THE GEOMETRY OF NONLINEAR
SCHRÖDINGER STANDING WAVES

By

Paul K. Newton
and
Shinya Watanabe

IMA Preprint Series # 849
August 1991
The Geometry of Nonlinear Schrödinger Standing Waves

Paul K. Newton
Department of Mathematics
and Center for Complex Systems Research
University of Illinois, Urbana, IL 61801

Shinya Watanabe *
Department of Mathematics
University of Illinois, Urbana, IL 61801

Abstract. A numerical method to study radially symmetric standing wave solutions to the nonlinear Schrödinger equation is described and used to study several new geometric features of these waves. The method is based on locating the basin boundary between two attracting invariant lines in a three dimensional phase space. When projected onto a two dimensional subspace, the basin boundary forms an unstable manifold emanating from the origin. This solution trajectory is located by ‘squeezing’ it between two adjacent stable trajectories. Of particular interest is the number and location of zeroes of the eigenfunctions. A case in which solutions are known not to exist is also studied and contrasted with the cases where existence is guaranteed.

1. Introduction

A numerical method to study radially symmetric standing wave solutions to the nonlinear Schrödinger equation (NLS) is described [1]. Our focus is on the nodal properties of the solutions in analogy to the well developed oscillation theory for linear Sturm-Liouville eigenvalue problems [2]. The NLS, written as:

\[ i\psi_t + \Delta \psi + f(|\psi|^2)\psi = 0 \]
\[ \psi : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{C}, \quad f : \mathbb{R} \rightarrow \mathbb{R} \]

is by now a well established canonical amplitude equation arising in such diverse physical contexts as water wave theory [3], nonlinear optics [4], nonlinear plasma physics [5], vortex filament dynamics [6], and molecular biology [7]. Typically it arises as a first order solvability condition in a weak nonlinear perturbation analysis around a bifurcation point as a dimensionless parameter is increased. From this point of view, the higher order solvability conditions are ignored. These higher order conditions can be viewed as small correction

*Current address: Department of Mathematics, MIT, Cambridge, MA 02139
terms to the NLS, hence it is of great interest to study the effects of these terms on the long
time dynamics of the solution. To accomplish this, one first needs to understand the geometric
properties of the unperturbed problem (1.1), in particular the standing wave solutions,
then ask how these properties deform under perturbation. Our main goal in this paper is to
describe a numerical method to accurately compute the radially symmetric standing wave
solutions of (1.1), and to show several detailed geometrical features.

Standing wave solutions are obtained by writing \( \psi = \exp(i\lambda t)u(r) \), \( \lambda > 0 \), where \( r \) is the
radial variable. (1.1) then reduces to the nonlinear eigenvalue problem:

\[
\frac{d^2u}{dr^2} + \frac{n-1}{r} \frac{du}{dr} + f(u^2)u - \lambda u = 0
\]  

(1.2)

Here, \( n \) is the underlying spatial dimension. We augment (1.2) with the boundary conditions:

\[
\begin{align*}
  u(0) &= H \geq 0 \\
  \frac{du}{dr}(0) &= 0 \\
  u &\to 0 \quad \text{as} \quad r \to \infty \\
  \frac{d^2u}{dr^2} &\to 0 \quad \text{as} \quad r \to \infty
\end{align*}
\]  

(1.3)

This gives a two parameter eigenvalue problem in the parameters \((\lambda, H)\).

This problem has been studied in the past from several different points of view. Using
variational techniques, Strauss [8] and Berestycki and Lions [9] proved the existence of
infinitely many solutions. These results hold if \( s = \frac{4}{(n-2)} \) where \( f(u^2) = u^s \). In analogy
whether these solutions could be ordered according to the number of zeroes they contain.
This question motivated Jones and K"{u}pper to study this question further [12,13] and
introduce a very useful dynamical systems technique. This technique forms the basis of our
numerical 'squeezing method' described in section 4. Their main result establishes the existence
of solutions with a prescribed number of zeroes. This problem was also studied more

The starting point in Jones and K"{u}pper's proof and our numerical method is to write
(1.2), (1.3), as the first order nonautonomous system:

\[
\begin{align*}
u' &= v \\
v' &= -\frac{n-1}{r}v - f(u^2)u + \lambda u
\end{align*}
\]  

(1.4)

\[
\begin{align*}
u(0) &= H, v(0) = 0, u, v \to 0 \quad \text{as} \quad r \to \infty \\
v'(0) &= H\{\lambda - f(H^2)\}/n
\end{align*}
\]  

(1.5)

The first term on the right side of the second equation acts as a dissipation term for
\( n > 1 \) with decreasing dissipation as \( r \to \infty \). In the limit \( r = \infty \) the system is Hamiltonian
and equivalent to the one dimensional problem \( n = 1 \). As \( r \to 0 \), the system is singular.
The last condition in (1.5) is obtained by Taylor expanding (1.2) about the origin.
Equivalently, (1.4) can be written in autonomous form by introducing \( w = r \), then the system reads:

\[
\begin{align*}
    u' &= v \\
    v' &= -\frac{n-1}{w} v - f(u^2)u + \lambda u \\
    w' &= 1
\end{align*}
\]  

(1.6)

In this three-dimensional phase space, standing wave solutions satisfying the boundary conditions (1.3) are trajectories which start on the \( u \) axis for \( r = 0 \), wind around the \( w \) axis a prescribed number of times in a corkscrew like fashion as \( r \) increases, and converge to the \( w \) axis in the limit \( r \to \infty \). The number of times the trajectory intersects the \((v, w)\) plane corresponds to the number of zeroes of the standing wave.

We will describe a numerical method which locates these solutions to arbitrary accuracy, and hence study several geometric properties of the solutions for four different nonlinearities \( f(u^2) \). This is accomplished without actually solving for the desired solution, but by solving for two adjacent trajectories which straddle the solution. These two adjacent trajectories are global attractors hence easier to calculate. By squeezing these trajectories arbitrarily close to each other, the desired solution, which is the boundary separating the basins of attraction of the two attractors, is obtained to arbitrary accuracy.

The paper is organized as follows. In section 2 we first describe a subproblem in which the dissipation coefficient in (1.4) is fixed, thus reducing the problem to a two dimensional autonomous phase space. Although the subproblem is not equivalent to the three dimensional problem (1.6) it is nonetheless a useful idealization. In section 3 we extend some of the results of this subsystem to the true system. Section 4 contains a detailed description of the squeezing algorithm. Sections 5-8 contain our main numerical results. The results are organized around the four different cases:

(i) case 1: \( n = 2, \ f(u^2)u = u^3 \)
(ii) case 2: \( n = 3, \ f(u^2)u = u^3 \)
(iii) case 3: \( n = 2, \ f(u^2)u = u^5 \)
(iv) case 4: \( n = 2, \ f(u^2)u = u \sin(u^2) \)

The cases are chosen to exhibit the dependence of solutions on the underlying dimension, the power of the nonlinearity, and the behavior for non pure power nonlinearities. Odd powers typically arise in applications to ensure the invariance \( \psi \to -\psi \). We have found it clearer to focus our initial descriptions of techniques on case 1, then generalize the numerical results to the other cases. After computing the first five eigenfunctions for each of the above cases hence computing the eigensets \((\lambda, H)\), we describe in section 5 several characteristic properties of the eigenfunctions. These include the distribution of zeroes and peaks of the eigenfunctions for each of the cases as a function of \( \lambda \) and \( m \) (the number of zeroes) and the height of the peaks as a function of \( \lambda \) and \( m \). In section 6 we describe the behavior of the invariants \( E \) (energy) and \( M \) (mass) along each curve in the \((\lambda, H)\) plane. Section 7 contains
a general description of the separating surface, or basin boundary for the full system. The behavior of this separating surface, as \( r \to 0 \) is crucial to the existence or non-existence of solutions. This is shown in section 8 for a case where a non-existence theorem can be proven.

2. Frozen coefficient subsystem

We first examine a 'subsystem' of (1.1) in which the dissipation coefficient, \( \frac{1}{r} \) is 'frozen' to a fixed value. As mentioned, for the description of the numerical method, we will focus on the cubic nonlinearity. Hence in this section we consider the problem:

\[
\begin{align*}
\frac{du}{dr} &= v \\
\frac{dv}{dr} &= \frac{-(n-1)}{w} v - u^3 + \lambda u \quad (n > 1)
\end{align*}
\]  

(2.1) (2.2)

Here \( w \) is treated as a positive parameter. Because this subsystem is a two-dimensional cross section \( (w = r) \) of the original three dimensional problem, it is easier to analyze than the full system, however it still contains many of its essential features. Moreover, it will be shown that the qualitative features of the subsystem are independent of the parameter \( w \), thus every cross section of the phase space of the full system is qualitatively similar.

We summarize here the main features of (2.1), (2.2) (damped Duffing system) which will be used in understanding the full system. There are three equilibrium points of (2.1), (2.2) given by \((u, v) = (0, 0), (\pm \sqrt{\lambda}, 0)\). Their local stability properties are determined by the eigenvalues of the linearized system, thus:

**Lemma 1:**

(i) The origin \((u, v) = (0, 0)\) is an unstable saddle with eigenvalues \(\sigma_{\pm}^{(0)} = \frac{-(n-1)}{2w} \pm \sqrt{(\frac{n-1}{2w})^2 + \lambda}\) and eigenvectors \(\bar{u}_+ = (\frac{1}{\sigma_+^{(0)}}, \frac{1}{\sigma_-^{(0)}})\), \(\bar{u}_- = (\frac{1}{\sigma_+^{(0)}}, \frac{1}{\sigma_-^{(0)}})\).

(ii) The equilibrium points \((u, v) = (\pm \sqrt{\lambda}, 0)\) are asymptotically stable. The eigenvalues are given by \(\sigma_{\pm}^{(1)} = \frac{-(n-1)}{2w} \pm \sqrt{(\frac{n-1}{2w})^2 - 2\lambda}\). Therefore when \(0 < w \leq \frac{(n-1)}{\sqrt{8\lambda}}\) they are asymptotically stable nodes and when \(w > \frac{(n-1)}{\sqrt{8\lambda}}\) they are asymptotically stable foci.

The next step is to prove that all trajectories on the phase plane converge to one of the equilibria. To do this, it is convenient to consider the energy function \(\tilde{H}(u, v; w)\) given by:

\[
\tilde{H}(u, v; w) = \frac{v^2}{2} + \frac{u^4}{4} - \frac{\lambda}{2} u^2
\]

(2.3)

This serves as the Hamiltonian for the system (2.1), (2.2) in the dissipationless limit \(w \to \infty\). \(\tilde{H}\) assumes minima at the nonzero equilibrium points of the subsystem and has saddle
structure at the origin. The level set $\tilde{H} = 0$ is a leaf shaped figure shown in figure 2.1. The derivative of the energy function along a trajectory is given by:

\[
\frac{d\tilde{H}}{dr} = \frac{\partial \tilde{H}}{\partial u} \frac{du}{dr} + \frac{\partial \tilde{H}}{\partial v} \frac{dv}{dr}
\]

\[
= (u^3 - \lambda u)v + v\left(\frac{-(n-1)v}{w} - u^3 + \lambda u\right)
\]

\[
= \frac{-(n-1)v^2}{w} \leq 0
\]

Equality holds in (2.4) only if $n = 1$, or $v = 0$, or $w = \infty$. Thus, every trajectory of the subsystem ($n > 1$) converges either to one of the minima of $\tilde{H}$ or to some other stationary point along the $u$ axis, the only one being the origin. These statements are summarized in:

**Theorem 1:**

Every trajectory of the subsystem (2.1), (2.2) ($0 < w < \infty$) converges to one of the three equilibrium points as $r \to \infty$.

We next address the more detailed question on the domain of attraction of the equilibria. Since the origin is locally a saddle, there are two pairs of convergent trajectories (one pair as $r \to +\infty$, the other as $r \to -\infty$) to the point. The trajectories of each pair are connected smoothly at the origin and their tangent vectors correspond to the eigenvectors $\tilde{u}_\pm$ in lemma 1. The ‘inset’ corresponds to the curve with tangent vector $\tilde{u}_-$ and the ‘outset’ corresponds to the curve with tangent vector $\tilde{u}_+$. The other two equilibria ($\pm \sqrt{\lambda}, 0$) are both asymptotically stable, so the two trajectories on the outset of the origin converge to these points. Since there are no other equilibria or periodic orbits, the two trajectories on the inset of the origin come from infinity, hence the inset divides the phase plane. For this reason, the inset is also referred to as the separatrix. Although on the separatrix the flow is towards the saddle, the separatrix itself is unstable in the sense that any trajectory arbitrarily near it for some $r$ eventually diverges from it as $r \to \infty$. Thus, the two trajectories corresponding to the separatrix are the unique ones convergent to the origin as $r \to \infty$, and act as the basin boundary between the basins of attraction for the right and left fixed points, a fact which will be used for the squeezing method described in section 4.

Typical phase portraits of the subsystem for increasing values of the parameter $w$ are shown in figure 2.2 with $\lambda = .04$ and $n = 2$ (case 1). The separatrix is drawn as a dashed curve while other trajectories are drawn as solid curves. The equilibrium points for $\lambda = .04$ are $(0, 0)$ and $(\pm 0.2, 0)$. When $w \leq \sqrt{8\lambda} \approx 1.77$ the stable equilibria are locally nodes and the separatrix looks nearly straight near the origin (see figure 2.2a), however on a larger scale it is found to be a pair of spirals (see figure 2.2b). As $w$ increases to values greater than $w = 1.77$, the stable equilibria change to foci and the separatrix spiral winds tighter (figure 2.2c). For increasing $w$, the qualitative character of the phase portrait remains the same, although the density of the spiral increases causing convergence to the origin to slow down (figure 2.2d), a fact consistent with the decrease in the damping coefficient $\frac{1}{w}$ as $w$ increases.

In the limit $w \to \infty$, the subsystem (2.1), (2.2) reduces to a conservative Hamiltonian system with Hamiltonian given by (2.3). The limiting shape of the separatrix spiral is the
level set of the Hamiltonian passing through the origin, i.e. with zero energy. The parametric representation is given by:

\[ u = \pm \sqrt{2\lambda} \text{sech}(t) \]  \hspace{1cm} (2.5)

\[ v = \sqrt{2\lambda} \text{sech}(t) \tanh(t) \quad -\infty < t < \infty \]  \hspace{1cm} (2.6)

The curve surrounds two leaf shaped regions symmetrical with respect to both the u and v axes (see Figure 2.1). We will call the region in the positive u plane the positive leaf and the other the negative leaf.

We next prove that each leaf represents a domain of attraction for the equilibrium point it encloses. In each leaf we consider a separate Liapunov function as discussed in [15]. In the positive leaf we take the Liapunov function:

\[ V_+(x, y; w) = \frac{y^2}{2} + \frac{x^4}{4} + \sqrt{\lambda} x^3 + \lambda x^2 + \nu (\frac{(n-1)x^2 + xy}{2w}) \]  \hspace{1cm} (2.7)

where \( x = u - \sqrt{\lambda} \), \( y = v \), and \( \nu \) is a constant such that \( 0 < \nu < \frac{(n-1)}{w} \). The function \( V_+ \) has the properties that \( V_+(0, 0; w) = 0 \), \( V_+ \to \infty \) as \( \sqrt{x^2 + y^2} \to \infty \) and for \( (x, y) \neq (0, 0) \) we have \( \frac{x^2}{2} + \nu (\frac{(n-1)x^2 + xy}{2w}) > 0 \) and \( \frac{x^4}{4} + \sqrt{\lambda} x^3 + \lambda x^2 \geq 0 \). Therefore, \( V_+(x, y; w) \) is positive definite. Its derivative along a trajectory of the subsystem is:

\[ \frac{dV_+}{dr} = [x^3 + 3\sqrt{\lambda} x^2 + 2\lambda x + \nu (\frac{x(n-1)}{w} + y)] y + \\
(\nu x [\frac{-y(n-1)}{w} - (x + \sqrt{\lambda})^3 + \lambda (x + \sqrt{\lambda})] = -(\frac{(n-1)}{w} - \nu)y^2 - \nu x^2 F(x) \]  \hspace{1cm} (2.8)

where \( F(x) = x^2 + 3\sqrt{\lambda} x + 2\lambda = (x + \sqrt{\lambda})(x + 2\sqrt{\lambda}) \). The positive leaf is always in a region \(-\sqrt{\lambda} < x < (\sqrt{2} - 1)\sqrt{\lambda}\) hence \( F(x) > 0 \) in this region which implies \( \frac{dV_+}{dr} < 0 \). Similarly, for the negative leaf we can use the Liapunov function:

\[ V_-(x, y; w) = \frac{y^2}{2} + \frac{x^4}{4} - \sqrt{\lambda} x^3 + \lambda x^2 + \nu (\frac{(n-1)}{2w}x^2 + xy) \]  \hspace{1cm} (2.9)

where \( x = u + \sqrt{\lambda} \), \( y = v \). By the same reasoning as above, it is easy to show that \( \frac{dV_-}{dr} < 0 \) in the negative leaf. This is summarized as:

**Theorem 2:**

Each leaf of the Hamiltonian (2.3) represents the domain of attraction for the equilibrium point it encloses. Thus, every trajectory in the positive leaf converges asymptotically to \((\sqrt{\lambda}, 0)\) as \( r \to \infty \) under the flow (2.1), (2.2) and every trajectory in the negative leaf converges asymptotically to \((-\sqrt{\lambda}, 0)\) as \( r \to \infty \) under the flow (2.1), (2.2)

An important feature of the two leaves is that their shape is independent of \( w \), thus these two regions remain as cores of attraction on any finite cross section of the full system, a fact which will be useful for the squeezing method. Another point worth mentioning is that the leaves enclose all the points in the plane which stay on one side of the separatrix for
all $u$. Conversely, any given point outside of the leaves will lie on alternating sides of the separatrix (hence will converge to either the positive fixed point or the negative fixed point alternatingly) as $u$ increases since the spiral winds tighter as $u$ increases.

We next consider the eigenvalue subproblem given by (2.1), (2.2) with the auxiliary conditions (1.5). It is useful to examine, for example, figure 2.2c. Any trajectory starting on the $u$ axis at point $u = H$ will converge to one of the three equilibria by theorem 1. Only those which converge to the origin are solutions to the eigenvalue subproblem. There are infinitely many distinct intersections between the separatrix and the positive $u$ axis labeled outward $P_0, P_1, \ldots P_n, \ldots$. If $H$ is located between the origin and $P_0$, the trajectory will converge to the positive equilibrium point $(\sqrt{\lambda}, 0)$. If $P_0 < H < P_1$, the trajectory will converge to $(-\sqrt{\lambda}, 0)$. In general, if $P_{2n} < H < P_{2n+1}$ the trajectory converges to $(-\sqrt{\lambda}, 0)$ while if $P_{2n+1} < H < P_{2n+2}$ it converges to $(\sqrt{\lambda}, 0)$. By continuity of the flowfield, on each boundary between adjacent regions there is a solution to the eigenvalue problem, i.e. a trajectory which converges to the origin. These are the trajectories which start at $P_m$ ($m = \text{integer}$) for $r = 0$ in figure 2.2c. This set of points is discrete on the $u$-axis and each one corresponds to a solution of the eigenvalue problem with $m$ zeroes in $u$. $H$ can be viewed as an eigenvalue parameter ($\lambda = 0.04$ is fixed in figure 2.2), whose spectrum is discrete ($H_0 = P_0 < H_1 = P_1 < H_2 = P_2 < \ldots$). The eigenfunctions can be ordered according to the number of zeroes they contain.

Alternatively, when $H$ is fixed instead, as $\lambda$ monotonically changes, the separatrix spiral changes continuously and the discrete values of $\lambda$ which force an intersection of the spiral with $(H, 0)$ are the eigenvalues $\{\lambda_0, \lambda_1, \lambda_2, \ldots\}$. As before, the trajectory associated with each $\lambda_m$ corresponds to a solution of the boundary value problem with $m$ zeroes. Thus, an eigenvalue problem with discrete spectrum can be formed in either of the two parameters $(\lambda, H)$. This general set-up holds for the full system which we describe in the next section and is the basis for the squeezing algorithm.

As a final observation of figure 2.2c, it is possible to count the number of times a given trajectory crosses the $u$-axis before converging to one of the stable equilibria. For example, if a trajectory starting at $P^*$ in figure 2.2c is chosen, it crosses the $u$-axis twice, hence has two zeroes in $u$. Considering each trajectory in this way, an index associated with it can be defined according to the number of times it crosses the $u$-axis. This index is accompanied by a ‘+’ if the trajectory converges to the positive equilibrium point and by a ‘−’ if it converges to the negative equilibrium point (here, +0 and −0 should be distinguished). Examining figure 2.2c, the index is uniquely determined a priori by the position of the initial point, hence one can think of the index as being attached to the initial point. In this way, every point on the phase plane has its index which can be used to divide the whole plane into domains, as shown in figure 2.3. From this figure, it is understood, for example, that there is a point on the boundary between the domain of the index +0 and the one with −1 corresponding to a solution trajectory of the boundary value problem with no zeroes, i.e. the ground state solution. The alternating structure of the phase space for the subsystem in this way is completely visualized.
3. Extension to full system

The nonautonomous system (1.4) contains the same equilibrium points as the subsystem, their locations being independent of $r$. The same eigenvalues as given in lemma 1 determine the local flowfield on this cross section just by replacing $w$ with $r$. Since the eigenvalues $\sigma^{(0)}_{\pm}$ never change sign for any $r$, the character of the origin remains an unstable saddle as before. However a slightly subtle problem arises when analyzing the other equilibria. Although the real parts of the eigenvalues $\sigma^{(1)}_{\pm}$ are negative for any finite $r$, they tend to zero as $r \to \infty$ which causes an ambiguity as to the limiting character of the points.

The same subtlety also arises when trying to extend the global analysis of the subsystem described in theorems 1 and 2. One starts by using the energy function (2.3). The derivative of $\hat{H}$ along a trajectory of the full system is:

$$\frac{d\hat{H}}{dr} = \frac{\partial \hat{H}}{\partial r} + \frac{\partial \hat{H}}{\partial u} \frac{du}{dr} + \frac{\partial \hat{H}}{\partial v} \frac{dv}{dr}$$

$$= 0 + (u^3 - \lambda u)v + v\left(\frac{-(n-1)v}{r} - u^3 + \lambda u\right)$$

$$= \frac{-(n-1)v^2}{r} \leq 0 \quad (3.1)$$

Thus, if a trajectory crosses one of the level curves of $\hat{H}$ at some finite $r$, the nonpositive derivative implies that it is confined inside this region for all larger $r$. This gives:

**Theorem 3:**
Consider the two leaves of the Hamiltonian (2.3) shown in figure 2.1. Once a trajectory of the system (1.1) enters a leaf for some finite $r$, it never escapes.

However, when the limit set of the trajectories is considered the same subtlety arises as in the local analysis. Since the system is a nonautonomous, the limit set need not only be equilibrium points. There remains the possibility of a trajectory converging to a periodic orbit due to the fact that the decay rate $\frac{d\hat{H}}{dr} \to 0$ as $r \to \infty$. The crucial aspect of the problem is the decay rate of the damping factor $\frac{(n-1)v}{r}$. The subtlety is described in the following:

**Theorem 4:**
Consider the nonautonomous system:

$$\frac{du}{dr} = v \quad (3.2)$$

$$\frac{dv}{dr} = \frac{(n-1)}{r^\alpha} v - u^3 + \lambda u \quad (n > 1, \lambda > 0) \quad (3.3)$$

If $\alpha > 1$, there exist trajectories which do not converge to one of the three equilibria $(0,0)$, $(\pm \sqrt{\lambda},0)$ as $r \to \infty$.

**Proof:**
Consider a trajectory which lies on the level curve $\hat{H} = K_1$ at some $r_0$ (see figure 3.1). The key question is how much energy is dissipated under the flow (3.2), (3.3) from $r = r_0$ to $r = \infty$. Thus:
\[
\left| \int_{r_0}^{\infty} \frac{d\tilde{H}}{dr} \, dr \right| = \left| \int_{r_0}^{\infty} \frac{(n-1)v^2}{r^\alpha} \, dr \right| \leq (n-1) \int_{r_0}^{\infty} \frac{v^2}{r^\alpha} \, dr \\
\leq (n-1) \left( 2K_1 + \frac{\lambda^2}{2} \right) \int_{r_0}^{\infty} \frac{dr}{r^\alpha} = (n-1) \left( 2K_1 + \frac{\lambda^2}{2} \right) \left[ \frac{1}{(\alpha-1)} r^{(1-\alpha)} \right]_{r_0}^{\infty}
\]

Therefore, if \( K_1 > \frac{(n-1)\left(2K_1 + \frac{\lambda^2}{2}\right)}{(\alpha-1)r_0^{(\alpha-1)}} \), the trajectory never enters into one of the leaves of the Hamiltonian (\( \tilde{H} = 0 \)) hence it must converge to a periodic orbit with positive energy. Solving the above inequality for \( K_1 \) gives the criterion:

\[
K_1 > \frac{\frac{\lambda^2}{2} (n-1)}{(\alpha-1)r_0^{(\alpha-1)} - 2(n-1)}.
\]

where \( r_0 \) satisfies \( r_0 > \left\{ \frac{2(n-1)}{(n-1)} \right\}^{\frac{1}{(\alpha-1)}} > 0 \). One can always find such an \( r_0 \) and \( K_1 \) if \( \alpha > 1 \).

The power \( \alpha \) determines the rate at which \( \frac{d\tilde{H}}{dr} \to 0 \). When \( \alpha \) is large (\( \alpha > 1 \)) the rate is rapid enough so that the trajectory does not travel down the potential surface \( \tilde{H} \) to one of the minima, but ends up in a periodic oscillatory state. Therefore, the determination of the limit set of trajectories of the full system involves slight ambiguity. However, for the purposes of the squeezing algorithm described in the next section, a sufficient ingredient is the result of theorem 3, that once a trajectory enters one of the leaves of the Hamiltonian, it never escapes.

To understand the eigenvalue problem (1.3) for the full system we must describe the analogue of the separatrix of the previous section. If we use here the three dimensional autonomous system (1.6) expressed by the flow in \( (u,v,w) \) space, any trajectory forms a curve extending along the \( w \)-axis in a corkscrew like fashion. As in the subsystem, the solutions to (1.6) with (1.5) are ones with initial positions on the positive \( u \)-axis that converge to the \( w \) axis as \( r \to \infty \). These solutions will wind around the \( w \) axis a prescribed number of times [13], the number of crossings with the \( (v,w) \) plane corresponding to the number of zeroes in the eigenfunction. See figure 3.2 for several computed trajectories of the system for case 1 shown in the phase space \( (u,v,w) \).

The trajectories which qualify as solutions to the eigenvalue problem lie on a surface in the \( (u,v,w) \) space which we call the ‘separatrix surface’. This surface is the analogue of the separatrix described in the previous section. If a cross section \( (w = \text{constant}) \) of the phase space is taken, the intersection of the separating surface with the cross plane forms a curve \( C \). See figure 3.3 for an example of this curve. Each point on \( C \) is an intersection of a solution trajectory with the cross section. The curve \( C \) always passes through the origin \( (u,v) = (0,0) \) of the
cross plane since the $w$ axis itself qualifies as the trivial solution trajectory. Since the system is symmetric under the transformation $u \rightarrow -u, \ v \rightarrow -v$ the curve $C$ and thus the separating surface inherits this symmetry. This proves useful in cutting down the computations involved in visualizing the separating surface.

A first naive expectation is that the curve $C$ corresponds exactly to the separatrix of the subsystem corresponding to the same $w$ value as the cross section (i.e. see figures 2.2). The separating surface thus would be formed by joining the spiral shaped separatrices of figure 2.2 for each $w$. The surface formed in this way would be increasingly tightly wound as $w$ increases. However, it is easy to see that such a surface is not invariant, i.e. trajectories do not remain on this surface under the flow (1.6). This is because the $(u,v)$ velocity components on any cross section $w =$ constant of such a surface are tangential to the curve, thus will not stay on such a surface as $w$ increases. It turns out that although this naive expectation is false, the true separating surface is not qualitatively different from it, thus it serves as a useful prototype.

The key to the squeezing algorithm and the existence (and non-existence) of solutions to (1.2), (1.3) are the intersections of the separating surface with the positive $u$ axis at $r = 0$. Consider a trajectory starting on the positive $u$ axis at $r = 0$. If the initial point does not coincide with the intersection of the separating surface and the $u$-axis then the trajectory will not converge to the $w$ axis as $r \rightarrow \infty$ but instead will be trapped either in the positive or negative leaf tube (set of the leaves for all $w > 0$) around the stable invariant axes $u = \pm \sqrt{\lambda}$. The separating surface is the boundary which separates the trajectories convergent to $\sqrt{\lambda}$ from those convergent to $-\sqrt{\lambda}$, thus it contains all the solutions of the eigenvalue problem (1.2), (1.3).

4. The squeezing algorithm

In this section we describe the squeezing algorithm and study case 1 in some detail. As described in the previous section, we are interested in trajectories in the autonomous $(u,v,w)$ space with initial data along the positive $u$ axis which converge to the $w$ axis as $r \rightarrow \infty$. For this purpose it is convenient to view the projection of the trajectory onto the $(u,v)$ plane and look for solution curves which converge to the origin. Since the origin is unstable, however, these trajectories are difficult to compute directly. Most trajectories converge to one of the stable fixed points $(\pm \sqrt{\lambda}, 0)$, hence they enter one of the leaves of the Hamiltonian (2.3). To locate the initial data $(H, 0)$ corresponding to a solution trajectory we need only determine to which leaf a given trajectory will converge. If we locate a trajectory with initial data $(u,v) = (A, 0)$ converging to the positive fixed point and another with initial data $(C, 0)$ converging to the negative fixed point, then by continuity there must exist a solution trajectory with initial data $(B, 0)$ where $A < B < C$. If the interval $[A,C]$ contains only one solution then we can obtain a solution trajectory within arbitrary accuracy by ‘squeezing’ the unstable solution between the two stable trajectories. In figure 3.2 trajectories
for various values of $H$ are shown for $\lambda = 0.04$. Their projection onto the $(u,v)$ and $(w,u)$ planes are shown in figure 4.1. We focus on figure 4.1a. When $H$ is small (trajectories labeled $A, B$ in figure 4.1a, b) the initial point $(H,0)$ is in the positive leaf, hence the trajectories converge to the positive equilibrium point $(\sqrt{\lambda},0) = (.2,0)$. As $H$ increases, the corresponding trajectory $C$ converges to the opposite equilibrium point $(-\sqrt{\lambda},0) = (-.2,0)$ after crossing the $v$ axis once. This trajectory therefore has one zero in $u$, although it is not a solution. As $H$ increases further, (trajectory labeled $D$) the convergent point switches back to the positive equilibrium point. In this case, the trajectory loops around the origin once, thus it has two zeroes in $u$. This process continues as $H$ increases, each switch signals an additional zero in $u$. Between each pair of trajectories whose convergent points are in opposite leaves lies a solution trajectory convergent to the origin with $m$ zeroes. For instance, between trajectories $B$ and $C$ in figure 4.1 there exists a solution (labeled $S$ and shown as dashed) with no zeroes, i.e. the ground state eigenfunction. Similarly, between $C$ and $D$ another solution with $m = 1$ exists.

Thus, for each fixed $\lambda$, the initial points on the $u$-axis are grouped together in segments according to the number of zeroes of the corresponding trajectory. This number $m$ increases from one segment to the next, the closest segment to the origin corresponding to $m = 0$. The boundary of every two segments corresponds to a solution trajectory with the same $m$ as the segment to its left. We label the segments by integer $m$. The boundary points are labeled $P_0, P_1, P_2, \ldots$

We compute the boundary points $P_m$ according to the following algorithm. Suppose a point $Q_m$ in the $m^{th}$ segment and a point $Q_{m+1}$ in the $(m+1)^{th}$ segment have been obtained. Using these two points, $P_m$ can be calculated as follows. By continuity, we know $P_m \in (Q_m, Q_{m+1})$. For any arbitrary test point $T_0 \in (Q_m, Q_{m+1})$ it can be determined whether $P_m \in (Q_m, T_0)$ or $P_m \in (T_0, Q_{m+1})$ in the following way. Produce a trajectory starting at $T_0$ and integrate forward in $r$ until it enters one of the two leaves of the Hamiltonian. The number of zeroes of the trajectory is recorded, this number being either $m$ or $m+1$. If $m$, then $T_0 \in (Q_m, P_m)$ thus $P_m \in (T_0, Q_{m+1})$. Otherwise $T_0 \in (P_m, Q_{m+1})$ hence $P_m \in (Q_m, T_0)$. In either case, the length of the interval in which $P_m$ lies is decreased. A binary search, i.e. choosing $T_0$ as the midpoint between $Q_m$ and $Q_{m+1}$ is the most efficient method of decreasing the interval. $T_0$ has now become an endpoint for a new test interval and the method is repeated by choosing $T_1$ as the midpoint of this interval. Continuing this procedure, the interval becomes arbitrarily small hence we locate $P_m$ without ever having to compute the unstable trajectory directly. In practice, if the interval is $(T_L, T_R)$ after a finite number of such procedures where $(T_R - T_L) < 2\epsilon$ and $\epsilon > 0$ is a preset accuracy threshold, the $u$-coordinate $H$ of $P_m$ is approximated as $H = \left(\frac{T_R + T_L}{2}\right) \pm \epsilon$. $H$ can be obtained as precisely as needed by making $\epsilon$ as small as necessary.

We now give a brief discussion of how we obtain the initial endpoints of the test interval $(Q_m, Q_{m+1})$. This constitutes the first half of the algorithm followed
by the iterative squeezing method described above. $Q_0$ is found easily. In fact the positive equilibrium point $(u, v) = (\sqrt{\lambda}, 0)$ is suitable since the trajectory ‘converges’ to the right leaf with no zeroes in $u$. Thus this point lies in the $0^{th}$ segment on the $u$ axis. Starting from $Q_0$, the other endpoints are obtained successively. $Q_1$ is chosen by traversing an arbitrary amount from $Q_0$ in the positive $u$ direction until a point with a trajectory having one zero is found. Since the number of zeroes increases as $H$ is moved in the positive $u$ direction, in practice it is easy to find $Q_1, Q_2, \ldots$ For the calculations in this report, the first guess for $Q_1$ is located at $D_0 = (2\sqrt{\lambda}, 0)$. If the test point $D_0$ must be moved further away from $Q_0$, the next test point $D_1$ is chosen so that $D_1Q_0 = \frac{3}{2}D_0Q_0$. If the point $D_0$ corresponds to a trajectory with too many zeroes, $D_1$ is chosen by $D_1Q_0 = \frac{1}{2}D_0Q_0$. If $D_1$ is still not in the expected interval, successive test points $D_2, D_3, \ldots$ are selected in the same way. After $Q_1$ is found, the next endpoint $Q_2$ is found by repeating the above procedure.

To summarize the method, in order to obtain the switching points $P_0, P_1, \ldots$, a coarse slicing of intervals $(Q_0, Q_1), (Q_1, Q_2), \ldots$ is performed to make sure that each interval has just one switching point. Next, an accurate determination is carried out for each interval sliced and the points $P_m$ are approximated within an arbitrary accuracy $\epsilon$. Each value $P_m$, together with its corresponding value of $\lambda = \lambda_0$ corresponds to a point on the eigenset in the $(\lambda, H)$ plane along the vertical line $\lambda = \lambda_0$.

5. Nodal properties

We next turn to a more detailed description of the eigenfunctions for the four cases cited in section 1. In particular, we will show the computed projection of the solution trajectories projected on the $(u, v)$ plane and the $(u, w)$ plane, the eigenset diagrams in the $(\lambda, H)$ plane, the location of the zeroes of the eigenfunctions as a function of $\lambda$ and $m$ and the magnitude and location of the peaks of eigenfunctions as a function of $\lambda$ and $m$.

As a first step we present a scaling result linking the eigenvalue parameters $\lambda$ and $H$. Hence, consider the eigenvalue problem for pure power nonlinearities:

\[ \frac{d^2 u}{d r^2} + \frac{n-1}{r} \frac{d u}{d r} + u^{s+1} - \lambda u = 0 \]  \hspace{1cm} (5.1)

with auxiliary conditions (1.5). Introduce the new variables $\bar{r} = \sqrt{\lambda} r, \bar{u} = \lambda^{-\frac{1}{s}} u$ so that (5.1) becomes:

\[ \frac{d^2 \bar{u}}{d \bar{r}^2} + \frac{n-1}{\bar{r}} \frac{d \bar{u}}{d \bar{r}} + \bar{u}^{s+1} - \bar{u} = 0 \]  \hspace{1cm} (5.2)

\[ \bar{u}(0) = \frac{H}{\lambda^{\frac{1}{s}}}, \quad \bar{u} \to 0 \text{ as } \bar{r} \to \infty, \quad \frac{d \bar{u}}{d \bar{r}}(0) = 0 \]  \hspace{1cm} (5.3)

From this, we can conclude that for the case of pure power nonlinearities, the two parameters $\lambda, H$ scale so that $H \propto \lambda^{\frac{1}{s}}$. Thus, the eigenfunctions with a fixed
number of zeroes (say \( m \)) all lie on a single algebraic curve in the \((\lambda, H)\) plane. To find the particular curve for a given \( m \) and hence the eigenset \((\lambda, H)\), it is only necessary to compute the proportionality constant as a function of \( m \). The curves are shown in figure 5.1 for the first three cases described earlier. Cases 1 and 2 shown in figures 5.1a,b are families of square root curves since \( s = 2 \), while case 3 shown in figure 5.1c is a family of fourth root curves. We show the eigenset curves corresponding to the first five eigenfunctions, \( i.e., m = 0, 1, 2, 3, 4 \). Table 1 shows the proportionality constant relating \( \lambda^2 \) and \( H \) as a function of \( m \) for the first 5 eigenfunctions. The results are accurate to three decimal places.

The eigenfunctions projected onto the \((u, v)\) plane and the \((u, w)\) plane are shown for the first three cases in figures 5.2 - 5.4. Figure 5.2 shows the eigenfunctions in case 1 for \( \lambda = 0.01 \). In practice, as described earlier, most trajectories eventually converge either to \( \sqrt{\lambda} \) or \(-\sqrt{\lambda}\) since these are the stable fixed points. Thus, in the figures shown, we ‘squeeze’ the eigenfunction between the two stable trajectories in neighboring segments so that it is correct to three decimal places. The calculation is terminated at the \( w \) value where \( u^2 + v^2 \) is minimal, after the last peak in \( u \). Cases 2 and 3 are shown in figures 5.3, 5.4 with the same \( \lambda \) value. It is interesting to compare the shapes of the eigenfunctions for the different cases. Looking at figure 5.3 the eigenfunctions for case 2 in which the dimension of the space is changed from 2 to 3, the sharpness of the center peak is seen to increase. The initial height is larger and the decay of the amplitude is more rapid. In figure 5.4 the sharpness of the peaks of the eigenfunctions for case 3, in which the dimension is 2 but the nonlinearity is quintic, is similar to case 1, however the amplitude is larger.

Since there is no scaling theory available for case 4 we describe it separately. The nontrivial equilibria for this case are obtained by solving \( \sin(u^2) = \lambda \) where we restrict \( \lambda \in (0, 1) \). Solving for \( u \) gives:

\[
\pm \sqrt{2k\pi + \gamma} \quad (\text{group A}) \\
\pm \sqrt{(2k + 1)\pi - \gamma} \quad (\text{group B})
\]

where \( k = 0, 1, \ldots \) and \( \gamma = \arcsin(\lambda), \quad 0 < \gamma < \frac{\pi}{2} \). This yields an infinite number of equilibria on the \( u \) axis, those in group B are all saddles while those in group A are asymptotically stable with the eigenvalues of the linearized coefficient matrix having negative real parts for all \( r > 0 \). The origin is locally a saddle. The equilibria on the positive \( u \) axis are labeled as \( R_0, R_1, \ldots, R_k, \ldots \) starting closest to the origin. Even suffixes denote points in group A (stable) whereas odd suffixes represent points in group B (saddles). The equilibria on the negative \( u \) axis are labeled as \( R_{-0}, R_{-1}, \ldots, R_{-k}, \ldots \). Figure 5.5 shows four trajectories projected onto the \((u, v)\) plane corresponding to four different values of \( H \). In the figure, \( \lambda = 0.25 \), hence the \( u \) coordinate of \( R_0 \) is \( \sqrt{\gamma} = \sqrt{\arcsin(0.25)} \cong 0.503 \) and \( R_1 \) is \( \sqrt{\pi - \gamma} \cong 1.70 \). When the starting point of a trajectory \((H, 0)\) is located between 0 and \( R_0 \) or larger than but sufficiently close to \( R_0 \), then the trajectory converges to \( R_0 \) (see trajectories labeled A, B in the figure). When the initial point is increased further towards \( R_1 \), the corresponding trajectory...
(labeled C) converges to \( R_{-0} \) with one zero in \( u \). Increasing \( H \) further towards \( R_1 \) causes the trajectory (labeled 0) to switch to \( R_0 \) but with two zeroes in \( u \). This switching of the convergent point continues as do the number of zeroes as \( H \) approaches \( R_1 \). When \( H \) is greater than \( R_1 \), the corresponding trajectory either converges to one of the stable equilibria \( R_2, R_4, \ldots \) or diverges. In either case, such a trajectory never re-enters the interval \((R_{-1}, R_1)\). Thus the structure described earlier in case 1 for \( u \in (-\infty, \infty) \) is preserved, but compressed in a region \((R_{-1}, R_1)\) for case 4. Thus, there exist an infinite number of switching points between \( R_0 \) and \( R_1 \) which lie on trajectories converging to the unstable origin. Since all initial points are packed in a region \((R_0, R_1)\) on the \( u \) axis, the \( u \) coordinate \( H \) of the switching points must be smaller than the \( u \)-coordinate of \( R_1 \). As \( \lambda \) changes, the \( u \) coordinate of \( R_1 \) changes, hence we can draw on the \((\lambda, H)\) parameter plane a curve corresponding to its location. In figure 5.6 this curve, \( H = \sqrt{\pi - \arcsin(\lambda)} \) is drawn on the parameter plane as a solid curve together with a dashed one denoting the location of \( R_0 \), i.e. \( \sqrt{\arcsin(\lambda)} \). As \( \lambda \to 1 \) the two curves coincide. Figure 5.7 shows the eigencurves (solid lines) for case 4. These eigencurves cannot cross the boundary curve which is shown as a dot-dashed curve. The dashed curves are the square root curves from case 1 which asymptotically match the solid curves for small \( H \). This is expected since case 4 approaches case 1 when \( u \) is sufficiently small. An interesting aspect of these eigencurves is that they are not monotonic. Although they are single valued functions with respect to \( \lambda \), they are not so with respect to \( H \). This implies nonuniqueness when \( H \) is fixed in problem (1.2),(1.3) and \( \lambda \) varies. Two solutions exist having the same number of zeroes, \( m \), and the same height, \( H \), at the center. This is not true of the previous three cases.

Since the phase space structure of case 4 is similar to the other cases, the same squeezing method can be used to calculate the eigenvalues \((\lambda, H)\). Table 2 shows the results. Since there is no scaling theory in this case, \( H \) must be calculated for each \( \lambda \). Figure 5.7 shows the eigencurves from the table 2. The eigenfunctions for case 4 are shown in figure 5.8 - 5.12 for increasing values of \( \lambda \). Note that the top of the solutions become flat as \( H \) approaches the bounding height which is shown in each figure as a dashed line.

There are several other 'characteristic properties' of the solutions worth examining in greater detail. We describe a number of these properties in some detail for case 1 first, and then show the other cases. Figure 5.13 shows the distribution of zeroes, peaks, and 'decay point' for case 1. The decay point is defined to be the value of \( r \) when \( |u| \) is decreased to 10% of a typical height scale of the wave during the decay stage (after the peak furthest from the origin) of the solution. The initial height of the wave \( H \) may be chosen as the typical height scale, however, the distance, \( d_0 \), between the stable equilibrium and origin (i.e. \( d_0 = \sqrt{\lambda} \) in case 1) makes more sense since it is independent of \( m \). Figure 5.13 shows the dependence on \( m \) of the locations of the zeroes, the peaks and the decay point of each solution shown earlier. The dot-dashed curve at the right
connects the points \((w, m)\) where \(w\) is the decay point of the wave with \(m\) zeroes. The dashed curve next to the decay curve connects the points corresponding to the first peak, defined as the peak furthest from the origin. Since the wave with \(m = 0\) has only one peak at \(w = 0\), the curve starts at the origin. The solid curve next to the first peak curve denotes the location of the first zero, \(i.e.,\) the zero furthest from the origin, for each \(m\). Since the wave with \(m = 0\) does not have any zeroes, the curve starts from \(m = 1\). The next dashed curve denotes the second peak and its next solid curve the second zero and so on. Only the wave with \(m = 4\) has the fourth zero and the fifth peak so that the last two curves appear as single data points. From this graph it is easily seen that the ‘characteristic points’ evolve linearly in \(w\) as well as in parallel, \(i.e.,\) with nearly the same slope, especially when \(w\) is large. This means that as \(m\) increases, ‘old’ peaks and zeroes as well as the decay point are shifted to larger \(w\) values, keeping their relative distances while a new zero and peak are formed in the vacant region at the center created by the shift. At the inner region of the waves (smaller \(w\)) each curve has a slight curvature. However the nonlinear effect seems to diminish as the nodes are shifted outward.

In figure 5.14 the dependence of these same characteristic locations on \(\lambda\) is shown for the wave with \(m = 4\) for case 1. In a similar manner as in figure 5.13, the dot-dashed curve at the far right represents the decay point and, to its left, five curves of peaks (as dashed ) and four curves of zeroes (solid) are ordered alternatively. The fifth peak curve is overlapped on the \(\lambda\) axis since this peak denotes the initial peak of the wave at its center. As predicted in the result of the dimensional analysis, \(w\) depends on \(\lambda\) like \(w = r = \frac{1}{\sqrt{\lambda}} \tilde{r}\) where \(\tilde{r}\) is a dimensionless location which is constant on each curve. The calculated result agrees with this analysis.

The height of the peaks is examined next. In figure 5.15 the dependence on \(m\) of the absolute values of the \(u\)-coordinates of the peaks is shown for \(\lambda = 0.01\). Since \(|u|\) is examined, both peaks and troughs are included. The lowest dashed curve denotes the height of the first peak, the furthest peak from the center. Since \(m = 0\) has only one peak at the center, the height corresponds to \(H\) in this case (\(\approx 0.22\)). When \(m = 1\), the first peak is a trough, hence the height is negative (\(\approx -0.16\)), although we plot its absolute value. For larger \(m\) the sign alternates similarly from peak to trough. Surprisingly, the curve becomes nearly constant after \(m = 1\). Thus, for every mode except the ground state, the wave front has a near constant peak level independent of \(m\). Considering the ‘linearity’ discussed earlier and shown in figure 5.13, one may guess that the shape of the front itself is kept almost constant for different \(m\) although mirror images with respect to the \(w\)-axis have to be taken between odd and even \(m\). This feature is essentially correct as seen in figure 5.2, the eigenfunctions of case 1. The second lowest curve of figure 5.15 denotes the level of the second peak. Since the \(m = 0\) solution has no such peak, the curve starts at \(m = 1\). The level changes rapidly for small \(m\), but becomes almost constant again as \(m\) increases. The second curve seems to approach the first curve as \(m\) increases. The other three curves
show similar features.

The dependence of the magnitude of the peaks on $\lambda$ is shown in figure 5.16 for the $m = 4$ solutions. The lowest curve denotes the first peak while the highest curve is the fifth peak. A simple scaling argument shows that all curves must behave like $|u| \propto \sqrt{\lambda}$ which agrees with the computations.

Finally, the magnitude of the peaks and troughs in one particular wave is also curious. In figure 5.17, the wave with $m = 4$ ($\lambda = 0.01$) is drawn by a dashed curve and the peaks are connected by solid lines. As one can see in the figure, the decay of the amplitude becomes very small when $w$ is large so that the first and second peaks have similar heights.

Figure 5.18 - 5.20 show the dependence of the location of zeroes, peaks, and decay point of solutions on $m$ and on $\lambda$ for the other three cases. Figure 5.18 shows cases 2 and 3. Only $\lambda = 0.01$ is considered for the dependence on $m$ since other values can be derived through rescaling. The linear behavior described earlier for case 1 (figure 5.13) is seen in these other cases as well. For the dependence on $\lambda$, only the $m = 4$ solutions are considered and the proportionality on $1/\sqrt{\lambda}$ is confirmed. Figure 5.19 shows the same graphs for case 4. To see how $\lambda$ affects the dependence on $m$, the graphs are shown for four values of $\lambda$ in figure 5.19. It is remarkable that the linear evolution of waves with different $m$ can be seen even in this case, independent of $\lambda$. In figure 5.20 only the $m = 4$ solutions are considered. When $\lambda$ is small, the characteristic locations depend on $\lambda$ proportionally to $1/\sqrt{\lambda}$. When $\lambda$ get larger ($\sim .2$), this relationship seems to break down.

The dependence of the magnitude of the peaks on $m$ and $\lambda$ are shown in figures 5.21 - 5.23. They should be compared with figures 5.15, 5.16 of case 1. Cases 2 and 3 are shown together in figure 5.21 while case 4 is shown in figure 5.22, 5.23. As before, the magnitude of the peaks remote from the center become almost constant for large $m$. This phenomenon is independent of the nonlinearity or on $\lambda$. The graphs showing the dependence on $\lambda$ are also similar as in case 1 except in case 4 when $\lambda$ becomes sufficiently large (figure 5.23). In this case the curve denoting the position of the fifth peak is bent by the bounding curve discussed earlier.

The decay of the magnitude of the peaks of the waves are shown in figures 5.24 - 5.26 for cases 2-4. In case 2, the decay is rapid, while case 3 is more similar to case 1. Finally, figure 5.27 shows the dependence of the height at the center, i.e. the eigenvalue $H$ on $m$ for the four cases. Since cases 1-3 can be scaled, the proportionality constant $H$ is used instead of $H$ for these three cases. For case 4 the dependence of $H$ on $m$ for several values of $\lambda$ is displayed.

6. Invariants

Two important invariant quantities associated with (1.1) are given by:

$$M = \int_{\mathbb{R}^n} |\psi|^2 d\bar{x}$$

(6.1)
\[ E = \int_{\mathbb{R}^n} \{ |\nabla \psi|^2 - g(|\psi|^2) \} d\tilde{x} \]  
(6.2)

Here, \( f(\xi) = g'(\xi) \). \( M \) can be interpreted as the mass of the wave, while \( E \) is the total energy. \( E \) can be decomposed into \( E = T + U \) where:

\[ T = \int_{\mathbb{R}^n} |\nabla \psi|^2 d\tilde{x} \quad \text{(kinetic energy)} \]  
(6.3)

\[ U = -\int_{\mathbb{R}^n} g(|\psi|^2) d\tilde{x} \quad \text{(potential energy)} \]  
(6.4)

As described earlier, each curve shown in the eigenset diagram of figure 5.1 for the pure power nonlinearities has a common dimensionless eigenfunction \( \tilde{u}(\tilde{r}) \). It is interesting to examine how the mass \( M \) and energy \( E \) vary on each of these curves. For this, we need to relate \( M \) and \( \tilde{M} \), \( E \) and \( \tilde{E} \), and \( E \) and \( M \) for each case. The results are summarized as:

**Theorem 5:**

(i) For case 1 we have:
\[ M = \tilde{M} \]  
(6.5)

\[ E = \lambda \tilde{E} \]  
(6.6)

\[ E \equiv 0 \]  
(6.7)

(ii) For cases 2, 3 we have:
\[ M = \tilde{M}/\sqrt{\lambda} \]  
(6.8)

\[ E = \sqrt{\lambda} \tilde{E} \]  
(6.9)

\[ E = \lambda M \]  
(6.10)

**Proof:**

(i) Case 1:

The mass of the dimensionless eigenfunction written in spherical coordinates for case 1 is given by:
\[ \tilde{M} = 2\pi \int_0^\infty \tilde{u}^2 \tilde{r} d\tilde{r} \]  
(6.11)

Relating this to the mass \( M \) can be done by introducing dimensionless variables:
\[ M = 2\pi \int_0^\infty u^2 r dr = 2\pi \int_0^\infty (\sqrt{\lambda} \tilde{u})^2 (\frac{\tilde{r}}{\sqrt{\lambda}}) (\frac{d\tilde{r}}{\sqrt{\lambda}}) \]  
(6.12)

The dimensionless energy is:
\[ \tilde{E} = 2\pi \int_0^\infty \left\{ (\frac{d\tilde{u}}{d\tilde{r}})^2 - \frac{\tilde{u}^4}{2} \right\} \tilde{r} d\tilde{r} \]  
(6.13)
Thus, the relationship between $E$ and $\bar{E}$ is given by:

\[
E = 2\pi \int_0^\infty \left\{ \frac{d\hat{u}}{dr} \right\}^2 - \frac{u^4}{2} \right\} r dr \\
= 2\pi \int_0^\infty \left\{ \sqrt{\lambda}/\sqrt{\lambda} \frac{d\hat{u}}{dr} \right\}^2 - \frac{1}{2} \left( \sqrt{\lambda}\hat{u} \right)^4 \right\} \left( \frac{\hat{r}}{\sqrt{\lambda}} \right) \left( \frac{d\hat{r}}{\sqrt{\lambda}} \right) \\
= 2\pi \lambda \int_0^\infty \left\{ \frac{d\hat{u}}{d\hat{r}} \right\}^2 - \frac{\hat{u}^4}{2} \right\} \hat{r} d\hat{r} = \lambda E
\]  
(6.14)

To derive the relationship between $M$ and $E$ we can use the relationships in [5] valid for $f(u^2) = u^2$:

\[T + 2U + \lambda M = 0\]  
(6.15)

and

\[\frac{n\lambda}{2} M + \left( \frac{n}{2} - 1 \right) T + \frac{n}{2} U = 0\]  
(6.16)

Eliminating $T$ from these gives:

\[U = \frac{2\lambda}{n-4} M\]  
(6.17)

Therefore from (6.15)

\[E = T + U = -U - \lambda M = \frac{n-2}{4-n} \lambda M\]  
(6.18)

For $n = 2$, this yields (6.7), while for $n = 3$, case 2 it yields (6.10).

(ii) Case 2, 3:

For case 2 in spherical coordinates with $n = 3$ we get:

\[M = 4\pi \int_0^\infty u^2 r^2 dr, \quad E = 4\pi \int_0^\infty \left\{ v^2 - \frac{u^4}{2} \right\} r^2 dr\]  
(6.19)

Since the nonlinearity is the same as case 1, the dimensionless scaling is identical, and when used on (6.19) yields (6.8), (6.9). The relationship between $E$ and $M$ is given by (6.18) which for $n = 3$ yields (6.10).

For case 3, $f(\xi) = \xi^2$, hence $g(|\psi|^2) = \frac{|\psi|^4}{3}$. This gives:

\[M = 2\pi \int_0^\infty u^2 r dr, \quad E = 2\pi \int_0^\infty \left\{ v^2 - \frac{u^6}{3} \right\} r dr\]  
(6.20)

The dimensionless variables for this case are given by

\[u = \sqrt{\lambda} \hat{u}, \quad r = \frac{\hat{r}}{\sqrt{\lambda}}\]  
(6.21)

Using these in (6.20) gives the result (6.8), (6.9). Since $\int_{\mathbb{R}^n} f(u^2)u^2 dx = -\int_{\mathbb{R}^n} u^6 dx = 3U$, the analogue of (6.15) for case 3 is:

\[T + 3U + \lambda M = 0\]  
(6.22)
(6.16) still holds for case 3 which together with (6.22) yields:
\[ E = T + U = \frac{n-1}{3-n} \lambda M \]  
(6.23)

For \( n = 2 \) one gets the result (6.10).

For case 4, no dimensional analysis can be done. The invariant quantities can be written as:
\[ M = 2\pi \int_0^\infty u^2 r dr, \quad E = 2\pi \int_0^\infty \{ v^2 + \cos(u^2) \} r dr \]  
(6.24)

We expect that the mass \( M \) must asymptotically approach that of case 1 as \( |u| \to 0 \).

We show in figures 6.1 - 6.4 numerical plots of the above relationships for all four cases. The results of the numerical integration based on the eigenfunctions obtained in the previous section are shown as dots. When the theory exists, the theoretical (solid) curves are fitted. Otherwise, the calculated points are just connected by dashed lines. For case 1 shown in figure 6.1a \( M \) is constant on each eigencurve, consistent with (6.5). Figure 6.1b shows the energy \( E \), which is zero in agreement with (6.7). Figure 6.1c shows the dependence of \( \tilde{M} = M \) on \( m \) which is more interesting since it cannot be predicted analytically. The curve is independent of \( \lambda \) for case 1 and shows that the growth of \( \tilde{M} \) increases with increasing \( m \) in a nonlinear fashion.

The numerical results for cases 2-4 are shown in figures 6.2 - 6.5. The dependence of \( M \) and \( E \) on \( \lambda \) are shown in figures 6.2, 6.3. In all cases, when \( M \) decreases with \( \lambda \), \( E \) increases. In case 4, for small \( \lambda \), \( M \) approaches a constant in figure 6.2d and its level matches that of case 1 which agrees with the theory.

The dependence of \( M \) and \( E \) on \( m \) is displayed in figures 6.4, 6.5. For cases 2 and 3 the above non-dimensionalization is possible so that the dimensionless quantities \( \tilde{M} \) and \( \tilde{E} \) can be incorporated. Then a unique curve independent of \( \lambda \) is obtained. For case 4, curves for various \( \lambda \) values are drawn. The dependence of \( \tilde{M} \) on \( m \) is surprisingly similar in all cases. The dependence of \( \tilde{E} \) on \( m \) varies with case, except cases 2 and 3 are similar. In case 1, \( \tilde{E} \) vanishes identically (figure 6.1b), in cases 2 and 3, \( \tilde{E} = \tilde{M} \) (figures 6.4, 6.5) which agrees with the computation.

7. Separating surface

We return to a more detailed discussion of the separating surface described briefly in section 3. As mentioned, this surface contains all the trajectories in the phase space which are solutions to the eigenvalue problem. The geometry of the surface will give us the crucial information on the existence or non-existence of solutions.

The surface can be obtained by finding an arbitrary cross section, say \( r = w_0 \). This intersection is a curve \( C \), shown for example in figure 3.3. Different points on
the curve represent different solution trajectories. The surface can be obtained from this cross section by simply forward integrating or backward integrating the equations using as initial data a point on $C$. Thus, for example, a new cross sectional curve can be obtained at $r = w_1 (> w_0)$ by forward integrating the equations of motion to $r = w_1$ using as initial data the points on the curve $C$ for $r = w_0$. The value of $w_0$ for the initial cutting plane does not affect the separating surface. We do not describe in detail how the initial curve $C$ is obtained except to say that it is based on the squeezing algorithm.

For case 1 with $\lambda = 0.01$, the separating surface was shown in figure 3.3. The calculated points are shown as dots. Only one spiral is calculated up to one and a half rotations and another spiral is produced using the symmetry. From the calculated points a backward integration to $r = 10^{-5}$ is carried out to obtain the cross section shown in figure 7.1. Note that the scales for the axes are different in figures 3.3 and 7.1. The later curve is still a pair of spirals, but more vertical than the one in 3.3.

The cross sectional curves at $w = 10^{-5}, 10^{-3}, 10^{-2}$ and $10^{-1}$ are shown in figures 7.2 with the same scales. Figure 7.2a shows a partial view of figure 7.1. Figure 7.2d is the same as figure 3.3 with different scaling. Visualizing the surface from the cross sections for increasing $w$ is fairly clear. When $r$ is small, the spirals are so large that the cross sections look like a set of vertical lines. As $r$ becomes large, they tilt and shrink. Figures 7.3 show the cross sections at $w = 10^{-1}, 10^{-1/2}, 1$ and $10^{1/2}$ produced by forward integration in the same scales. A similar transition can be observed.

It is important to observe the transition as $r \to 0^+$. Since the spirals get larger and more upright as $r$ becomes small, their limiting shape should be a set of straight vertical lines. This agrees both with the phase flow as $r \to 0^+$ and with the limit of the eigenvector $\vec{u}_-$ at the origin. Although the spirals extend rapidly in the $v$-direction as $r \to 0$, they remain similar in the $u$-direction. Looking at the intersections between the spirals and the $u$-axis, one notices that they converge to a set of fixed points as $r \to 0$. Consider these limiting points on the positive $u$ axis. These points all lie on the separating surface hence trajectories starting at these points are solutions to the eigenvalue problem (1.2, 1.3). The intersection point $u(0)$ for each trajectory is the eigenvalue $H$. Thus, the determination of the eigenvalues $H$ described earlier is seen to be identical with the determination of the intersection between the positive $u$ axis and the limiting shape of the separating surface as $r \to 0$. From table 1, $H$ for case 1 with $\lambda = 0.01$ is calculated as 0.22062, 0.33318, 0.41500, etc. These values agree with the $u$-coordinates of the vertical lines in figure 7.2a. Hence the geometric meaning of the eigenvalues is made clear. This interpretation will be useful when the non-existence case is discussed in the next section.

We next examine the separating surfaces for the other three cases. Cases 2 and 3 are similar to case 1, and several cross sections of the separating surface are shown for $\lambda = 0.01$. Figure 7.4 shows the cross sections at $w = 10^{-3}, 10^{-2}, 10^{-1}, 10^{-1/2}$ for case 2. The same qualitative behavior as that
in case 1 is shown. The \( u \)-coordinates of the intersections in 7.4a agree with the calculated values \( H = 0.43372, 1.41041, 2.91316 \) from table 1. The surfaces for case 3 for \( w = 10^{-3}, 10^{-2}, 10^{-1}, 1 \) are shown in figure 7.5. The \( u \)-coordinates in 7.5a at the intersection are given by \( H = 0.63252, 0.97819 \) and 1.2263.

Case 4 must be treated separately. As described earlier, if one analyses the frozen coefficient subsystem for case 4, the nontrivial equilibrium points \( R_k \) are obtained by solving \( \sin(u^2) = \lambda, v = 0 \). For any \( w > 0 \), the stability of the fixed points alternates, with the even indexed points \( (R_{\pm 0}, R_{\pm 2}, \ldots) \) being saddles. A qualitative plot of the phase flow for the subsystem is shown in figure 7.6a. This qualitative picture holds for any \( w \), hence it is helpful in understanding the full system. In analogy to the discussion of case 1, one can think of the surfaces for the full system as a surface which joins each cross section of the subsystem for increasing \( w \). As mentioned earlier, the surface formed in such a manner is not an invariant surface since trajectories of the full system do not stay on such a surface. It is nonetheless a useful model.

On examining figure 7.6a, one notices that only trajectories between \( R_1 \) and \( R_{-1} \) can reach the origin. Thus, in the phase space of the full system, trajectories must be confined by 'boundary surfaces' passing through these points. The solution trajectories must start at a point between \( R_{\pm 1} \) on the \( u \)-axis, hence the eigenvalues \( H \) are found only in the interval \( (R_{-1}, R_1) \).

Figure 7.6b illustrates a cross-sectional view of the confined separating surface between the two boundary surfaces. Qualitatively, one sees a pair of spirals packed into a band. The eigenvalues are given by the intersection of the separating surface with the \( u \)-axis. Figure 7.7 shows calculations of the separating surface for case 4 for \( \lambda = 0.15 \) with \( w = 10^{-3}, 10^{-2}, 10^{-1} \) and 1. At this value of \( \lambda \), the spirals are similar to the pure power case. Figure 7.8 shows the case for \( \lambda = 0.45 \). The same values of \( w \) are used. Here, the differences between the two cases are clear. Although the spirals tend to become larger as \( \lambda \) increases, they cannot cross the unstable equilibrium points \( R_1 \) and \( R_{-1} \) closest to the origin. One can see the result of this deformation clearly.

8. Non-existence

For pure power nonlinearities, \( f(u^2)u = u^{s+1} \), it is a well established result that solutions to (1.2), (1.3) exist only if \( s < 4/(n-2) \). For the cubic nonlinearity for example, \( s = 2 \), hence the existence condition is \( n < 4 \). It is informative to examine the separating surfaces in the case where the existence condition is violated. Hence, figure 8.1 shows cross sections in phase space for the cubic nonlinearity, \( n = 5 \). There is no qualitative change in the analysis of case 1. In fact, since \( n \) is larger the damping effect is four times stronger than in case 1, so convergence is enhanced. The cross sections shown in figure 8.1 are for \( w = 10^{-2}, 10^{-1}, 1, 10 \). Similar contraction of the spirals is seen. Looking at the change from (b) to (c), it is seen that the contraction speed is so fast that the intersections between the spirals and the \( u \)-axis move inward rapidly even
when \( w \) is small. Conversely, as \( w \) decreases, the spirals move rapidly outward on the \( u \)-axis. As \( w \to 0 \), they are not bounded as in the previous cases, but diverge to infinity. In (a), only one intersection is seen, corresponding to the trivial solution. Since the other intersection points diverge, there are no other solutions to the eigenvalue problem. Thus, the damping effect must be strong enough to have convergence of trajectories as \( r \to \infty \), but weak enough not to contract so rapidly as \( r \) increases. The existence of eigenfunctions seems to stand on the balance of these effects.

9. Conclusion

Our main goal in this paper has been to describe a numerical method to compute nonlinear Schrödinger standing waves with a prescribed number of zeroes. Although the method is directly applicable only to NLS, we believe that the ideas will be useful in more general contexts. As a particular example of this, if one considers the 'incompressible' limit of the Zakharov system in plasma physics [16, 17] one arrives at a singularly perturbed NLS where the perturbation term is due to a finite acoustic speed which leads to non-trivial acoustic-plasma interactions. An interesting question for this system is what effect a symmetry breaking acoustic wave (e.g. plane wave) will have on the nodal properties of the NLS plasma standing wave. To this end, we have collected here several detailed properties of the NLS standing waves and chosen four cases to isolate how these properties vary with the power of the nonlinearity and the dimension of the underlying space. Thus, we have computed the first five eigenfunctions for each of the four cases and studied the distribution of zeroes and peaks of the eigenfunctions, the height of the peaks and the mass and energy as a function of the eigenvalue parameters \((\lambda, H)\). We also show that a key to the existence or non-existence of the standing waves is the existence of a separating surface (or basin boundary) which intersects the \( u \) axis as \( r \to 0 \). This surface is shown for the four cases as well as for a case in which it is known that no standing waves exist.

Acknowledgements

This work was supported in part by the National Science Foundation under NSF DMS 90-00593. Part of the work was done while the first author was visiting the Institute for Mathematics and its Applications. Their hospitality is greatly appreciated.
References


*case 1*

\[
m = 0 \quad 1 \quad 2 \quad 3 \quad 4 \\
2.2062 \quad 3.3318 \quad 4.1500 \quad 4.8297 \quad 5.4241
\]

*case 2*

\[
m = 0 \quad 1 \quad 2 \quad 3 \quad 4 \\
4.3372 \quad 14.1041 \quad 29.1316 \quad 49.3604 \quad 74.7723
\]

*case 3*

\[
m = 0 \quad 1 \quad 2 \quad 3 \quad 4 \\
2.0002 \quad 3.0933 \quad 3.8779 \quad 4.5268 \quad 5.0929
\]

Table 1. Proportionality constant relating \(H\) and \(\lambda^{1/3}\). Results are accurate to three decimal places.

---

\[H\text{ in case 4}\]

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(m = 0)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>(upper bound)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.2206</td>
<td>0.3332</td>
<td>0.4148</td>
<td>0.4824</td>
<td>0.5415</td>
<td>1.76963</td>
</tr>
<tr>
<td>0.02</td>
<td>0.3120</td>
<td>0.4709</td>
<td>0.5856</td>
<td>0.6801</td>
<td>0.7619</td>
<td>1.76680</td>
</tr>
<tr>
<td>0.03</td>
<td>0.3821</td>
<td>0.5762</td>
<td>0.7155</td>
<td>0.8288</td>
<td>0.9252</td>
<td>1.76397</td>
</tr>
<tr>
<td>0.04</td>
<td>0.4412</td>
<td>0.6646</td>
<td>0.8233</td>
<td>0.9503</td>
<td>1.0565</td>
<td>1.76113</td>
</tr>
<tr>
<td>0.05</td>
<td>0.4933</td>
<td>0.7418</td>
<td>0.9162</td>
<td>1.0534</td>
<td>1.1649</td>
<td>1.75829</td>
</tr>
<tr>
<td>0.15</td>
<td>0.8553</td>
<td>1.2427</td>
<td>1.4523</td>
<td>1.5702</td>
<td>1.6372</td>
<td>1.72946</td>
</tr>
<tr>
<td>0.25</td>
<td>1.1055</td>
<td>1.5031</td>
<td>1.6343</td>
<td>1.7775</td>
<td>1.6921</td>
<td>1.69968</td>
</tr>
<tr>
<td>0.35</td>
<td>1.3073</td>
<td>1.6165</td>
<td>1.6609</td>
<td>1.6674</td>
<td>1.6684</td>
<td>1.66854</td>
</tr>
<tr>
<td>0.45</td>
<td>1.46958606</td>
<td>1.62996608</td>
<td>1.63529233</td>
<td>1.63548270</td>
<td>1.63548967</td>
<td>1.635489931</td>
</tr>
</tbody>
</table>

Table 2. \(H\) as a function of \(\lambda\) and \(m\) for case 4 where no scaling theory is available.
Figure 2.1 The "leaves" of attraction.
Figure 2.2  Trajectories of the subsystem (case 1, $\lambda = 0.04$). The separatrices are drawn as dashed curves, other typical trajectories as solid curves, and the leaves as shaded areas.
Figure 2.3 Division of phase plane by an index.
Figure 3.1  Level curve $H = K_1$ of Hamiltonian (2.3).
Figure 3.2 Trajectories of case 1 (\( \lambda = 0.04 \)).

Figure 3.3 Cross-section of separating surface (case 1, \( \lambda = 0.01, w=0.1 \)).
Figure 4.1a  Projection of fig.3.2 onto \((u,v)\)-plane.

Figure 4.1b  Projection of fig.3.2 onto \((w,u)\)-plane.
Figure 5.1a-c  The eigensets for cases 1-3.
Figure 5.2a-d  Solution trajectories of case 1, ($\lambda = 0.01$).
Figure 5.3a-d  Solution trajectories of case 2 ($\lambda = 0.01$).
Figure 5.4a-d  Solution trajectories of case 3 ($\lambda = 0.01$).
Figure 5.5  Projection of trajectories of case 4 ($\lambda = 0.25$).

Figure 5.6  The bounding curve for case 4 on the parameter plane.
Figure 5.7 The eigensets for case 4.
Figure 5.8 Solution trajectories of case 4 ($\lambda = 0.05$).
Figure 5.9 Solution trajectories of case 4 ($\lambda = 0.15$).
(3a) $(v, u)$-projection $(m = 0 \sim 2)$

(3b) $(w, u)$-projection $(m = 0 \sim 2)$

(3c) $(v, u)$-projection $(m = 3 \sim 4)$

(3d) $(w, u)$-projection $(m = 3 \sim 4)$

Figure 5.10 Solution trajectories of case 4 $(\lambda = 0.25)$. 
Figure 5.11 Solution trajectories of case 4 \((\lambda = 0.35)\).
Figure 5.12  Solution trajectories of case 4 ($\lambda = 0.45$).
Figure 5.13  Dependence of characteristic locations on $m$ (case 1).

Figure 5.14  Dependence of characteristic locations on $\lambda$ (case 1).
Figure 5.15  Dependence of $|u|$ on $m$ (case 1).

Figure 5.16  Dependence of $|u|$ on $\lambda$ (case 1).
Figure 5.17  Decay of peak height (case 1, $\lambda = 0.01$, $m = 4$)
Figure 5.18  Dependence of characteristic locations on $m$ and on $\lambda$ (cases 2, 3).
Figure 5.19  Dependence of characteristic locations on $m$ (case 4).
Figure 5.20  Dependence of characteristic locations on $\lambda$ (case 4, $m = 4$).
Figure 5.21  Dependence of magnitude of peaks on m and on λ (case 2-3).
Figure 5.22 Dependence of magnitude of peaks on $m$ (case 4).
Figure 5.23  Dependence of magnitude of peaks on $\lambda$ (case 4, $m = 4$).
Figure 5.24 Decay of peak height in $m = 4$ solution (case 2, $\lambda = 0.01$).

Figure 5.25 Decay of peak height in $m = 4$ solution (case 3, $\lambda = 0.01$).
Figure 5.26  Decay of peak height in $m = 4$ solutions (case 4).
Figure 5.27  Dependence of height at the center $H$ on $m$ for cases 1-4.
Figure 6.1a  Dependence of $M$ on $\lambda$ in case 1.

Figure 6.1b  Dependence of $E$ on $\lambda$ in case 1.

Figure 6.1c  Dependence of $\bar{M}$ on $m$ in case 1.
Figure 6.2 Dependence of mass $M$ on $\lambda$.  

(b) case 2  

(c) case 3  

(d) case 4
Figure 6.3  Dependence of energy $E$ on $\lambda$. 
Figure 6.4  Dependence of $\bar{M}$ or $M$ on $m$. 
Figure 6.5  Dependence of $\bar{E}$ or $E$ on $m$. 

(b)  case 2  $\bar{E}$ vs $m$ ($\bar{E} = \frac{E}{\sqrt{\lambda}} = \overline{M}$)

(c)  case 3  $\bar{E}$ vs $m$ ($\bar{E} = \frac{E}{\sqrt{\lambda}} = \overline{M}$)

(d)  case 4  $E$ vs $m$ (various $\lambda$)
Figure 7.1  Cross-section of separating surface at $w = 10^{-5}$. 
Figure 7.2  Cross-sections of separating surface (case 1, $\lambda = 0.01$).
Figure 7.3 Cross-sections of separating surface (case 1, $\lambda = 0.01$).
Figure 7.4  Cross-sections of separating surface (case 2, $\lambda = 0.01$).
Figure 7.5  Cross-sections of separating surface (case 3, $\lambda = 0.01$).
Figure 7.6a  Illustration of phase flow of subsystem of case 4.

Figure 7.6b  Illustration of cross-section of separating surface in case 4.
Figure 7.7  Cross-sections of separating surface (case 4, $\lambda = 0.15$).
Figure 7.8 Cross-sections of separating surface (case 4, $\lambda = 0.45$).
Figure 8.1  Cross-sections of separating surface \( n = 5, f(u^2)u = u^3, \lambda = 0.01 \).
<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>774</td>
<td>L.A. Peletier &amp; W.C. Troy</td>
<td>Self-similar solutions for infiltration of dopant into semiconductors</td>
</tr>
<tr>
<td>775</td>
<td>H. Scott Dumas and James A. Ellison</td>
<td>Nehoroshev’s theorem, ergodicity, and the motion of energetic charged particles in crystals</td>
</tr>
<tr>
<td>776</td>
<td>Stathis Filippas and Robert V. Kohn</td>
<td>Refined asymptotics for the blowup of $u_t - \Delta u = u^p$.</td>
</tr>
<tr>
<td>777</td>
<td>Patricia Bauman, Nicholas C. Owen and Daniel Phillips</td>
<td>Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity</td>
</tr>
<tr>
<td>778</td>
<td>Patricia Bauman, Nicholas C. Owen and Daniel Phillips</td>
<td>Maximal smoothness of solutions to certain Euler–Lagrange equations from nonlinear elasticity</td>
</tr>
<tr>
<td>779</td>
<td>Jack Carr and Robert Pego</td>
<td>Self-similarity in a coarsening model in one dimension</td>
</tr>
<tr>
<td>780</td>
<td>J.M. Greenberg</td>
<td>The shock generation problem for a discrete gas with short range repulsive forces</td>
</tr>
<tr>
<td>781</td>
<td>George R. Sell and Mario Taboada</td>
<td>Local dissipativity and attractors for the kuramoto-sivashinsky equation in thin 2D domains</td>
</tr>
<tr>
<td>782</td>
<td>T. Subba Rao</td>
<td>Analysis of nonlinear time series (and chaos) by bispectral methods</td>
</tr>
<tr>
<td>783</td>
<td>Nicholas Baumann, Daniel D. Joseph, Paul Mohr and Yuriko Renardy</td>
<td>Vortex rings of one fluid in another free fall</td>
</tr>
<tr>
<td>784</td>
<td>Oscar Bruno, Avner Friedman and Fernando Reitich</td>
<td>Asymptotic behavior for a coalescence problem</td>
</tr>
<tr>
<td>785</td>
<td>Johannes C.C. Nitsche</td>
<td>Periodic surfaces which are extremal for energy functionals containing curvature functions</td>
</tr>
<tr>
<td>786</td>
<td>F. Abergel and J.L. Bona</td>
<td>A mathematical theory for viscous, free-surface flows over a perturbed plane</td>
</tr>
<tr>
<td>787</td>
<td>Gunduz Caginalp and Xinfu Chen</td>
<td>Phase field equations in the singular limit of sharp interface problems</td>
</tr>
<tr>
<td>788</td>
<td>Robert P. Gilbert and Yongzhi Xu</td>
<td>An inverse problem for harmonic acoustics in stratified oceans</td>
</tr>
<tr>
<td>789</td>
<td>Roger Fosdick and Eric Volkmann</td>
<td>Normality and convexity of the yield surface in nonlinear plasticity</td>
</tr>
<tr>
<td>790</td>
<td>H.S. Brown, I.G. Kevrekidis and M.S. Jolly</td>
<td>A minimal model for spatio-temporal patterns in thin film flow</td>
</tr>
<tr>
<td>791</td>
<td>Chao–Nien Chen</td>
<td>On the uniqueness of solutions of some second order differential equations</td>
</tr>
<tr>
<td>792</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>The thermistor problem for conductivity which vanishes at large temperature</td>
</tr>
<tr>
<td>793</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>The thermistor problem with one-zero conductivity</td>
</tr>
<tr>
<td>794</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>Separation of variables for the Dirac equation in Kerr Newman space time</td>
</tr>
<tr>
<td>795</td>
<td>E. Knobloch, M.R.E. Proctor and N.O. Weiss</td>
<td>Finite-dimensional description of doubly diffusive convection</td>
</tr>
<tr>
<td>796</td>
<td>V.V. Pukhnachov</td>
<td>Mathematical model of natural convection under low gravity</td>
</tr>
<tr>
<td>797</td>
<td>M.C. Knaap</td>
<td>Existence and non-existence for quasi-linear elliptic equations with the $p$-laplacian involving critical Sobolev exponents</td>
</tr>
<tr>
<td>798</td>
<td>Stathis Filippas and Wenxiong Liu</td>
<td>On the blowup of multidimensional semilinear heat equations</td>
</tr>
<tr>
<td>799</td>
<td>A.M. Meirmanov</td>
<td>The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution</td>
</tr>
<tr>
<td>800</td>
<td>Bo Guan and Joel Spruck</td>
<td>Interior gradient estimates for solutions of prescribed curvature equations of parabolic type</td>
</tr>
<tr>
<td>801</td>
<td>Hi Jun Choe</td>
<td>Regularity for solutions of nonlinear variational inequalities with gradient constraints</td>
</tr>
<tr>
<td>802</td>
<td>Peter Shi and Yongzhi Xu</td>
<td>Quasistatic linear thermoelasticity on the unit disk</td>
</tr>
<tr>
<td>803</td>
<td>Satyanad Kichenassamy and Peter J. Olver</td>
<td>Existence and non-existence of solitary wave solutions to higher order model evolution equations</td>
</tr>
<tr>
<td>804</td>
<td>Dening Li</td>
<td>Regularity of solutions for a two-phase degenerate Stefan Problem</td>
</tr>
<tr>
<td>805</td>
<td>Marek Fila, Bernhard Kawohl and Howard A. Levine</td>
<td>Quenching for quasilinear equations</td>
</tr>
<tr>
<td>806</td>
<td>Yoshikazu Giga, Shun'ichi Goto and Hitoshi Ishii</td>
<td>Global existence of weak solutions for interface equations coupled with diffusion equations</td>
</tr>
<tr>
<td>807</td>
<td>Mark J. Friedman and Eusebius J. Doedel</td>
<td>Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study</td>
</tr>
<tr>
<td>808</td>
<td>Mark J. Friedman</td>
<td>Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds</td>
</tr>
<tr>
<td>809</td>
<td>Peter W. Bates and Songmu Zheng</td>
<td>Inertial manifolds and inertial sets for the phase-field equations</td>
</tr>
<tr>
<td>810</td>
<td>J. López Gómez, V. Márquez and N. Wolanski</td>
<td>Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition</td>
</tr>
<tr>
<td>811</td>
<td>Xinfu Chen and Fahuai Yi</td>
<td>Regularity of the free boundary of a continuous casting problem</td>
</tr>
<tr>
<td>813</td>
<td>Jose–Francisco Rodrigues and Boris Zaltzman</td>
<td>On classical solutions of the two-phase steady-state Stefan problem in strips</td>
</tr>
<tr>
<td>814</td>
<td>Viorel Barbu and Srdjan Stojanovic</td>
<td>Controlling the free boundary of elliptic variational inequalities on a variable domain</td>
</tr>
<tr>
<td>815</td>
<td>Viorel Barbu and Srdjan Stojanovic</td>
<td>A variational approach to a free boundary problem arising in electro-photography</td>
</tr>
<tr>
<td>816</td>
<td>B.H. Gilding and R. Kersner</td>
<td>Diffusion-convection-reaction, free boundaries, and an integral equation</td>
</tr>
</tbody>
</table>
Shoshana Kamin, Lambertus A. Peletier and Juan Luis Vazquez, On the Barenblatt equation of elasto-plastic filtration

Avner Friedman and Bei Hu, The Stefan problem with kinetic condition at the free boundary

M.A. Grinfeld, The stress driven instabilities in crystals: mathematical models and physical manifestations

Bei Hu and Lihe Wang, A free boundary problem arising in electrophotography: solutions with connected toner region

Yongzhi Xu, T. Craig Poling, and Trent Brundage, Direct and inverse scattering of time harmonic acoustic waves in an inhomogeneous shallow ocean

Steven J. Altschuler, Singularities of the curve shrinking flow for space curves

Steven J. Altschuler and Matthew A. Grayson, Shortening space curves and flow through singularities

Tong Li, On the Riemann problem of a combustion model

L.A. Peletier & W.C. Troy, Self-similar solutions for diffusion in semiconductors


Minkyu Kwak, Finite dimensional description of convective reaction-diffusion equations

Minkyu Kwak, Finite dimensional inertial forms for the 2D Navier-Stokes equations

Victor A. Galaktionov and Sergey A. Posashkov, On some monotonicity in time properties for a quasilinear parabolic equation with source

Victor A. Galaktionov, Remark on the fast diffusion equation in a ball

Hi Jun Choe and Lihe Wang, A regularity theory for degenerate vector valued variational inequalities

Vladimir I. Oliker and Nina N. Uraltseva, Evolution of nonparametric surfaces with speed depending on curvature, II. The mean curvature case.

S. Kamin and W. Liu, Large time behavior of a nonlinear diffusion equation with a source

Shoshana Kamin and Juan Luis Vazquez, Singular solutions of some nonlinear parabolic equations

Bernhard Kawohl and Robert Kersner, On degenerate diffusion with very strong absorption

Avner Friedman and Fernando Reitich, Parameter identification in reaction-diffusion models

E.G. Kalnin, H.L. Manocha and Willard Miller, Jr., Models of q-algebra representations I. Tensor products of special unitary and oscillator algebras

Robert J. Sacker and George R. Sell, Dichotomies for linear evolutionary equations in Banach spaces

Oscar P. Bruno and Fernando Reitich, Numerical solution of diffraction problems: a method of variation of boundaries

Oscar P. Bruno and Fernando Reitich, Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain

Victor A. Galaktionov and Juan L. Vazquez, Asymptotic behaviour for an equation of superslow diffusion. The Cauchy problem

Josephus Hulshof and Juan Luis Vazquez, The Dipole solution for the porous medium equation in several space dimensions

Shoshana Kamin and Juan Luis Vazquez, The propagation of turbulent bursts

Miguel Escobedo, Juan Luis Vazquez and Enrike Zuazua, Source-type solutions and asymptotic behaviour for a diffusion-convection equation

Marco Biroli and Umberto Mosco, Discontinuous media and Dirichlet forms of diffusion type

Stathis Filippas and Jong-Shenq Guo, Quenching profiles for one-dimensional semilinear heat equations

H. Scott Dumas, A Nehoroshev-like theory of classical particle channeling in perfect crystals

R. Natalini and A. Tesei, On a class of perturbed conservation laws

Paul K. Newton and Shinya Watanabe, The geometry of nonlinear Schrödinger standing waves

S.S. Sritharan, On the nonsmooth verification technique for the dynamic programming of viscous flow

Mario Taboada and Yuncheng You, Global attractor, inertial manifolds and stabilization of nonlinear damped beam equations

Shigeru Sakaguchi, Critical points of solutions to the obstacle problem in the plane

F. Abergel, D. Hilhorst and F. Issard-Roch, On a dissolution-growth problem with surface tension in the neighborhood of a stationary solution

Erasmus Langer, Numerical simulation of MOS transistors

Haim Brezis and Shoshana Kamin, Sublinear elliptic equations in \( \mathbb{R}^n \)

Johannes C.C. Nitsche, Boundary value problems for variational integrals involving surface curvatures

Chao-Nien Chen, Multiple solutions for a semilinear elliptic equation on \( \mathbb{R}^N \) with nonlinear dependence on the gradient

D. Brochet, X. Chen and D. Hilhorst, Finite dimensional exponential attractor for the phase field model

Joseph D. Fehribach, Mullins-Sekerka stability analysis for melting-freezing waves in helium-4

Walter Schempp, Quantum holography and neurocomputer architectures

D.V. Anosov, An introduction to Hilbert's 21st problem

Herbert E Huppert and M Grae Worster, Vigorous motions in magma chambers and lava lakes
<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>774</td>
<td>L.A. Peletier &amp; W.C. Troy</td>
<td>Self-similar solutions for infiltration of dopant into semiconductors</td>
</tr>
<tr>
<td>775</td>
<td>H. Scott Dumas and James A. Ellison</td>
<td>Nekhoroshev’s theorem, ergodicity, and the motion of energetic charged particles in crystals</td>
</tr>
<tr>
<td>776</td>
<td>Statth Filippas and Robert V. Kohn</td>
<td>Refined asymptotics for the blowup of $u_t - \Delta u = \nu^b$.</td>
</tr>
<tr>
<td>777</td>
<td>Patricia Bauman, Nicholas C. Owen and Daniel Phillips</td>
<td>Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity</td>
</tr>
<tr>
<td>778</td>
<td>Patricia Bauman, Nicholas C. Owen and Daniel Phillips</td>
<td>Maximal smoothness of solutions to certain Euler–Lagrange equations from nonlinear elasticity</td>
</tr>
<tr>
<td>779</td>
<td>Jack Carr and Robert Pego</td>
<td>Self-similarity in a coarsening model in one dimension</td>
</tr>
<tr>
<td>780</td>
<td>J.M. Greenberg</td>
<td>The shock generation problem for a discrete gas with short range repulsive forces</td>
</tr>
<tr>
<td>781</td>
<td>George R. Sell and Mario Taboada</td>
<td>Local dissipativity and attractors for the Kuramoto–Sivashinsky equation in thin 2D domains</td>
</tr>
<tr>
<td>782</td>
<td>T. Subba Rao</td>
<td>Analysis of nonlinear time series (and chaos) by bispectral methods</td>
</tr>
<tr>
<td>783</td>
<td>Nicholas Baumann, Daniel D. Joseph, Paul Mohr and Yuriko Renardy</td>
<td>Vortex rings of one fluid in another free fall</td>
</tr>
<tr>
<td>784</td>
<td>Oscar Bruno, Avner Friedman and Fernando Reitich</td>
<td>Asymptotic behavior for a coalescence problem</td>
</tr>
<tr>
<td>785</td>
<td>Johannes C.C. Nitsche</td>
<td>Periodic surfaces which are extremal for energy functionals containing curvature functions</td>
</tr>
<tr>
<td>786</td>
<td>F. Abergel and J.L. Bona</td>
<td>A mathematical theory for viscous, free-surface flows over a perturbed plane</td>
</tr>
<tr>
<td>787</td>
<td>Gunduz Caginalp and Xinu Chen</td>
<td>Phase field equations in the singular limit of sharp interface problems</td>
</tr>
<tr>
<td>788</td>
<td>Robert P. Gilbert and Yongzhi Xu</td>
<td>An inverse problem for harmonic acoustics in stratified oceans</td>
</tr>
<tr>
<td>789</td>
<td>Roger Fosdick and Eric Volkman</td>
<td>Normality and convexity of the yield surface in nonlinear plasticity</td>
</tr>
<tr>
<td>790</td>
<td>H.S. Brown, I.G. Kevrekidis and M.S. Jolly</td>
<td>A minimal model for spatio-temporal patterns in thin film flow</td>
</tr>
<tr>
<td>791</td>
<td>Chao–Nien Chen</td>
<td>On the uniqueness of solutions of some second order differential equations</td>
</tr>
<tr>
<td>792</td>
<td>Xinu Chen and Avner Friedman</td>
<td>The thermistor problem for conductivity which vanishes at large temperature</td>
</tr>
<tr>
<td>793</td>
<td>Xinu Chen and Avner Friedman</td>
<td>The thermistor problem with one-zero conductivity</td>
</tr>
<tr>
<td>794</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>Separation of variables for the Dirac equation in Kerr Newman space time</td>
</tr>
<tr>
<td>795</td>
<td>E. Knobloch, M.R.E. Proctor and N.O. Weiss</td>
<td>Finite-dimensional description of doubly diffusive convection</td>
</tr>
<tr>
<td>796</td>
<td>V.V. Pukhnachov</td>
<td>Mathematical model of natural convection under low gravity</td>
</tr>
<tr>
<td>797</td>
<td>M.C. Knaap</td>
<td>Existence and non-existence for quasi-linear elliptic equations with the p-laplacian involving critical Sobolev exponents</td>
</tr>
<tr>
<td>798</td>
<td>Statth Filippas and Wenxiong Liu</td>
<td>On the blowup of multidimensional semilinear heat equations</td>
</tr>
<tr>
<td>799</td>
<td>A.M. Meirmanov</td>
<td>The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution</td>
</tr>
<tr>
<td>800</td>
<td>Bo Guan and Joel Spruck</td>
<td>Interior gradient estimates for solutions of prescribed curvature equations of parabolic type</td>
</tr>
<tr>
<td>801</td>
<td>Hi Jun Choe</td>
<td>Regularity for solutions of nonlinear variational inequalities with gradient constraints</td>
</tr>
<tr>
<td>802</td>
<td>Peter Shi and Yongzhi Xu</td>
<td>Quasistatic linear thermoelasticity on the unit disk</td>
</tr>
<tr>
<td>803</td>
<td>Satyanad Kichenassamy and Peter J. Olver</td>
<td>Existence and non-existence of solitary wave solutions to higher order model evolution equations</td>
</tr>
<tr>
<td>804</td>
<td>Dening Li</td>
<td>Regularity of solutions for a two-phase degenerate Stefan Problem</td>
</tr>
<tr>
<td>805</td>
<td>Marek Fila, Bernhard Kawohl and Howard A. Levine</td>
<td>Quenching for quasilinear equations</td>
</tr>
<tr>
<td>806</td>
<td>Yoshikazu Giga, Shun’ichi Goto and Hitoshi Ishii</td>
<td>Global existence of weak solutions for interface equations coupled with diffusion equations</td>
</tr>
<tr>
<td>807</td>
<td>Mark J. Friedman and Eusebius J. Doedel</td>
<td>Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study</td>
</tr>
<tr>
<td>808</td>
<td>Mark J. Friedman</td>
<td>Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds</td>
</tr>
<tr>
<td>809</td>
<td>Peter W. Bates and Songmu Zheng</td>
<td>Inertial manifolds and inertial sets for the phase-field equations</td>
</tr>
<tr>
<td>810</td>
<td>J. López Gómez, V. Márquez and N. Wolanski</td>
<td>Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition</td>
</tr>
<tr>
<td>811</td>
<td>Xinu Chen and Fahuai Yi</td>
<td>Regularity of the free boundary of a continuous casting problem</td>
</tr>
<tr>
<td>812</td>
<td>Eden, A., Foias, C., Nicolaenko, B. and Temam, R.</td>
<td>Inertial sets for dissipative evolution equations Part I: Construction and applications</td>
</tr>
<tr>
<td>813</td>
<td>Jose–Francisco Rodrigues and Boris Zaltzman</td>
<td>On classical solutions of the two-phase steady-state Stefan problem in strips</td>
</tr>
<tr>
<td>814</td>
<td>Viorel Barbu and Srdjan Stoianovic</td>
<td>Controlling the free boundary of elliptic variational inequalities on a variable domain</td>
</tr>
<tr>
<td>815</td>
<td>Viorel Barbu and Srdjan Stoianovic</td>
<td>A variational approach to a free boundary problem arising in electro-photography</td>
</tr>
<tr>
<td>816</td>
<td>B.H. Gilding and R. Kersner</td>
<td>Diffusion-convection-reaction, free boundaries, and an integral equation</td>
</tr>
</tbody>
</table>
Shoshana Kamin, Lambertus A. Peletier and Juan Luis Vazquez, On the Barenblatt equation of elastoplastic filtration
Avner Friedman and Bei Hu, The Stefan problem with kinetic condition at the free boundary
M.A. Grinfeld, The stress driven instabilities in crystals: mathematical models and physical manifestations
Bei Hu and Lihe Wang, A free boundary problem arising in electrophotography: solutions with connected toner region
Yongzhi Xu, T. Craig Poling, and Trent Brundage, Direct and inverse scattering of time harmonic acoustic waves in an inhomogeneous shallow ocean
Steven J. Altschuler, Singularities of the curve shrinking flow for space curves
Steven J. Altschuler and Matthew A. Grayson, Shortening space curves and flow through singularities
Tong Li, On the Riemann problem of a combustion model
L.A. Peletier & W.C. Troy, Self-similar solutions for diffusion in semiconductors
Minkyu Kwak, Finite dimensional description of convective reaction-diffusion equations
Minkyu Kwak, Finite dimensional inertial forms for the 2D Navier–Stokes equations
Victor A. Galaktionov and Sergey A. Posashkov, On some monotonicity in time properties for a quasilinear parabolic equation with source
Victor A. Galaktionov, Remark on the fast diffusion equation in a ball
Hi Jun Choe and Lihe Wang, A regularity theory for degenerate vector valued variational inequalities
Vladimir I. Oliker and Nina N. Uraltseva, Evolution of nonparametric surfaces with speed depending on curvature. II. The mean curvature case.
S. Kamin and W. Liu, Large time behavior of a nonlinear diffusion equation with a source
Shoshana Kamin and Juan Luis Vazquez, Singular solutions of some nonlinear parabolic equations
Bernhard Kawohl and Robert Kersner, On degenerate diffusion with very strong absorption
Avner Friedman and Fernando Reitich, Parameter identification in reaction-diffusion models
E.G. Kalnins, H.L. Manocha and Willard Miller, Jr., Models of q-algebra representations I. Tensor products of special unitary and oscillator algebras
Robert J. Sacker and George R. Sell, Dichotomies for linear evolutionary equations in Banach spaces
Oscar P. Bruno and Fernando Reitich, Numerical solution of diffraction problems: a method of variation of boundaries
Oscar P. Bruno and Fernando Reitich, Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain
Victor A. Galaktionov and Juan L. Vazquez, Asymptotic behaviour for an equation of superslow diffusion. The Cauchy problem
Josephus Hulshof and Juan Luis Vazquez, The Dipole solution for the porous medium equation in several space dimensions
Shoshana Kamin and Juan Luis Vazquez, The propagation of turbulent bursts
Miguel Escobedo, Juan Luis Vazquez and Enrike Zuazua, Source-type solutions and asymptotic behaviour for a diffusion-convection equation
Marco Biroli and Umberto Mosco, Discontinuous media and Dirichlet forms of diffusion type
Statthias Filippas and Jong-Shenq Guo, Quenching profiles for one-dimensional semilinear heat equations
H. Scott Dumas, A Nekhoroshev-like theory of classical particle channeling in perfect crystals
R. Natalini and A. Tesei, On a class of perturbed conservation laws
Paul K. Newton and Shinya Watanabe, The geometry of nonlinear Schrödinger standing waves
S.S. Sritharan, On the nonsmooth verification technique for the dynamic programming of viscous flow
Mario Taboada and Yuncheng You, Global attractor, inertial manifolds and stabilization of nonlinear damped beam equations
Shigeru Sakaguchi, Critical points of solutions to the obstacle problem in the plane
F. Abergel, D. Hilhorst and F. Issard-Roch, On a dissolution-growth problem with surface tension in the neighborhood of a stationary solution
Erasmus Langer, Numerical simulation of MOS transistors
Haim Brezis and Shoshana Kamin, Sublinear elliptic equations in $\mathbb{R}^n$
Johannes C.C. Nitsche, Boundary value problems for variational integrals involving surface curvatures
Chao–Nien Chen, Multiple solutions for a semilinear elliptic equation on $\mathbb{R}^N$ with nonlinear dependence on the gradient
D. Brochet, X. Chen and D. Hilhorst, Finite dimensional exponential attractor for the phase field model
Joseph D. Fehribach, Mullins-Sekerka stability analysis for melting-freezing waves in helium-4
Walter Schempp, Quantum holography and neurocomputer architectures
D.V. Anosov, An introduction to Hilbert's 21st problem
Herbert E Huppert and M Grae Worster, Vigorous motions in magma chambers and lava lakes