
</tr>
<tr>
<td>813</td>
<td>Jose–Francisco Rodrigues and Boris Zaltzman</td>
<td>On classical solutions of the two-phase steady-state Stefan problem in strips</td>
</tr>
<tr>
<td>814</td>
<td>Viorel Barbu and Srdjan Stojanovic</td>
<td>Controlling the free boundary of elliptic variational inequalities on a variable domain</td>
</tr>
<tr>
<td>815</td>
<td>Viorel Barbu and Srdjan Stojanovic</td>
<td>A variational approach to a free boundary problem arising in electrophotography</td>
</tr>
<tr>
<td>816</td>
<td>B.H. Gilding and R. Kersner</td>
<td>Diffusion-convection-reaction, free boundaries, and an integral equation</td>
</tr>
<tr>
<td>817</td>
<td>Shoshana Kamin, Lambertus A. Peletier and Juan Luis Vazquez</td>
<td>On the Barenblatt equation of elastoplastic filtration</td>
</tr>
<tr>
<td>818</td>
<td>Avner Friedman and Bei Hu</td>
<td>The Stefan problem with kinetic condition at the free boundary</td>
</tr>
<tr>
<td>819</td>
<td>M.A. Grinfeld</td>
<td>The stress driven instabilities in crystals: mathematical models and physical manifestations</td>
</tr>
<tr>
<td>820</td>
<td>Bei Hu and Lihe Wang</td>
<td>A free boundary problem arising in electrophotography: solutions with connected toner region</td>
</tr>
</tbody>
</table>
| 821| Yongzhi Xu, T. Craig Poling, and Trent Brundage | Direct and inverse scattering of time harmonic }