SOURCE-TYPE SOLUTIONS AND ASYMPTOTIC BEHAVIOUR FOR A DIFFUSION-CONVECTION EQUATION

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Introduction

In this paper we consider the equation

\[ u_t = u_{xx} - u^{q-1}u_x \]

posed in the domain \( Q = \{(x,t) : x \in \mathbb{R}, t > 0\} \). The exponent \( q \) will be a constant larger than 1. This equation represents a very simple model of diffusion and convection, as proposed for instance in [RK], [B] or [P].

It is known that for every \( u_0 \in L^1(\mathbb{R}) \) there exists a unique solution \( u \in C^\infty(Q) \) of equation (E) such that \( u(\cdot,t) \to u_0 \) in \( L^1(\mathbb{R}) \). Moreover, physical considerations lead us to assume that \( u_0(x) \geq 0 \) in which case \( u(x,t) \geq 0 \) in \( Q \). It is a basic rule of parabolic equations that the properties of general classes of solutions can be explained in terms of the properties of some special solutions. The most important of such particular solutions is the one corresponding to initial data a Dirac mass, \( M\delta(x) \), namely a solution \( u(x,t) \) such that for every test function \( \phi \in C^\infty(\mathbb{R}) \), \( \phi \) bounded,

\[ \int u(x,t)\phi(x)\,dx \to M\phi(0) \quad \text{as} \quad t \to 0. \]

Such a solution is called a fundamental solution in the linear theory and we will retain this name here; sometimes it is also called a source-type solution. It is well-known that fundamental solutions play a crucial role in describing the large-time behaviour of the class of solutions of parabolic linear equations whose initial data decay fast enough as \( |x| \to \infty \). It has been recently proved that the same phenomenon is true for important examples of nonlinear equations.

The asymptotic behaviour of the solutions of (E) as \( t \to \infty \) was been studied by Escobedo and Zuazua [EZ] when the exponent \( q \) is critical, \( q = 2 \), or supercritical, \( q > 2 \). In the former case a explicit self-similar solution exists, which in one space dimension is explicitly given by the formula

\[ \hat{U}(x,t) = t^{-1/2} F(x/t^{1/2}) \quad \text{with} \quad F(\eta) = \frac{e^{-\eta^2/4}}{K + \frac{1}{2} \int_0^\eta e^{-\xi^2/4} \,d\xi}, \]

for some \( K > 0 \), a function of \( M \). \( \hat{U} \) is the unique fundamental solution of (E) for \( q = 2 \). It represents the asymptotic behaviour of all solutions of (E) with integrable data in this critical case.

The situation is a bit different for \( q > 2 \). In this case the effect of the convection term completely disappears from the asymptotic form of the solution, and the solutions look for large times like the fundamental solution of the heat equation with initial data, \( M\delta(x) \) with \( M = \int u_0(x)\,dx \).
We study in this paper the asymptotic behaviour of the solutions of (E) in the subcritical exponent range, $1 < q < 2$. We prove that the solutions with integrable data look as $t \to \infty$ like the entropy solution of the purely convective equation

\[(C) \quad u_t + (u^q/q)_x = 0\]

with initial data $M\delta(x)$, and the effect of the diffusive term $u_{xx}$ completely disappears in the limit. This completes the study of the asymptotic behaviour for equation (E). We recall that solutions of such conservation laws are in general discontinuous and that uniqueness holds only for a special class of solutions called entropy solutions, see [L], [D], [O]. Let us introduce the fundamental solution of the convective equation (C): according to [LP] there exists only one entropy solution of (E) with initial data $M\delta(x)$ and is given by the formula

\begin{equation}
U(x,t;M) = \begin{cases} 
(x/t)^{1/(q-1)} & \text{if} \quad 0 < x < r(t) \\
0 & \text{otherwise},
\end{cases}
\end{equation}

with

\begin{equation}
r(t) = cM^{(q-1)/q}t^{1/q} \quad \text{and} \quad c = \left(\frac{q}{q-1}\right)^{\frac{q-1}{q}}.
\end{equation}

(Function of this form usually appearing in conservation laws are called $N$-wave profiles: actually, in this case we find only a half of an inverted $N$ and that for the variable $u^{q-1}$). We will prove that as $t \to \infty$ the solution $u$ increasingly resembles $U$. Convergence holds in all spaces $L^p(\mathbb{R})$, $1 \leq p < \infty$ (with appropriate scale factor depending on $t$ if $p > 1$, see below). It is natural to look for uniform convergence. However, since $U$ is discontinuous, the convergence cannot be uniform near the discontinuity (shock line), because this would preclude any deviation in the position of the maxima of both functions to be compared, an unreasonable requirement. In fact, a usual recourse in such situations arising frequently in conservation laws is to measure graph deviation in the appropriate rescaled variables. Therefore, we introduce the new variables

\begin{equation}
\bar{u} = t^{\alpha}u, \quad \bar{U} = t^{\alpha}U, \quad \text{and} \quad y = \frac{x}{t^{1/q}},
\end{equation}

where $\alpha = 1/q$. Observe that $\bar{U}$ is stationary, i.e. time independent (given by $y^{1/(q-1)}$ for $0 < y < y_0$, $y_0 = y_0(q, M) > 0$). Our asymptotic result can be stated as follows.

**Theorem 1.** Let $u$ be a nonnegative solution of equation (E) with initial data $u_0 \in L^1(\mathbb{R})$ of mass $M > 0$, and assume that $1 < q < 2$. Then, as $t \to \infty$

\begin{equation}
\|u(\cdot,t) - U(\cdot,t;M)\|_1 \to 0.
\end{equation}

Moreover, with the notations introduced in (0.4),

\begin{equation}
\bar{u}(y,t) \to \bar{U}(y) \quad \text{as} \quad t \to \infty,
\end{equation}

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and the convergence is uniform in the sense of graphs in \( \mathbb{R}^2 \).

We would like to remind the reader that this graph convergence is equivalent to uniform convergence away from the shock line of \( U \) plus a control from above for the maxima of the family \( \{u(-,t)\} \). On the other hand, it is clear that from this statement one can derive convergence statements in \( L^p \)-norms for every \( p > 1 \), with decay rate \( o(t^{\alpha_p}) \) with \( \alpha_p = (p - 1)/pq \).

A second subject of the paper is the existence and properties of fundamental solutions to the whole equation (E). In Section 3 we construct fundamental solutions for equation (E) for all exponents \( q > 1 \). We have

**Theorem 2.** For every \( q > 1 \) equation (E) admits a unique fundamental solution. In other words there exists a unique nonnegative function \( u(x,t) \in C((0,\infty) : L^1(\mathbb{R})) \cap C^\infty(Q) \) which solves (E) in the classical sense and takes on the initial data \( M\delta \) in the weak sense (0.1).

The existence of a fundamental solution was shown in [AE] in the range \( 1 < q < 2 \). The uniqueness of such solution was an open problem even in that range. It is interesting to remark that the fundamental solution in the range \( 1 < q < 2 \) converges as \( t \to 0 \) to the fundamental solution of the heat equation with the same mass. In other words, the fundamental solution can be viewed as a curve \( t \mapsto u(\cdot,t) \) in the space \( L^1(\mathbb{R}) \), which evolves in a continuous way from the heat equation kernel \( (t = 0) \) to the \( N \)-wave profile (0.3). More graphically, after a convenient rescaling it goes from the exponential profile \( \exp(y^2/4) \) to the \( N \)-wave profile \( U(y) \).

The plan of the paper is as follows. We devote Section 1 to state and prove the necessary estimates for the solutions of (E), in particular the entropy inequality

\[
(u^{q-1})_t \leq \frac{1}{t},
\]

which pays a crucial role in establishing the convergence to an \( N \)-wave. The asymptotic result, Theorem 1, is proved in Section 2, while Section 3 is devoted to the existence and uniqueness of fundamental solutions. Section 4 contains some properties of the fundamental solutions. Section 5 comments on the case \( q = 1 \) and the curiously different asymptotic behaviour of the solutions for \( q = 1 \) and \( q > 1 \), \( q \approx 1 \). Finally, we pay some attention to possible extensions: the restriction of nonnegativity on the solutions is eliminated in Section 6, while Section 7 considers the equation \( u_t = u_{xx} - \phi(u)_x \) under suitable assumptions on the continuous function \( \phi \).

1. Estimates

It is well known that for smooth initial data \( u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) there exists a unique classical solution \( u \) of equation (E) which is \( C^\infty \) smooth in \( Q \). Moreover, if \( u_0 \geq 0 \) then
$u$ is positive in $Q$. Using estimates as those given below plus standard quasilinear theory we can extend the class of initial data. Thus we obtain for every $u_0 \in L^1(\mathbb{R})$, $u_0 \geq 0$, a unique solution $u \in C((0, \infty) : L^1(\mathbb{R})) \cap C^\infty(Q)$.

We recall for convenience a number of well-known basic estimates that we shall be using. Some of the typical integral estimates of the heat equation apply literally to equation (E), since the convection term produces a contribution of the form $\int (u^r)_x \, dx$, which is 0 for any solution as above. Thus, integration with respect to $x$ in $\mathbb{R}$ gives the conservation of the integral $\int u(x, t) \, dt$, usually called the mass

\begin{equation}
\frac{d}{dt} \int u(x, t) \, dx = 0.
\end{equation}

Therefore we have for every $t > 0$ and every $n$

\begin{equation}
\int u_n(x, t) \, dx = M \quad \text{(conservation of mass)}.
\end{equation}

Besides, by the standard technique of multiplying by a sign function and integrating we get the following result on continuous dependence of solution with respect to the initial data: for any two solutions $u$ and $\hat{u}$ we have

\begin{equation}
\|u(\cdot, t) - \hat{u}(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u(\cdot, 0) - \hat{u}(\cdot, 0)\|_{L^1(\mathbb{R})}.
\end{equation}

Moreover, for every $p \in (1, \infty)$ and every $0 < \tau < t$ we have

\begin{equation}
\int |u(x, t)|^p \, dx + \frac{4(p - 1)}{p} \int_0^t \int_S |\nabla u|^{p/2} \, dx \, dt = \int |u(x, \tau)|^p \, dx
\end{equation}

where $S = \mathbb{R} \times (\tau, t)$. By using Sobolev's inequality and integrating the resulting ODE for $f(t) = \int |u(x, t)|^p \, dx$ we get for every $t > 0$

\begin{equation}
\int |u(x, t)|^p \, dx \leq C(t^{-\frac{p-1}{2}}),
\end{equation}

see [EZ]. Regarding the bound in sup norm, it is proved in [EZ] that for $q > 2$ there exists a constant $C = C(M)$ such that

\begin{equation}
u(x, t) \leq Ct^{1/2}
\end{equation}

By scaling considerations we can write $C$ as $cM$, $c > 0$. In view of the asymptotic results of [EZ] which show that for $q > 2$ solutions will tend to the fundamental solution of the heat equation and for $q = 2$ to some self-similar profile, these estimates are sharp in the range $q \geq 2$.

However, similar estimates are not be sharp for $1 < q < 2$, since according to Theorem 1 we expect the effect of convection to be dominant over that of diffusion for large times in that exponent range, thus giving a decay rate of the sup norm of the solution like $t^{-1/q}$, as in equation (C). We derive here below the necessary estimates in the range $1 < q < 2$.

The main estimate is a bound for the derivative $(u^{q-1})_x$ which is copied from the so-called entropy estimates known to be valid for the purely convective equation. Observe that the estimate is exact for the self-similar solution $U$ of (0.3).
Lemma 1.1. If $1 < q < 2$ we have for every solution as above

(1.5) \[(u^{q-1})_x \leq \frac{1}{t}\].

Proof: In order to perform our estimate we need to approximate our solution $u$ by uniformly positive and bounded solutions. Therefore, we ask that $u_0 \in C^\infty(\mathbb{R})$ and $0 < \varepsilon < u_0(x) \leq N$. Once estimate (1.5) is proved for such solutions, it will hold for our setting by a simple approximation and limit process.

We define the new variable $z = u^{q-1}$. In terms of $z$ equation (E) reads

(1.6) \[z_t + z z_x - \beta z^2_x = z_{xx}\]

where $\beta = (2 - q)/(q - 1)$. Differentiating in $x$ and putting $w = z_x$ we get

(1.7) \[w_t - w_{xx} + [z - 2\beta \frac{w}{z}]w_x + w^2 + \beta \frac{w^3}{z^2} = 0.\]

Now we observe that for $1 < q < 2$ we have $\beta > 0$, hence the last term in the first member of (1.7) has the same sign as $w$. We can verify that the function

\[W(t) = \frac{1}{t}\]

which is an exact solution of the ODE: $w_t + w^2 = 0$, is also a supersolution for equation (1.7). Since $W(0) = \infty$ we conclude from the Maximum Principle that $w(x, t) \leq W(t)$ for every $(x, t) \in Q$. #

From this estimate and the mass estimate we get a bound for $u$.

Lemma 1.2. For $1 < q < 2$ we have

(1.8) \[0 \leq u(x, t) \leq \left(\frac{qM}{(q - 1)t}\right)^{1/q} .\]

Proof: Let us fix a time $t > 0$ and let us consider a point $x_0$ and put $A = u(x_0, t)$. Then the value of $v = u^{q-1}$ is $B = A^{q-1}$. Thanks to Lemma 1.1 we have to the left of $x_0$

(1.9) \[u^{q-1}(x, t) \geq B - \frac{1}{t}(x_0 - x),\]

hence $u$ will be positive in the interval $(x_1, x_0)$ with $x_1 = x_0 - Bt$ and (1.9) can be written as

\[u^{q-1}(x, t) \geq \frac{1}{t}(x - x_1) \quad \text{for} \quad 0 \leq x - x_1 \leq Bt.\]

Integration of this inequality gives

\[M = \int_R u(x, t) \, dx \geq \int_0^{Bt} (x/t)^{1/(q-1)} \, dx = \frac{q-1}{q} A^q t,\]

From this (1.8) follows. #

We note another immediate consequence of Lemma 1.1.
LEMMA 1.3. Everywhere in $Q$ we have if $1 < q < 2$

\begin{equation}
(u^q)_x \leq \frac{qu}{(q-1)t},
\end{equation}

while for every $t > 0$

\begin{equation}
\int_{\mathbb{R}} |(u^q)_x| \, dx \leq \frac{2qM}{(q-1)t}.
\end{equation}

PROOF: (1.10) is immediate, while for (1.11) we use (1.10) and the equations $\int u(x, t) \, dx = M$ and $\int (u^q)_x \, dx = 0$. #

Finally we get an energy estimate for $u_x$ by multiplying equation (E) by $u$ and integrating by parts.

LEMMA 1.4. For every $0 < \tau < T$ we have

\begin{equation}
\int_{\tau}^{T} \int |u_x|^2 \, dx \, dt \leq \frac{1}{2} \int u^2(\tau, x) \, dx \leq CM^{(q-1)/q} T^{-1/q}
\end{equation}

if $1 < q < 2$. #

2. Asymptotic behaviour

In this section we prove Theorem 1, thus establishing the asymptotic behaviour of solutions to equation (E) in the subcritical case $1 < q < 2$. The proof will be divided for convenience into various steps which represent the different ideas needed to complete the result.

Step 1. Rescaling. Based on the expected decay rate that corresponds to estimate (1.8) we introduce a scaling transformation and define for every $\lambda > 0$ a new function

\begin{equation}
u_{\lambda}(x, t) = \lambda u(\lambda x, \lambda^q t).
\end{equation}

This function is not a solution of (E) but it satisfies the equation

\begin{equation}u_{\lambda,t} + u_{\lambda}^{q-1} u_{\lambda,x} = \lambda^{q-2} u_{\lambda,xx}
\end{equation}

with initial data

\begin{equation}u_{\lambda}(x, 0) = \lambda u_0(\lambda x).
\end{equation}

We immediately see that the mass is conserved in the transformation and that as $\lambda \to \infty$ the initial function converges to $M$ times the Dirac delta.
On the other hand, for $\lambda$ very large we can view equation (2.2) as a small perturbation of the convective equation (C): $u_t + u^{q-1}u_x = 0$. The proof of Theorem 2 will consist precisely in showing that in the limit $\lambda \to \infty$ we find the function $U$ given in (0.3) as limit of the family $u_\lambda$. Indeed, note that

\[(2.4) \quad u_\lambda(x,1) - U(x,1) = \lambda u(\lambda x, \lambda^q) - U(x,1) = \tau^{1/q}|u(y,\tau) - U(y/\tau)|\]

(where $y = \lambda x$ and $\tau = \lambda^q$). Therefore, if $u_\lambda(x,1)$ converges to $U(x,1)$ as $\lambda \to \infty$ in some norm, then we obtain the result (0.3) in the same norm.

The same is true if we replace $t = 1$ by any fixed $t_0 > 0$.

**Step 2. Passage to the limit.** It is easy to check that our main estimates, namely (1.8) on $u$, (1.5) on $(u^{q-1})_x$, and (1.10), (1.11) on $(u^q)_x$, as well as the mass estimate (1.2), are valid for all the functions $u_\lambda$ and indeed the constants are unchanged. On the other hand estimate (1.12) becomes

\[(2.5) \quad \int_\tau^T \int |u_{\lambda,x}|^2 \, dx \, dt \leq \frac{\lambda^{2-q}}{2} \int u_{\lambda}^2(\tau, x) \, dx = O(\lambda^{2-q}).\]

In order to understand the limit it will be useful to introduce a new family of variables:

\[(2.6) \quad v_\lambda(x,t) = \int_{-\infty}^x u_\lambda(y,t) \, dy.\]

The functions $v_\lambda$ will be bounded,

\[(2.7) \quad 0 \leq v_\lambda(x,t) \leq M,\]

because of the mass estimate. Clearly $v_{\lambda,x} = u_\lambda \geq 0$ so that we have another uniform estimate of the form

\[(2.8) \quad |v_{\lambda,x}(x,t)| \leq C t^{-1/q}.\]

Moreover, integration of (2.2) gives

\[(2.9) \quad v_{\lambda,t} = \lambda^{q-2} v_{\lambda,x} - \frac{1}{q} (v_{\lambda,x})^q = \lambda^{q-2} u_{\lambda,x} - \frac{1}{q} (u_\lambda)^q \text{ in } Q.\]

Therefore, by (2.5), $v_{\lambda,t}$ is uniformly bounded in $L^2_{loc}(Q)$. Using standard compactness results we may thus select a sequence $\lambda_j$ going to infinity so that the sequence $v_{\lambda_j}$ converges to a function $\bar{v}$ almost everywhere in $Q$ and locally in $L^p$ for any $p < \infty$.

For every fixed $t_0 > 0$ the family $u_{\lambda_j}^q(\cdot, t_0)$ is uniformly bounded in $W^{1,1}(\mathbb{R})$. This allows us to pass to the limit along a possibly finer subsequence and obtain a limit $\bar{w}(\cdot, t_0)$ locally in $L^p(\mathbb{R})$ for every $p < \infty$ and almost everywhere. It follows that for the same subsequence
$u_\lambda(\cdot, t_0)$ converges to $\overline{u}(\cdot, t_0) = (\overline{w}(\cdot, t_0))^{1/q}$ in the same sense. It is now immediate that $\overline{u}$ must be the derivative of $\overline{v}$, namely

$\overline{u}(x, t_0) = \overline{v}_x(x, t_0)$ \hspace{1em} for a.e. \hspace{0.5em} x \in \mathbb{R}.

(2.10)

Since the limit of $u^q_\lambda$ has been identified independently of the subsequence, the whole sequences $u_{\lambda_j}$ and $u^q_{\lambda_j}$ converge for every fixed $t_0 > 0$ to the functions $\overline{u}$ and $\overline{w}$ resp. These functions will have the bounds established above for the sequences.

We are now in a position to pass to the weak limit in the equation. Indeed, we have for every $0 < \tau < t$ and for every test function $\phi \in C_0^\infty(\mathbb{R})$

$$\int u_\lambda(x, t)\phi(x)\,dx - \int u_\lambda(x, \tau)\phi(x)\,dx$$

(2.11)

$$= \lambda^{q-2} \int_\tau^t \int_\mathbb{R} u_\lambda \phi_{xx} \,dx\,dt + \frac{1}{q} \int_\tau^t \int_\mathbb{R} u^q_\lambda \phi_x \,dx\,dt,$$

which in the limit gives

$$\int \overline{u}(x, t)\phi(x)\,dx - \int \overline{u}(x, \tau)\phi(x)\,dx = \frac{1}{q} \int_\tau^t \int_\mathbb{R} \overline{u}^q \phi_x \,dx\,dt. \quad (2.12)$$

(we can use the Dominated Convergence Theorem in the last integral). Equation (2.12) is a way of stating that $\overline{u}$ satisfies equation (C) in the sense of distributions.

**Step 3. Initial conditions.** We multiply equation (2.2) by a $C^\infty$-smooth test function $\phi(x)$ with compact support and such that $0 \leq \phi(x) \leq 1$. Integration in $S = \mathbb{R} \times (0, t)$ gives

$$\left| \int_\mathbb{R} u_\lambda(x, t)\phi(x)\,dx - \int_\mathbb{R} u_\lambda(x, 0)\phi(x)\,dx \right|$$

$$= \lambda^{q-2} \int_S u_\lambda(x, t)\phi_{xx}(x)\,dx + \frac{1}{q} \int_S u^q_\lambda(x, t)\phi_x(x)\,dx$$

$$\leq \lambda^{q-2} \|\phi_{xx}\|_\infty M t + \frac{1}{q} \|\phi_x\|_\infty \int_0^t \|u(\cdot, t)\|_1 \|u(\cdot, t)\|^{q-1}_\infty \,dt.$$
i.e. \( \lim_{t \to 0} \tilde{u}(x, t) = M\delta(x) \) in the weak sense of bounded nonnegative measures in \( \mathbb{R} \).

**Step 4. Identification of the limit.** We have thus obtained a limit function \( \tilde{u}(x, t) \in C((0, \infty) : L^1(\mathbb{R})) \cap L^\infty(\mathbb{R} \times (\tau, \infty)) \) which is a solution of equation (C) in the sense of distributions and takes on initial data \( M\delta(x) \). Since weak solutions of conservation laws like (C) are not unique, in order to be able to conclude that \( \tilde{u} \) coincides with the explicit solution \( U \) given in (0.3) we have to make sure that it satisfies a form of the entropy conditions. This is guaranteed by the first-order inequality

\[
(\tilde{u}^t)_x \leq C\tilde{u}/t
\]

which follows from (1.10) upon passing to the limit. This type of condition has been introduced in conservation laws by Oleinik [O]. Since (2.2) is a viscosity approximation to (C) it can also be checked that for every entropy pair in the sense used in [D], the corresponding inequality holds, or alternatively that Kruzhkov's integral conditions [K] hold. The uniqueness of the fundamental solution of (C) satisfying Kruzhkov's integral conditions is shown in [LP]. Finally, observe that since the limit has been uniquely identified, the whole family \( u_\lambda \) converges to \( U \).

**Step 5. Convergence in \( L^1(\mathbb{R}) \).** We have shown that \( u_\lambda \) converges locally in \( L^p \) to \( \tilde{u} \). It follows that the mass of \( \tilde{u} \) is equal or less than \( M \). But we have also proved that \( \tilde{u} \) is \( U \), a fundamental solution with initial data \( M\delta \), hence its mass is \( M \) for all times since conservation of mass holds for equation (C).

We can now prove that \( u_\lambda \) converges to \( U \) in \( L^1(\mathbb{R}) \). The argument is as follows: Fix a positive time, say \( t = 1 \). Then \( U \) is supported in a bounded subset of \( \mathbb{R} \), say in \([-r, r]\). We know that given \( \varepsilon > 0 \) there exists \( \lambda_0 \) such that for \( \lambda > \lambda_0 \)

\[
\int_{-r}^{r} |u_\lambda(x, 1) - U(x, 1)| \, dx \leq \varepsilon.
\]

Since \( \int u_\lambda(x, 1) \, dx = \int U(x, 1) \, dx = M \), this means that \( \int_{-r}^{r} u_\lambda(x, 1) \, dx \geq M - \varepsilon \), hence

\[
\int_{\{|x| > r\}} u(x, 1) \, dx \leq \varepsilon.
\]

Both (2.17) and (2.18) imply that

\[
\int_{\mathbb{R}} |u_\lambda(x, 1) - U(x, 1)| \, dx \leq 2\varepsilon.
\]

**Step 6. Convergence in sup norm in the sense of graphs.** We know that the sequence \( u_\lambda \) converges to \( U \) the fundamental solution of (C), in \( L^1_{loc}(\mathbb{R}) \) for fixed \( t \) that we can assume to be 1. We also know that the masses of \( u \) and \( U \) are the same, \( M > 0 \). Convergence of \( u_\lambda(\cdot, 1) \) towards \( U(\cdot, 1) \) in sup-graph norm is a consequence of the following calculus Lemma, which shows that \( U \) satisfies a certain extremal property among the nonnegative functions which satisfy estimate (1.10).
Lemma 2.1. \( f \in L^1(\mathbb{R}) \) be a nonnegative function such that \( \int f(x) \, dx = 1 \) and

\[
(f^q)_x \leq 1, \quad \text{where} \quad q > 1.
\]

Let \( F \) be the function defined as

\[
F(x) = \begin{cases} x^{1/q} & \text{for} \quad 0 \leq x \leq r \\ \text{otherwise}, \end{cases}
\]

where \( r > 0 \) is determined so that \( F \) satisfies \( \int F(x) \, dx = 1 \). Assume furthermore that

\[
\int_{-r}^{2r} |f(x) - F(x)| \, dx \leq \varepsilon
\]

Then the distance between the graphs of \( f \) and \( F \) can be estimated as a power of \( \varepsilon \).

The proof is just a calculus exercise. We consider a point \( x_0 \) where the graphs of \( f \) and \( F \) are at some distance \( d > 0 \). Then, thanks to inequality (2.18) and the fact that \( F \) satisfies (2.18) with equality sign, at least on one side of \( x_0 \) the functions will differ \( d/2 \) in a certain interval which is a function of \( d \). This allows to evaluate from below the integral (2.20) as a function of \( d \) if \( |x_0| \leq 3/2r \). For \( |x_0| > 3r/2 \) we evaluate the integral \( \int f(x) \, dx \) where \( I = \{x : |x| \geq 1\} \). As a consequence of our assumptions this integral has to be less than \( 2\varepsilon \). The details are left to the reader.

3. Existence and uniqueness of fundamental solutions

1. Existence. When \( q = 2 \) a fundamental solution is given by formula (0.2). We are thus left with the cases \( q > 2 \) and \( 1 < q < 2 \). The proof is divided in a series of steps.

(i) Approximation. Let \( \phi_n \in C_0^\infty(\mathbb{R}) \) be an approximation of the identity, for instance \( \phi_n(x) = n \phi(nx) \) where \( \phi \) is smooth, nonnegative \( C^\infty \) function with compact support such that \( \int \phi(x) \, dx = 1 \). It is known that for given \( M > 0 \) there exists a unique classical solution \( u_n(x,t) \) of equation (E) with initial data \( M\phi_n \), see for instance [AE], and the functions \( u_n(x,t) \) are positive and bounded in \( Q \).

(ii) Passage to the limit. In view of the estimates of Section 1 the family \( u_n \) is uniformly bounded in in any region \( S = \mathbb{R}^N \times (\tau, \infty) \). By standard quasilinear theory [LSU] our family will be uniformly bounded in \( C^{2+\alpha,1+\alpha/2}(K) \) for some \( \alpha > 0 \) and every \( K \) compact subset of \( Q \). Therefore, we can pass to the limit \( n \to \infty \) (along a suitable subsequence) and obtain a function \( u(x,t) \geq 0 \) in the same spaces, which is a classical solution of (E) in \( Q \). It follows easily that \( u \) is necessarily positive everywhere.

(iii) The initial data. We have to make sure that \( u \) takes on \( M\delta(x) \) as initial data. There are two ways in which we can justify that. The simplest consists in multiplying the equation
satisfied by the approximate solutions by a test function \( \zeta(x) \in C_0^\infty(\mathbb{R}) \) and integrate to obtain

\[
\int u_n(x,t)\zeta(x)\,dx = \int u_{0n}(x)\zeta(x)\,dx \\
+ \int \int u_n(x,t)\zeta_{xx}(x)\,dx\,dt + \int \int u_n^q(x,t)\zeta_x(x,t)\,dx\,dt. 
\]

(3.1)

Assume that \( 1 < q < 2 \). Using the conservation of mass plus estimate (1.8) we get

\[
| \int u_n(x,t)\zeta(x)\,dx - \int u_{0n}(x)\zeta(x)\,dx | \leq M\|\zeta_{xx}\|_\infty t + M\|\zeta_x\|_\infty \int_0^t (\sup u(\cdot,t))^{q-1}\,dt = C_1 t + C_2 t^{1/q}. 
\]

and passing to the limit \( n \to \infty \)

(3.2)

\[
| \int u(x,t)\zeta(x)\,dx - M\zeta(0) | \leq C_1 t + C_2 t^{1/q}. 
\]

Letting \( t \to 0 \) we get the desired result. When \( q \geq 2 \) we cannot use estimate (1.8). Instead we have an estimate of the form \( u = O(t^{-1/2}) \), so that the contribution of the convective term in (3.1) can be estimated as

(3.3)

\[
| \int u_n^q(x,t)\zeta_x(x,t)\,dx\,dt | \leq C \int_0^t t^{-(q-1)/2} \,dt, 
\]

that is finite for \( q < 3 \), in which case we complete as before the proof that the initial data is \( M\delta(x) \).

In order to treat the case \( q \geq 3 \) we introduce another technique that works for all \( q \) and has further uses. As in (2.6) we use the variables

(3.4)

\[
v(x,t) = \int u(y,t)\,dy, \quad v_n(x,t) = \int u_n(y,t)\,dy. 
\]

The equation solved by \( v \) and \( v_n \) is

(3.5)

\[
v_t = v_{xx} - (v_x)^q. 
\]

The \( v_n \) have initial data

(3.6)

\[
v_n(x,0) = \int_{-\infty}^{x} \phi_n(s)\,ds. 
\]

Fix \( n_0 > 1 \) and let \( n \) be much larger than \( n_0 \). Then there exists \( a > 0 \) such that

(3.7)

\[
v_{n_0}(x-a,0) \leq v_n(x,0) \leq v_{n_0}(x+a,0) 
\]
and \( a \) depends only on \( n_0 \) with \( a \to 0 \) as \( n_0 \to \infty \). It follows from the maximum Principle that for every \((x,t) \in Q\)

\[
(3.8) \quad v_{n_0}(x-a,t) \leq v_n(x,t) \leq v_{n_0}(x+a,t)
\]

Since for \( x < -a \) the solution \( v_{n_0} \) takes on continuously the initial data \( 0 \), given \( \varepsilon > 0 \) there exists \( \tau > 0 \) such that for \( 0 < t < \tau \)

\[
v_n(x,t) \leq v_{n_0}(x+a,t) \leq \varepsilon \quad \text{if} \quad x < -2a.
\]

Passing to the limit \( n \to \infty \) we conclude that \( v(x,t) \) takes on the data \( 0 \) continuously for \( x < 0 \). A similar argument allows us to conclude that \( v(x,t) \) takes on the data \( M \) continuously for \( x > 0 \). Therefore

\[
(3.9) \quad v(x,0) = M H(x),
\]

where \( H \) is Heaviside's function. Consequently, \( u \) takes on the initial data \( M\delta(x) \) weakly in the sense of measures.

2. UNIQUENESS. We establish next the uniqueness of the fundamental solution of equation (E). We need first a technical Lemma.

**Lemma 3.1.** Let \( u \) be a fundamental solution of (E) and for all \( \varepsilon > 0 \) define \( u_\varepsilon \) as the solution of (E) with initial data

\[
(3.10) \quad u_{0\varepsilon}(x) = u(x,\varepsilon)\chi(x)
\]

where \( \chi \) is the characteristic function of a ball \( B_r(0) \), \( r > 0 \), i.e. \( \chi(x) = 1 \) if \( |x| \leq r \), \( \chi(x) = 0 \) otherwise. Then, as \( \varepsilon \to 0 \), \( u_\varepsilon \) converges to \( u \) in \( C((0,\infty); L^1(\mathbb{R})) \).

**Proof:** (i) Assume that we know that the mass of the solution is \( M \). Then, because of the convergence of \( u(\cdot, t) \) towards \( M\delta \) we know that for every \( \eta \) and \( r > 0 \) there exists \( \tau > 0 \) such that

\[
(3.11) \quad \int_{|x|<r} u(x,t) \, dx > M - \eta,
\]

so that the mass lying outside the interval \((-r,r)\) is less than \( \eta \) for \( 0 < t < \tau \). It follows that

\[
(3.12) \quad \eta_\varepsilon = ||u_{0\varepsilon} - u(\cdot,\varepsilon)||_1 \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

By the \( L^1 \)-contraction property (1.5) we get

\[
(3.13) \quad ||u_\varepsilon(\cdot,t) - u(\cdot,t+\varepsilon)||_1 \leq \eta_\varepsilon
\]

which proves that \( u_\varepsilon \to u \).
(ii) We have to make sure that the mass of any fundamental solution is precisely $M$. It is easy to see that the mass is constant in time for $t > 0$. It is also obvious that the weak initial condition (0.1) implies that the limit of the mass as $t \to 0$ is $M$. Therefore, the mass is $M$ at all times. #

We can now proceed with the Proof of uniqueness of the fundamental solution. We consider two solutions $u$ and $\overline{u}$. We define the approximating sequences $u_\varepsilon$ and $\overline{u}_\varepsilon$ by cutting off the tails of their data at $t = \varepsilon$ as in in (3.10). We next introduce the integrated variables $v_\varepsilon$ and $\overline{v}_\varepsilon$ as in (3.5). We are dealing now with classical solutions of equation (3.5).

Observe next that the the masses of both $u_\varepsilon$ and $\overline{u}_\varepsilon$ tend increasingly to $M$ as $\varepsilon \to 0$. Hence, we can find sequences $\varepsilon_n \to 0$ and $\varepsilon'_n \to 0$ such that the masses of $u_{\varepsilon_n}$ and $\overline{u}_{\varepsilon'_n}$ are the same. We can also assume that their initial data are confined to the interval $(-r, r)$. Let us write $u_n = u_{\varepsilon_n}$ and $\overline{u}_n = \overline{u}_{\varepsilon'_n}$. It easily follows that

$$v_n(x - 2r, 0) \leq \overline{v}_n(x, 0) \leq v_n(x + 2r, 0).$$

Applying the Maximum Principle we conclude that

$$v_n(x - 2r, t) \leq \overline{v}_n(x, t) \leq v_n(x, t)$$

(3.14)

holds everywhere in $Q$. Passing to the limit $n \to \infty$ we get

$$v(x - 2r, t) \leq \overline{v}(x, t) \leq v(x + 2r, t) \quad \text{in} \quad Q.$$  

(3.15)

Since $r > 0$ is arbitrary we conclude that $v = \overline{v}$ in $Q$, hence $u = \overline{u}$. This completes the proof. #

Remark: We could have replaced the formulation (0.1) of the initial condition in the definition of fundamental solution by the more classical statement: $u(x, t)$ takes on continuously the value 0 at all points $(x, 0)$ with $x \neq 0$ and $\int u(x, t) \, dx = M$ for every $t > 0$. It can be shown that both statements are equivalent.

4. Some properties of the fundamental solutions

We list here some properties of the fundamental solutions that can be useful. Since they can be obtained by means of techniques that form more or less part of the folklore knowledge for solutions of nonlinear parabolic equations, we will allow ourselves to be rather succinct in the exposition.

First, the fundamental solutions have for every fixed $t > 0$ exponential decay in $x$, exactly the same as the kernel of the heat equation. This can be proved by viewing (E) as a perturbation the heat equation.
Second, for every fixed $t > 0$ the function $u(\cdot, t)$ has precisely a maximum $x_*(t)$, being increasing for $x < x_*$ and decreasing for $x > x_*$. This follows as a consequence of the lap number theory of Matano [M], which applies to this equation.

Moreover, our asymptotic behaviour shows that for $1 < q < 2$ the point $x_*(t)$ must tend to the maximum of the $N$-wave profile of equation (C), hence

\begin{equation}
\lim_{t \to \infty} \frac{x_*(t)}{t^{1/q}} = \left( \frac{qM}{q-1} \right)^{\frac{q-1}{q}}.
\end{equation}

For $q = 2$ we get from the asymptotic convergence towards the function $\tilde{U}$ of the Introduction

\begin{equation}
\lim_{t \to \infty} \frac{x_*(t)}{t^{1/2}} = \eta_*,
\end{equation}

where $\eta_*$ is the point of maximum of the function $F$ given in (0.2). Finally, for $q > 2$ from the results of [EZ] we only deduce that

\begin{equation}
\lim_{t \to \infty} \frac{x_*(t)}{t^{1/2}} = 0.
\end{equation}

Another interesting way of measuring the displacement of the bulk mass due to the convective effect is to estimate the evolution of the center of mass defined as

\begin{equation}
\ddot{x}(t) = \frac{1}{M} \int x u(x, t) \, dx.
\end{equation}

By the reflection inequality (4.1) we have $\ddot{x}(t) \geq 0$. Moreover, a straightforward computation shows that

\begin{equation}
\frac{d\ddot{x}(t)}{dt} = -\int x(u^q)_x \, dx = \int u^q \, dx.
\end{equation}

For large times we can use our previous asymptotic analysis to conclude that

\begin{equation}
\frac{d\ddot{x}(t)}{dt} \approx \|u(\cdot, t)\|_\infty.
\end{equation}

Let us now integrate (4.7) using the decay rates that have been obtained in the different exponent cases. We arrive at

**Theorem 3.** As $t \to \infty$ we have

\[ \ddot{x}(t) \approx \begin{cases} O(t^{1/q}) & \text{if } 1 < q \leq 2 \\ O(t^{(3-q)/2}) & \text{if } 2 < q < 3 \\ O(\log t) & \text{if } q = 3 \\ O(1) & \text{if } 3 < q. \end{cases} \]

In the last case the asymptotic center of mass $\ddot{x}_\infty = \lim_{t \to \infty} \ddot{x}(t)$ exists and is positive and finite.
5. The linear case \( q = 1 \)

We end our study with some comments on the linear equation

\[
(5.1) \quad u_t = u_{xx} - u_x,
\]
i.e. the particular case of (E) with \( q = 1 \). We are interested in comparing the behaviour of the solutions for \( q = 1 \) and \( q > 1 \). Equation (5.1) is easy to solve. Indeed, introduction of the moving frame: \( y = x - t \) transforms (5.1) into the heat equation,

\[
u_t = u_{yy}.
\]

With the assumptions on the initial data made above, the asymptotic behaviour of solutions to such equation is given by the heat kernel, which in the original variables reads

\[
(5.2) \quad \bar{U}(x,t) = \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{|x-t|^2}{4t}\right).
\]

At first sight this seems to be at strong variance with the convective behaviour proved in Theorem 1 for \( q > 1 \), and more in tune with the behaviour in the exponent range \( q > 2 \). However, let us take a look at the fundamental solutions \( U(x,t;M) \) of equation (C) given in (0.3). As \( q \to 1 \) the front \( r(t) = c(q)(M^{q-1}t)^{1/q} \) converges precisely to \( r(t) = x \), in accordance with the displacement of the solutions by the effect of convection pointed out for (5.1). It is also apparent that

\[
(5.3) \quad \lim_{q \to 1} (x/t)^{1/(q-1)} = 0
\]

for every point \((x,t)\) with \( x < t \), while the limit has some kind of singularity at \( x = t \). In fact, the limit of \( U(\cdot,t;M) \) as \( q \to 1 \) is \( M\delta(\cdot - t) \), a Dirac mass moving along the line \( x = t \). This is in accordance with formula (5.2) for the actual asymptotic behaviour, since at the scale \( x = O(t) \) the exponential profile (5.2) with typical length \( O(t^{1/2}) \) can be confused with the Dirac mass. Of course, formula (5.2) allows us to take into account the effect of diffusion, which was a secondary effect for \( q > 1 \), because of the presence of a leading \( N \)-wave, but is now the leading term in the determination of the asymptotic shape of the solution (it determines its height and width, though not its position). In fact, as shown in (5.3), the \( N \)-wave shape reduces in the limit to a mere point mark for the profile position.

6. Asymptotic behaviour for data with changing sign

We also want to consider solutions with changing sign. Of course we have to define in a precise way the convective term. Maybe the most common option is to consider the equation

\[
(5') \quad u_t = u_{xx} - |u|^{q-1}u_x.
\]
There is a unique viscosity solution for this problem, and the maps \( u_0 \mapsto u(\cdot, t) \) generate a semigroup of order-preserving contractions in \( L^1(\mathbb{R}) \), see for instance [LP]. Obviously, for \( u_0 \geq 0 \) we recover the viscosity solutions of (E).

The asymptotic behaviour of solutions with changing sign of (E') can be treated much in the same way that we have used for nonnegative solutions of (E) and the result stays true.

**Theorem 4.** The same asymptotic behaviour of Theorem 1 is true for solutions of (E') in the range \( 1 < q < 2 \) without the assumption \( u_0 > 0 \). We obtain however weaker convergence.

**Proof:** It is again divided into several steps.

**Step 1.** Let \( u \) be the solution of (E') with initial data \( u_0 \in L^1(\mathbb{R}) \), and \( u_0 \) changes sign. Let \( u_0^+ \) be the positive part of \( u_0 \), \( u_0^+(x) = \max\{u_0(x), 0\} \) and let \( v \) be the viscosity solution with initial data \( u_0^+ \). Similarly, let \( w \) be the solution with initial data \( u_0^-(x) = \min\{0, u_0(x)\} \). By the Maximum Principle we have

\[
(6.1) \quad w \leq u \leq v \quad \text{in} \quad Q.
\]

Using Lemma 1.2 we get the \( L^\infty \)-bound

\[
(6.2) \quad |u(x, t)| \leq C(q) \|u_0\|_1 t^{-\frac{1}{q}}.
\]

We also have

\[
(6.3) \quad \int u(x, t) \, dx \leq \int v(x, t) \, dx = \int u_0^+(x) \, dx \equiv M_0^+.
\]

and

\[
(6.4) \quad \int u(x, t) \, dx \geq \int w(x, t) \, dx = \int u_0^-(x) \, dx \equiv M_0^-.
\]

where we have \( M \leq M_0^+ + M_0^- \) and \( \|u_0\|_1 = M_0^+ - M_0^- \). It follows in particular that

\[
(6.5) \quad \int u(x, t) \, dx \leq \|u_0(x)\|_1
\]

for all \( t > 0 \). Finally, multiplying (E') by \( u \) and integrating we obtain as in Lemma 1.4

\[
(6.6) \quad \int_{\tau}^{T} \int |u_x|^2 \, dx \, dt \leq C \tau^{-\frac{1}{q}}.
\]

**Step 2.** We introduce rescaling (2.1) which produces a function \( u_\lambda \) solving an equation like (2.2):

\[
(6.7) \quad u_{\lambda, t} + |u_\lambda|^{q-1} u_{\lambda, x} = \lambda^{q-2} u_{\lambda, xx}.
\]
The family \( \{ u_\lambda \} \) is bounded in \( L^\infty(\mathbb{R} \times (\tau, T)) \), hence we can extract sequences \( \lambda_j \to \infty \) which converge in the weak*-topology of \( L^\infty \). We now use Theorem 26, page 202, of [T] to conclude that along such a sequence

\[
\begin{align*}
 u_{\lambda_j} & \to \bar{u} \quad \text{in} \quad L^\infty_{loc}(Q) - w^* \\
 u_{\lambda_j} & \to \bar{u} \quad \text{in} \quad L^p_{loc}(Q), \quad \forall p < \infty,
\end{align*}
\]

where \( u \) is an entropy solution of (E'). Contrary to Section 2 this means using big machinery.

**Step 3.** We now want to pass to the limit in the initial condition. Since we have not obtained convergence of \( u_{\lambda_j} \) for fixed \( t \), we have to slightly modify the argument given in Section 2, Step 2, using a test function \( \phi \) which depends both on \( x \) and \( t \). We also assume that \( \phi \in C^\infty(\mathbb{R} \times [0, \infty]) \) and vanishes for all large \( x \) and also for \( t \geq t_1 > 0 \). Then we have

\[
\begin{align*}
- \int_0^{t_1} \int_{\mathbb{R}} u_\lambda(x, s) \phi_t(x, s) \, dx \, ds - \int_{\mathbb{R}} u_\lambda(x, 0) \phi(x, 0) \, dx \\
= \lambda^{q-2} \int_0^{t_1} \int_{\mathbb{R}} u_\lambda(x, t) \phi_{xx}(x) \, dx + \frac{1}{q} \int_0^{t_1} \int_{\mathbb{R}} u_\lambda^q(x, t) \phi_x(x) \, dx
\end{align*}
\]

As in Section 2 the second member tends to 0 as \( t_1 \to 0 \) uniformly in \( \lambda \) while the first member tends to

\[
\int_0^{t_1} \int_{\mathbb{R}} \bar{u} \phi_t \, dx \, dt - M \phi(0).
\]

By taking \( \phi \) of the form \( \phi_1(x) \phi_2(t) \) with \( \phi_2(t) = 1 \) for \( 0 \leq t \leq t_1 - \varepsilon \) and letting \( \varepsilon \to 0 \) we conclude that our entropy solution takes as initial data \( M \, \delta(x) \). It is therefore Liu-Pierre's entropy solution, i.e. \( U \) of (0.3)-(0.4). Since the limit is uniquely determined, the whole family (and not only a sequence) converges to it. We have proved that

\[
(6.9) \quad u_\lambda \to \bar{u} \quad \text{in} \quad L^p_{loc}(\mathbb{R} \times (0, \infty))
\]

for every \( p < \infty \).

**Step 4. Better convergence.** Proceeding as in Step 5 of Section 2 and taking care of making an extra integration in time we obtain the convergence

\[
(6.10) \quad u_\lambda \to \bar{u} \quad \text{in} \quad L^1(\mathbb{R} \times (\tau, T))
\]

for every \( 0 < \tau < T \). Moreover, this implies that for almost every \( t_0 > 0 \) we have \( u_\lambda(x, t_0) \to U(x, t_0) \) in \( L^1(\mathbb{R}) \). Fix one of those \( t_0 \). Since the family \( u_\lambda(x, t_0) \) is also bounded in \( L^\infty \)-norm, we have

\[
u_\lambda(x, t_0) \to U(x, t_0) \quad \text{in} \quad L^p(\mathbb{R})
\]

for very \( p < \infty \). It follows that as \( t \to \infty \)

\[
(6.11) \quad t^{\frac{1}{4}}(1 - \frac{1}{q}) \| u(t) - U(t) \|_p \to 0
\]

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for every $1 < p < \infty$.  

**Remark:** Assume that the mass $M = \int u_0(x) \, dx$ is positive. An interesting consequence of Theorem 4 is the fact that

$$\int_\mathbb{R} u^+(x, t) \, dx \to M \quad \text{and} \quad \int_\mathbb{R} u^-(x, t) \, dx \to 0.$$  

In case $M < 0$ the situation would be reversed, $U_{-M} = -U_M$. Finally, if $M = 0$ we get zero limit for both integrals. Indeed, when $M = 0$ the function $U$ is identically 0.

### 7. Extension to other convective nonlinearities

The result about convergence towards a special solution of the Conservation Law (C), which holds for the solutions of equation (E) in the range $1 < q < 2$, can be extended in other ways. One of the most natural extensions is based on the observation that for large values of time the solution $u$ is close to zero, hence only the behaviour near zero of the convective nonlinearity should matter. Therefore, we would like to consider a diffusion-convection equation of the type

$$(\text{E''}) \quad u_t = u_{xx} - \phi(u)_x$$

where $\phi$ is a continuous real function with $\phi(0) = 0$ and $\phi(u) > 0$ for $u > 0$, and such that there exists a $q > 1$ such that

$$(\text{H1}) \quad \lim_{s \to 0} \frac{\phi(s)}{s^q} = c \quad 0 < c < \infty.$$  

(Anyway, by suitably rescaling the time and space variables we can always assume that the limit number $c$ is any number, say $1/q$ as in equation (E) of previous sections). Thus, the self-similar asymptotic behaviour of the solutions of (E') has been studied in [EZ] under condition (H1) for the critical exponent $q = (N + 1)/N$ and the result coincides with the one for the pure power case.

A similar remark applies to the superlinear case $q > (N + 1)/N$, where the effect of convection disappears in the limit. In fact, a priori decay estimates like (1.3) do not depend on the growth of the function $\phi$, and condition (H1) is the right expression to ensure that the convection term disappears when a rescaling of heat-equation type is performed in order to study the large-time behaviour.

In our range $1 < q < 2$ we can extend Theorem 1 to equation (E') under basically condition (H1). However, working with general nonlinearities involves cumbersome changes in the proofs, so, if we want to be able to preserve the line of proof, and in particular the crucial estimate (1.5), we have to make some additional hypothesis. Therefore, we ask that
(H2): \( \phi \in C^1([0, \infty)) \cap C^2(0, \infty) \) with \( \phi(0) = \phi'(0) = 0 \), \( \phi(s) > 0 \) for \( s > 0 \) and
\[
\lim_{s \to 0} \frac{\phi''(s)}{s^{q-2}} = c_1, \quad \text{for some} \quad 0 < c_1 < \infty.
\]

Let us remind the reader here that (H2) implies that \( \phi(s) \approx c s^q \) with \( c = c_1/(q(q-1)) \) and \( \phi'(s) \approx q c s^{q-1} \) for \( s > 0 \), \( s \approx 0 \) and \( q \) has to be larger than 1. Also, these conditions are close to the ones used in [LP] in the study of the asymptotic convergence of the entropy solutions equation \( u_t + \phi'(u)_x = 0 \). We have

**Theorem 5.** The convergence results of Theorem 1 are still valid for the solutions of the Cauchy problem for equation (E') with nonnegative initial data \( u_0 \in L^1(\mathbf{R}) \) if \( \phi \) satisfies hypothesis (H2) with \( 1 < q < 2 \).

**Proof:** The most important part consists in checking that for every \( \varepsilon > 0 \) we can get an estimate of the form

\[
(7.1) \quad (u^{q-1})_x \leq \frac{1 + \varepsilon}{t},
\]

valid for all large enough times \( t \). This can be done by a slight variation of the method used in Lemma 1.1 above. The rest of the proof is straightforward. 

A second direction of extension concerns the existence of source-type solutions. In that case there is no growth restriction on \( \phi \), which can be taken merely to be continuous, see for instance [BH] for an existence theory for general equations of the form \( u_t = \phi(u)_{xx} - \psi(u)_x + v \). Since there are no essential novelties, we leave the details to the interested reader.

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