ERROR ESTIMATES FOR A FINITE ELEMENT METHOD FOR THE DRIFT-DIFFUSION SEMICONDUCTOR DEVICE EQUATIONS

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Abstract. In this paper, optimal error estimates are obtained for a method for numerically solving the so-called unipolar model (a one-dimensional simplified version of the drift-diffusion semiconductor device equations). The numerical method combines a mixed finite element method using a continuous piecewise-linear approximation of the electric field with an explicit upwinding finite element method using a piecewise-constant approximation of the electron concentration. For initial and boundary data ensuring that the electron concentration is smooth, the $L^\infty(L^1)$-error for the electron concentration and the $L^\infty(L^\infty)$-error of the electric field are both proven to be of order Δx. The error analysis is carried out first in the zero diffusion case in detail and then extended to the full unipolar model.

Key words. semiconductor devices, conservation laws, finite elements, error estimates

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1. Introduction. In this paper, we obtain error estimates for a method for numerically solving the so-called unipolar model, see [2] and the references therein,

$$u_\tau + (u\,\beta)_x = 0 \qquad \tau > 0,\ x \in (0,1), \tag{1.1a}$$

$$u(\tau,0) = u_0(\tau), \qquad \text{if } \beta(\tau,0) > 0, \quad \tau \geq 0, \tag{1.1b}$$

$$u(\tau,1) = u_1(\tau), \qquad \text{if } \beta(\tau,1) < 0, \quad \tau \geq 0, \tag{1.1c}$$

$$u(0,x) = u_i(x), \qquad x \in (0,1), \tag{1.1d}$$

where

$$-\beta_x = 1 - u, \qquad x \in (0,1),\ \tau \geq 0, \tag{1.2a}$$

$$\beta = \phi, \qquad x \in (0,1),\ \tau \geq 0, \tag{1.2b}$$

$$\phi(\tau,0) = 0, \qquad \text{for } \tau \geq 0, \tag{1.2c}$$

$$\phi(\tau,1) = \phi_1(\tau), \qquad \text{for } \tau \geq 0, \tag{1.2d}$$

where ϕ is the (scaled) electric potential. The numerical method under consideration combines a mixed finite element method using a continuous piecewise-linear approximation of the electric field, $-\beta$, with an explicit upwinding finite element method using a piecewise-constant approximation of the electron concentration, u. The resulting scheme can be considered to be the counterpart of the monotone schemes for scalar conservation laws, the main difference being that in our case, the 'flux', β, depends globally on the solution u. In [1], convergence under a suitable CFL-condition was established. In [2], an approximation theory was obtained which was then used to prove that the $L^\infty(L^1)$-error for the electron concentration is of order $(\Delta x)^{1/2}$. This

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order of convergence is sharp, as the numerical results in [1] indicate, and reflects the fact that the solution u can display discontinuities which always are contact discontinuities. If the electron concentration does not display discontinuities, numerical results in [1, §2e] indicate that the $L^\infty(L^1)$-error for the electron concentration and the $L^\infty(L^\infty)$-error of the electric field are both of order Δx. The main objective of this paper is to prove that this is indeed true.

We thus restrict ourselves to a class of data for which the electron concentration u does not display discontinuities. It is important to emphasize the fact that it is not enough to require the initial and boundary data to be very smooth to guarantee the absence of discontinuities in the electron concentration. In fact, discontinuities might appear even if the data are very smooth, as happens in classical conservation laws. On the other hand, the system of equations (1.1) and (1.2) is not a classical conservation law, and under some conditions on the sign of the electric field near the boundaries it can be proven that the solution is smooth indeed. It is for the class of data that ensures the satisfaction of the above conditions that we prove that the $L^\infty(L^1)$-error for the electron concentration and the $L^\infty(L^\infty)$-error for the electric field are optimal, that is, of order Δx.

To obtain the error estimates, we use the approximation result obtained in [2], which is an extension to our framework of the Kuznetsov approximation result [5] for classical conservation laws. In this paper, we do not estimate the 'entropy production term' as we did in [2]. Instead, we take advantage of the compactness properties of the solution generated by the scheme under consideration to write the error in terms of the approximation errors in the data and in terms of the 'residue' of the approximate solution. The error estimates then follow from an estimate of the 'residue', which in turn follows from the regularity properties of the approximate solution only. We show that the smoothness assumptions on the exact solution do not play explicitly any role in the error estimate. This makes our approach suitable for an error analysis of an adaptive algorithm, which will be explored in a forthcoming paper.

The error analysis will be carried out first for the system (1.1) and (1.2), i.e., for the unipolar model with the diffusion term neglected. Then, it will be extended to the full model under a suitable assumption on the initial data. The results obtained here can be extended to other convection-dominated problems such as those for miscible displacement in porous media [3], [4].

The paper is organized as follows. In §2, we display and discuss the hypotheses on the data. In §3, we define our numerical scheme. In §4, we prove that, under suitable conditions on the sign of the exact electric field near the boundary, the approximate electron concentration satisfies new compactness properties; see Theorem 4.2. In §5, we state the approximation result obtained in [2] and show how to combine it with the compactness properties of the approximate solution to obtain our main result, the error estimates of Theorem 5.5. The results in §4 and §5 are proven in §6 and §7, respectively. In §8, the full unipolar model is analyzed and the corresponding results are stated; see Theorems 8.1, 8.2 and 8.3. We end in §9 with some concluding remarks.

2. The hypotheses on the data. We want to consider initial and boundary data for which the solution u does not display discontinuities. A necessary condition for this to happen is that the initial and boundary data be smooth. This requirement, however, is not sufficient. To see this, consider the boundary $x = 0$ and suppose that $\beta(0,\tau) < 0$ for $\tau \in [0,\tau^\star)$ and that $\beta(0,\tau) > 0$ for $\tau > \tau^\star$. In this case, for $\tau \in [0,\tau^\star)$, the electron concentration u is being convected out of the domain. This causes the appearance of a discontinuity precisely at the boundary which is then convected inside the domain since $\beta(0,\tau) > 0$ for $\tau > \tau^\star$. Thus, even if the solution u is very smooth up to $\tau = \tau^\star$, it displays a discontinuity thereafter. We have thus to select data for which the negative electric field given by the zero diffusion case satisfies the following properties:

$$\beta(\tau,0) > 0, \text{ for } \tau \in [0,T], \text{ or } \beta(\tau,0) < 0, \text{ for } \tau \in [0,T], \tag{2.1a}$$
$$\beta(\tau,1) < 0, \text{ for } \tau \in [0,T], \text{ or } \beta(\tau,1) > 0, \text{ for } \tau \in [0,T]. \tag{2.1b}$$

In Theorem 4.2 , we prove that for data satisfying the following properties with $u^\star \geq 1$:

$$u_0 \equiv 0, \tag{2.2a}$$
$$u_1 \equiv 1, \tag{2.2b}$$
$$u_i(x) \in [0,u^\star], \quad x \in [0,1], \tag{2.2c}$$
$$u_{ix} \in \mathrm{BV}(0,1), \tag{2.2d}$$
$$u_i(0) = 0 \text{ and } u_i(1) = 1, \tag{2.2e}$$
$$\phi_1(\tau) \in [0,\phi_1^\star], \quad \tau \in [0,T], \tag{2.2f}$$
$$\phi_1 \in \mathrm{W}^{1,\infty}(0,T), \tag{2.2g}$$

if the conditions (2.1) are satisfied, then the electron concentration u is indeed a smooth function. The properties (2.2a) and (2.2b) on the boundary data simplify greatly our analysis. However, as pointed out in §9, the results below apply to other sets of data. Properties (2.2d) and (2.2e) are necessary to prevent the appearance of discontinuities in the electron concentration. The property (2.2g) ensures that the negative electric field $\beta(\tau,x)$ is a Lipschitz-continuous function in τ for each $x \in [0,1]$. This implies, in particular, that conditions (2.1) are always satisfied for small enough values of T. The maximal value of T, $T^\star$, for which (2.1) hold, will not be investigated in this paper. Let us note, however, that it can be a very large number. For example, if $u_i(x) = x^2$ and $\phi_1 \equiv 1/8$, then $T^\star = \infty$.

3. The numerical scheme. Now, let us introduce our numerical scheme. Following [1], let $\{x_{j+\frac{1}{2}}\}_{i=0}^{n_x}$ be a uniform partition of $(0,1)$ such that $x_{\frac{1}{2}} = 0$ and $x_{n_x+\frac{1}{2}} = 1$. We also set $x_{-\frac{1}{2}} = -\Delta x$ and $x_{n_x+\frac{3}{2}} = 1 + \Delta x$ in order to define an auxiliary computational domain as $\Omega_h = (x_{-\frac{1}{2}}, x_{n_x+\frac{3}{2}})$. Let $\{\tau^n\}_{n=0}^{n_T}$ be a partition of $[0,T]$ with $\tau^0 = 0$ and $\tau^{n_T} = T$. Set $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \triangle x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \equiv \triangle x$, and $J^n = [\tau^n, \tau^{n+1}], \triangle\tau^n = \tau^{n+1} - \tau^n$. For simplicity, we take $\triangle\tau^n \equiv \triangle\tau$; all our results hold for variable $\triangle\tau^n$. Finally, we associate with these partitions the following spaces:

$$\begin{aligned}
V_{\triangle x} &= \{v \in C^0(0,1) : v|_{I_j} \in P^1(I_j), \quad j = 1,\cdots,n_x\},\\
W_{\triangle x} &= \{w \in L^\infty(\Omega_h) : w|_{I_j} \in P^0(I_j), \quad j = 0,\cdots,n_x+1\},\\
W_{\triangle\tau} &= \{w \text{ right-continuous} : w|_{J^n} \in P^0(J^n), \quad n = 0,\cdots,n_T-1\}.
\end{aligned}$$

For $v \in V_{\triangle x}$, $v_{j+\frac{1}{2}}$ denotes the quantity $v(x_{j+\frac{1}{2}})$. For $w \in W_{\triangle x}$, w_j denotes the constant value $w(x), x \in I_j$; w_0 and w_{n_x+1} denote the boundary values. Finally, if $w \in W_{\triangle\tau}$, w^n denotes the constant $w(t)$, $t \in J^n$.

To discretize (1.1) and (1.2), we first discretize the data by setting

$$\phi^n_{1,\Delta\tau} = \int_{J^n} \phi_1(\tau)\, d\tau/\Delta\tau, \tag{3.1a}$$

$$u_{0,\Delta\tau} = \int_{J^n} u_0(\tau)\, d\tau/\Delta\tau, \tag{3.1b}$$

$$u_{1,\Delta\tau} = \int_{J^n} u_1(\tau)\, d\tau/\Delta\tau, \tag{3.1c}$$

$$(u_{i,\Delta x})_j = \begin{cases} u_{0,\Delta\tau}, & \text{for } j = 0, \\ u_{1,\Delta\tau}, & \text{for } j = n_x + 1, \\ \int_{I_j} u_i(x)\, dx/\Delta x, & \text{otherwise.} \end{cases} \tag{3.1d}$$

We then define the approximate solution u_h to be the element in $W_{\Delta\tau} \otimes W_{\Delta x}$ satisfying the boundary conditions

$$u^n_0 = u^n_{0,\Delta\tau},\ u^n_{n_x+1} = u^n_{1,\Delta\tau}, \tag{3.2a}$$

and the equation

$$(u^{n+1}_j - u^n_j)/\Delta\tau + (f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}})/\Delta x = 0, \tag{3.2b}$$

where the numerical flux $f^n_{j+\frac{1}{2}} = f(u^n_j, u^n_{j+1}; \beta^n_{j+\frac{1}{2}})$ is given by

$$f^n_{j+\frac{1}{2}} = u^n_j\, \beta^{n^+}_{j+1/2} + u^n_{j+1}\, \beta^{n^-}_{j+1/2}. \tag{3.2c}$$

Finally, the function $(\beta_h, \phi_h) \in W_{\Delta\tau} \otimes V_{\Delta x} \times W_{\Delta\tau} \otimes W_{\Delta x}$ is defined by the following mixed finite element method:

(3.3a)
$$-\left(\, (\beta_h)_x(\tau^n), w_{\Delta x}\,\right) = \left(\, 1 - u_h(\tau^n), w_{\Delta x}\,\right), \quad \forall w_{\Delta x} \in W_{\Delta x},$$

(3.3b)
$$\left(\, \beta_h(\tau^n), v_{\Delta x}\,\right) + \left(\, \phi_h(\tau^n), (v_{\Delta x})_x\,\right) = \phi_{1,\Delta\tau}(\tau^n)\, v_{\Delta x}(1), \quad \forall v_{\Delta x} \in V_{\Delta x},$$

where $(\cdot,\cdot)$ is the $L^2(0,1)$–inner product.

Thus, the algorithm of our numerical method is:

(3.4a) Compute the functions $u_{0,\Delta\tau}, u_{1,\Delta\tau}, u_{i,\Delta x}$, and $\phi_{1,\Delta\tau}$ by (3.1);
(3.4b) Set $u_h(0,\cdot) = u_{i,\Delta x}(\cdot)$;
(3.4c) For $n = 0, \cdots, n_T - 1$ compute $u_h(\tau^{n+1},\cdot)$ as follows:
   (i) Compute $(\beta_h(\tau^n,\cdot), \phi_h(\tau^n,\cdot))$ by using the mixed finite element method (3.3);
   (ii) Compute $u_h(\tau^{n+1}, x)$ for $x \in (0,1)$ by using the scheme (3.2).

4. Compactness of the approximate solution. The goal of this section is to obtain some compactness properties of the approximate solution defined by our numerical scheme (3.4) under the hypotheses (2.1) and (2.2). We begin with the following result.

THEOREM 4.1. *Suppose that the hypotheses* (2.2) *on the initial and boundary data hold. Assume that the following* CFL *condition is satisfied:*

$$\Delta\tau^n \le \min\{\frac{1}{u^\star}, \frac{\Delta x}{(2\,u^\star - 1)\,\Delta x + \phi_1^\star + \max\{1, u^\star - 1\}/2}\}. \tag{4.1}$$

Then, there exists a constant C_0, depending solely on T and the initial and boundary data, such that

$$\|\, u - u_h \,\|_{L^\infty(0,T;L^1(0,1))} \le C_0\, \Delta x^{1/3},$$
$$\|\, \beta - \beta_h \,\|_{L^\infty(0,T;L^\infty(0,1))} \le C_0\, \Delta x^{1/3}.$$

Moreover, $\beta \in \mathcal{C}^0(0,T;W^{1,1}(0,1))$.

Proof. We can obtain the result by replacing in Theorem 2.3 in [2] the norm in $L^1(0,T;L^\infty(0,1))$ by the norm in $L^\infty(0,T;L^\infty(0,1))$. This can be done since in our case $\phi \in W^{1,\infty}(0,T)$, by (2.2g); see Theorem 2.1 in [2]. The property of β follows from Theorem 2.3 in [1], equations (1.2), and from the smoothness hypothesis on ϕ_1, (2.2g). This completes the proof. □

Since $\beta \in \mathcal{C}^0(0,T;W^{1,1}(0,1))$, then both $\beta(\cdot,0)$ and $\beta(\cdot,1)$ are continuos functions. Hence, there are numbers β_0 and β_1 such that

$$|\,\beta(\tau,0)\,| \ge |\,\beta_0\,| \quad \text{for } \tau \in [0,T], \tag{4.2a}$$
$$|\,\beta(\tau,1)\,| \ge |\,\beta_1\,| \quad \text{for } \tau \in [0,T], \tag{4.2b}$$

where, by the hypotheses (2.1), the numbers β_0 and β_1 are not equal to zero. Hence, by the error estimates of Theorem 4.1, if we have

$$\Delta x \le \Big(\min\{|\,\beta_0\,|, \beta_1\,|\}/(2\,C_0)\Big)^3, \tag{4.3a}$$

then,

$$\text{sign}(\beta_h(\cdot,0)) = \text{sign}(\beta(\cdot,0)) \text{ and } |\,\beta_h(\cdot,0)\,| \ge |\,\beta_0\,|/2, \tag{4.2a'}$$
$$\text{sign}(\beta_h(\cdot,1)) = \text{sign}(\beta(\cdot,1)) \text{ and } |\,\beta_h(\cdot,1)\,| \ge |\,\beta_1\,|/2. \tag{4.2b'}$$

Moreover, if we require Δx to be such that

$$\Delta x \le \min\{|\,\beta_0\,|, |\,\beta_1\,|\}/(4\,(u^\star - 1)), \tag{4.3b}$$

then we have

$$\text{sign}(\beta_h(\cdot,\Delta x)) = \text{sign}(\beta_h(\cdot,2\Delta x)) = \text{sign}(\beta(\cdot,0)), \tag{4.2c'}$$

(4.2d')

$$\text{sign}(\beta_h(\cdot,1-\Delta x)) = \text{sign}(\beta_h(\cdot,1-2\Delta x)) = \text{sign}(\beta(\cdot,1)),$$

as will be proven in §6. Using these facts, we can prove the following result.

THEOREM 4.2. (Compactness of the approximate solution). *Suppose that the hypotheses on the data* (2.1) *and* (2.2) *and the* CFL *condition* (4.1) *are satisfied. Then,*

$$u_h(\tau,x) \in [0,u^\star], \quad (\tau,x) \in [0,T]\times[0,1], \tag{4.4a}$$
$$\|\,\beta_h\,\|_{L^\infty(0,T;L^\infty(0,1))} \le m^\star, \tag{4.4b}$$
$$|\,u_h\,|_{L^\infty(0,T;BV(0,1))} \le C_1, \tag{4.4c}$$
$$\max_{0\le n\le n_T-1} \|\,u_h^{n+1} - u_h^n\,\|_{L^1(0,1)} \le C_2\,\Delta\tau, \tag{4.4d}$$

where $m^\star = \phi_1^\star + \max\{1, u^\star - 1\}/2$, C_1 *depends on* T *and* $|\,u_i\,|_{BV(0,1)}$, *and* C_2 *depends on* C_1, $m^\star$, *and* $u^\star$. *Moreover, for* Δx *satisfying the conditions* (4.3), *the following properties hold:*

$$\sum_{n=0}^{n_T-1} \beta_h^{n^+}(0)\,|\,u_1^n - u_0^n\,|\,\Delta\tau \le C_3\,\Delta x^2, \tag{4.4e}$$

$$\sum_{n=0}^{n_T-1} \beta_h^{n^-}(1)\,|\,u_{n_x+1}^n - u_{n_x}^n\,|\,\Delta\tau \le C_4\,\Delta x^2, \tag{4.4f}$$

$$\sum_{j=1}^{n_x} |\,\beta_{j+1/2}^{n^+}(u_{j+1}^n - u_j^n) - \beta_{j-1/2}^{n^+}(u_j^n - u_{j-1}^n)\,| \le C_5\,\Delta x, \tag{4.4g}$$

$$\sum_{j=1}^{n_x} |\,\beta_{j+1/2}^{n^-}(u_{j+1}^n - u_j^n) - \beta_{j-1/2}^{n^-}(u_j^n - u_{j-1}^n)\,| \le C_5\,\Delta x, \tag{4.4h}$$

$$\sum_{n=0}^{n_T-1}\sum_{j=0}^{n_x} |\,(u_{j+1}^{n+1} - u_j^{n+1}) - (u_{j+1}^n - u_j^n)\,| \le C_6, \tag{4.4i}$$

where

$$C_3 = \begin{cases} m^\star\,\|\,(u_i)_x\,\|_{L^\infty(0,1)}/\beta_0, & \text{if } \beta(\cdot,0) > 0, \\ 0, & \text{if } \beta(\cdot,0) < 0, \end{cases}$$
$$C_4 = \begin{cases} m^\star\,\|\,(u_i)_x\,\|_{L^\infty(0,1)}/|\beta_1|, & \text{if } \beta(\cdot,1) < 0, \\ 0, & \text{if } \beta(\cdot,1) > 0, \end{cases}$$

and C_5 *and* C_6 *depend on* $T, u^\star, m^\star, |\,(u_i)_{xx}\,|_{BV(0,1)}, \|\,(u_i)_x\,\|_{L^\infty(0,1)}, C_1, C_3,$ *and* C_4.

The properties (4.4a), (4.4b), (4.4c) and (4.4d) have been proven in [1]. The remaining properties will be proven in §6. Next, we show how to use the above compactness properties of the approximate solution to get our error estimates.

5. The error estimates. To obtain error estimates, we use an extension of the Kuznetsov approximation theory for conservation laws [5, Lemma 2] to our framework obtained in [2]. This approximation result gives a measure of the closedness of two arbitrary pairs of functions (u, β) and (v, η) satisfying the following regularity requirements:

(5.1a)

(u, β) and (v, η) are right-continuous function from $[0, T)$ to $L^1(0,1) \times W^{1,\infty}(0,1)$ and have limits from the left on $(0, T]$,

(5.1b)

$$u, v \in L^\infty(0, T; BV(0,1)) \cap L^\infty(0, 1; L^1(0, T)),$$

in terms of the following smoothness-measuring quantities:

(5.2)
$$\begin{aligned}
\nu^+_{x,0}(\epsilon, v;\eta) &= \sup_{0\le\Delta\le\epsilon}\int_0^T |\, v(\tau,\Delta) - v(\tau,0-)\,|\,\eta^+(\tau,0)\,d\tau,\\
\nu^-_{x,1}(\epsilon, v;\eta) &= \sup_{0\le\Delta\le\epsilon} -\int_0^T |\, v(\tau,1-\Delta) - v(\tau,1+)\,|\,\eta^-(\tau,1)\,d\tau,\\
\nu^+_{\tau,0}(\epsilon_0, v) &= \sup_{0\le\Delta\le\epsilon_0} \|\, v(\Delta) - v(0)\,\|_{L^1(0,1)},\\
\nu^-_{\tau,T}(\epsilon_0, v) &= \sup_{0\le\Delta\le\epsilon_0} \|\, v(T-\Delta) - v(T)\,\|_{L^1(0,1)},\\
\nu_\tau(\epsilon_0, v) &= \sup_{\substack{|\,\tau-\tau'\,|\le\epsilon_0\\ \tau,\tau'\in[0,T]}} \|\, v(\tau) - v(\tau')\,\|_{L^1(0,1)},\\
\upsilon_\tau(\epsilon_0,\eta) &= \sup_{|\,\Delta\,|\le\epsilon_0}\int_0^T \chi(\tau+\Delta)\left|\int_0^1 (\eta(\tau,x)-\eta(\tau+\Delta,x))\,dx\right| d\tau,
\end{aligned}$$

where ϵ_0 and ϵ are arbitrary positive numbers, $v(\cdot,0-)$ denotes the boundary data for v at $x=0$, $v(\cdot,1+)$ the boundary data for v at $x=1$, and χ is the characteristic function of the interval $[0,T]$, and in terms of the entropy form $E^{\epsilon_0,\epsilon}(v,u;\eta)$

$$E^{\epsilon_0,\epsilon}(v,u;\eta) = \int_0^T\int_0^1 \Theta(v,u(\tau,x);\eta;\varphi(\tau,x;\cdot,\cdot))\,dx\,d\tau, \tag{5.3a}$$

where

(5.3b)
$$\begin{aligned}
\Theta(v,c;\eta;\varphi(\tau,x;\cdot,\cdot)) = &-\int_0^T\int_0^1 |v(\tau',x')-c|\,\varphi_{\tau'}(\tau,x;\tau',x')\,dx'\,d\tau'\\
&-\int_0^T\int_0^1 |v(\tau',x')-c|\,\eta(\tau',x')\,\varphi_{x'}(\tau,x;\tau',x')\;dx'\,d\tau'\\
&+\int_0^1 |v(T,x')-c|\,\varphi(\tau,x;T,x')\,dx'\\
&-\int_0^1 |v(0,x')-c|\,\varphi(\tau,x;0,x')\,dx'\\
&+\int_0^T G(v(\tau',1-)-c, v(\tau',1+)-c\,;\,\eta(\tau',1))\,\varphi(\tau,x;\tau',1)\,d\tau'\\
&-\int_0^T G(v(\tau,0-)-c, v(\tau',0+)-c\,;\,\eta(\tau',0))\,\varphi(\tau,x;\tau',0)\,d\tau'\\
&-\int_0^T\int_0^1 \eta_{x'}(\tau',x')\,V(v(\tau',x'),c)\,\varphi(\tau,x;\tau',x')\;dx'\,d\tau',
\end{aligned}$$

where the 'entropy' flux G and the function V are defined by

$$G(v_{left}, v_{right};\eta) = |v_{left}|\,\eta^+ + |v_{right}|\,\eta^-, \tag{5.3c}$$

$$V(v,c) = |v-c| - v\ \mathrm{sign}(v-c), \tag{5.3d}$$

and $\varphi(\tau,x;\tau',x') = \omega_{\epsilon_0}(\tau-\tau')\,\omega_\epsilon(x-x')$, where $\omega_\nu(s) = \omega(s/\nu)/\nu$, $\forall s\in\mathbb{R}$. The function $\omega:\mathbb{R}\to\mathbb{R}$ is an even nonnegative $\mathcal{C}^\infty(\mathbb{R})$ with support included in $[-1,1]$

and such that $\int_{-1}^{1} \omega\, dx = 1$. We can now state our approximation result, which is a particular case of Theorem 2.1 in [2] (the case '$M = \infty$').

THEOREM 5.1 ([2]). *Let (u, β) and (v, η) be functions satisfying the regularity conditions* (5.1). *Then, there exists a constant $\mathcal{C}$ such that*

$$
\begin{aligned}
\| v(T) - u(T) \|_{L^1(0,1)} \le \| v(0) - u(0) \|_{L^1(0,1)} \\
&+ \mathcal{C} \Big\{ \| v(1+) - u(1+) \|_{L^1(0,T)} + \| v(0-) - u(0-) \|_{L^1(0,T)} \\
&+ \int_0^T \left| \int_0^1 \big(\eta(\tau, x) - \beta(\tau, x)\big)\, dx \right| d\tau \\
&+ \int_0^T \| \partial_{x'} \eta(\tau') + (1 - v(\tau')) \|_{L^\infty(0,1)}\, d\tau' \\
&+ \int_0^T \| \partial_x \beta(\tau) + (1 - u(\tau)) \|_{L^\infty(0,1)}\, d\tau \\
&+ \{ \nu^-_{x,1}(\epsilon, u; \beta) + \nu^-_{x,1}(\epsilon, v; \eta) + \nu^+_{x,0}(\epsilon, u; \beta) + \nu^+_{x,0}(\epsilon, v; \eta) \} \\
&+ \{ \nu^-_{\tau,T}(\epsilon_0, u) + \nu^-_{\tau,T}(\epsilon_0, v) + \nu^+_{\tau,0}(\epsilon_0, u) + \nu^+_{\tau,0}(\epsilon_0, v) \} \\
&+ \{ \nu_\tau(\epsilon_0, u) + \nu_\tau(\epsilon_0, v) + \upsilon_\tau(\epsilon_0, \beta) \} \\
&+ E^{\epsilon_0,\epsilon}(v, u; \eta) + E^{\epsilon_0,\epsilon}(u, v; \beta) + \epsilon + \epsilon_0 \Big\},
\end{aligned}
$$

and, for $\tau \in [0, T]$,

$$
\begin{aligned}
\| \eta(\tau) - \beta(\tau) \|_{L^\infty(0,1)} \le \| v(\tau) - u(\tau) \|_{L^1(0,1)} \\
&+ \| \partial_x \eta(\tau) + (1 - v(\tau)) \|_{L^1(0,1)} \\
&+ \| \partial_x \beta(\tau) + (1 - u(\tau)) \|_{L^1(0,1)} \\
&+ \left| \int_0^1 \big(\eta(\tau, x) - \beta(\tau, x)\big)\, dx \right|.
\end{aligned}
$$

We take (u, β) equal to the exact solution of (1.1) and (1.2) and (v, η) equal to the function $(\tilde{u}_h, \beta_h)$, where $\tilde{u}_h$ is an interpolation of u_h. We define $\tilde{u}_h$ to be a piecewise bilinear, continuous (in $[0, T] \times [0, 1]$) function determined by the following values at 'the interpolation nodes':

$$
\left\{
\begin{aligned}
\tilde{u}_h(\tau^n, 0) &= u^n_0, \\
\tilde{u}_h(\tau^n, x_j) &= u^n_j, \qquad \text{where } x_j = (x_{j-1/2} + x_{j+1/2})/2, \\
\tilde{u}_h(\tau^n, 1) &= u^n_1,
\end{aligned}
\right.
\tag{5.4}
$$

for $n = 0, \cdots, n_T$ and $j = 1, \cdots, n_x$. The following simple result states that $\tilde{u}_h$ and u_h are reasonably close.

LEMMA 5.2. *Suppose that the hypotheses on the data* (2.2) *and the* CFL *condition* (4.1) *are satisfied. Then,*

$$
\| \tilde{u}_h - u_h \|_{L^\infty(0,T;L^1(0,1))} \le (C_1 + C_2)\, \Delta x.
$$

The next result displays the smoothness properties of $\tilde{u}_h$ relevant for our error estimates. It follows directly form Theorem 4.2, the definition of $\tilde{u}_h$, and the definitions (5.2).

Lemma 5.3. *Assume that the hypotheses on the data* (2.1) *and* (2.2) *and the* CFL *condition* (4.1) *are satisfied. Then, for Δx satisfying the conditions* (4.3),

$$\begin{aligned}
\nu^+_{x,0}(\epsilon, \tilde{u}_h; \beta_h) &\le C_7\,(\epsilon + \Delta x),\\
\nu^-_{x,1}(\epsilon, \tilde{u}_h; \beta_h) &\le C_8\,(\epsilon + \Delta x),\\
\nu^+_{\tau,0}(\epsilon_0, \tilde{u}_h) &\le C_2\,(\epsilon_0 + \Delta x),\\
\nu^-_{\tau,T}(\epsilon_0, \tilde{u}_h) &\le C_2\,(\epsilon_0 + \Delta x),\\
\nu_{\tau}(\epsilon_0, \tilde{u}_h) &\le C_2\,(\epsilon_0 + \Delta x),
\end{aligned}$$

where $C_7 = (C_1\,\max\{1, u^\star - 1\} + C_3 + T\,C_5 + T\,C_6)$ *and* $C_8 = (C_1\,\max\{1, u^\star - 1\} + C_4 + T\,C_5 + T\,C_6)$.

Finally, the following result shows that the entropy form $E^{\epsilon_0,\epsilon}(u_h, u; \beta_h)$ is suitably bounded. In our case, this is a direct consequence of the fact that the 'residue' is small enough.

Lemma 5.4 (Bound on the entropy form $E^{\epsilon_0,\epsilon}(\tilde{u}_h, u; \beta_h)$). *Assume that the hypotheses on the data* (2.1) *and* (2.2) *and the* CFL *condition* (4.1) *are satisfied. Then, for Δx satisfying the conditions* (4.3),

$$\begin{aligned}
E^{\epsilon_0,\epsilon}(u_h, u; \beta_h) &= \int_0^T\!\!\int_0^1\!\!\int_0^T\!\!\int_0^1 \mathrm{residue}(\tau', x')\,\bar{\varphi}(\tau, x; \tau', x')\,dx'\,d\tau'\,dx d\tau,\\
&\le C_9\,\Delta x,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{residue}(\tau', x') &= (\tilde{u}_h)_{\tau'} + (\beta_h\,\tilde{u}_h)_{x'},\\
\bar{\varphi}(\tau, x; \tau', x') &= \mathrm{sign}(\tilde{u}(\tau', x') - u(\tau, x))\,\varphi(\tau, x; \tau', x'),
\end{aligned}$$

and $C_9 = \max\{1, u^\star - 1\}\,T\,(C_1 + C_2) + 2C_5 + (2 + \max\{1, u^\star - 1\}/2)\,C_6$.

We can now state and prove our main result.

Theorem 5.5. (Error Estimates). *Suppose that the hypotheses on the data* (2.1) *and* (2.2) *and the CFL condition* (4.1) *are satisfied. Then, for Δx satisfying the conditions* (4.3),

$$\begin{aligned}
\|u - u_h\|_{L^\infty(0,T;L^1(0,1))} &\le C\Delta x,\\
\|\beta - \beta_h\|_{L^\infty(0,T;L^\infty(0,1))} &\le C\Delta x,
\end{aligned}$$

where the constant C depends on the constants C_i, $i = 1, \cdots, 9$, of Theorem 4.2.

Proof. By Lemma 5.2, we have

$$\|u - u_h\|_{L^\infty(0,T;L^1(0,1))} \le (C_1 + C_2)\,\triangle x + \|u - \tilde{u}_h\|_{L^\infty(0,T;L^1(0,1))}.$$

To estimate the second term of the right hand side, we use Theorem 5.1 with $(v, \eta) = (\tilde{u}_h, \beta_h)$. Thus, proceeding as in [2], we get

$$\begin{aligned}
\|\tilde{u}_h(0) - u(0)\|_{L^1(0,1)} &\le |\,u_i\,|_{BV(0,1)}\triangle x,\\
\|\tilde{u}_h(1+) - u(1+)\|_{L^1(0,T)} &= 0,\\
\|\tilde{u}_h(0-) - u(0-)\|_{L^1(0,T)} &= 0,\\
\int_0^T \left| \int_0^1 (\beta_h(\tau, x) - \beta(\tau, x))dx \right| dt &\le |\phi_1|_{BV(0,T)}\triangle\tau,\\
\int_0^T \|\partial_{x'}\beta_h(\tau') + (1 - \tilde{u}_h(\tau'))\|_{L^\infty(0,1)}d\tau' &= \int_0^T \|u_h(\tau') - \tilde{u}_h(\tau')\|_{L^\infty(0,1)}d\tau'\\
&\le T\,(C_1 + C_2)\,\triangle x,\\
\int_0^T \|\partial_x\beta(\tau) + (1 - u(\tau))\|_{L^\infty(0,1)}d\tau &= 0.
\end{aligned}$$

By Lemma 5.3,

$$\begin{aligned}
\nu^-_{x,1}(\epsilon, \tilde u_h; \beta_h) + \nu^+_{x,0}(\epsilon, \tilde u_h; \beta_h) &\le (C_7 + C_8)\,(\epsilon + \Delta x),\\
\nu^+_{\tau,0}(\epsilon_0, \tilde u_h) + \nu^-_{\tau,T}(\epsilon_0, \tilde u_h) &\le 2\,C_2\,(\epsilon_0 + \Delta\tau),\\
\nu_\tau(\epsilon_0, \tilde u_h) &\le C_2\,(\epsilon_0 + \Delta\tau).
\end{aligned}$$

As a consequence of the above inequalities and Theorem 4.1, we have

$$\begin{aligned}
\nu^-_{x,1}(\epsilon, u; \beta) + \nu^+_{x,0}(\epsilon, u; \beta) &\le (C_7 + C_8)\,\epsilon,\\
\nu^+_{\tau,0}(\epsilon_0, u) + \nu^-_{\tau,T}(\epsilon_0, u) &\le 2\,C_2\,(\epsilon_0 + \Delta\tau),\\
\nu_\tau(\epsilon_0, u) &\le C_2\,(\epsilon_0 + \Delta\tau).
\end{aligned}$$

By (5.2), (1.2), and (2.2g),

$$v_\tau(\epsilon_0, \beta) \le |\,\phi_1\,|_{BV(0,1)}\epsilon_0.$$

Finally, we have,

$$\begin{aligned}
E^{\epsilon_0,\epsilon}(u, u_h; \beta) &\le 0,\\
E^{\epsilon_0,\epsilon}(u_h, u; \beta_h) &\le C_9\,\Delta x, \text{ by Lemma 5.4.}
\end{aligned}$$

Combining all the above results, we get

$$\|u_h(T) - u(T)\|_{L^1(0,1)} \le C(\triangle x + \epsilon + \epsilon_0),$$

where the constant C depends on the constants C_i, $i = 1, \cdots, 9$ and is independent of the parameters ϵ and ϵ_0. Thus, the first estimate follows by letting ϵ_0 and ϵ go to zero. The second estimate can now be easily obtained from Theorem 5.1. This completes the proof. □

6. Proof of Theorem 4.2. In this section we shall complete the proof of Theorem 4.2.

a. Proof of (4.4e) and (4.4f). Let us prove property (4.4e). Property (4.4f) can be proven in a similar way. Since condition (4.3a) is satisfied, if $\beta(\cdot, 0) < 0$, then (4.4e) follows trivially from (4.2a'). If $\beta(\cdot, 0) > 0$, then, by (4.2a'), we have $\beta_h(\cdot, 0) \ge \beta_0/2$ and (using (4.2a'), (4.2c'), (2.2a), (3.1b) and the CFL condition (4.1)) we can write

$$\begin{aligned}
u^n_1 - u^n_0 &= (1 - \Delta\tau\,\beta^{n-1}_{1/2}/\Delta x)\,(u^{n-1}_1 - u^{n-1}_0)\\
&\le (1 - \Delta\tau\,\beta_0/(2\Delta x))\,(u^{n-1}_1 - u^{n-1}_0),\\
&\le (1 - \Delta\tau\,\beta_0/(2\Delta x))^n\,(u^0_1 - u^0_0),\\
&\le (1 - \Delta\tau\,\beta_0/(2\Delta x))^n\,\|\,(u_i)_x\,\|_{L^\infty(0,1)}\,\Delta x/2,
\end{aligned}$$

by (3.1d) and (2.2d). Hence,

$$\begin{aligned}
\sum_{n=0}^{n_T-1} \beta^{n+}_h(0)\,|\,u^n_1 - u^n_0\,|\,\Delta\tau &\le m^\star \sum_{n=0}^{n_T-1} |\,u^n_1 - u^n_0\,|\,\Delta\tau\\
&\le (m^\star\,\|\,(u_i)_x\,\|_{L^\infty(0,1)}/\beta_0)\,(\Delta x)^2.
\end{aligned}$$

This completes the proof of property (4.4e).

b. Proof of (4.4g) and (4.4h). To prove properties (4.4g) and (4.4h), we need several preliminary lemmas.

LEMMA 6.1 ([1]). *We have, for $n = 0, \cdots, n_T - 1$, the following equalities:*

(6.1) $$\beta^n_{j+\frac{1}{2}} - \beta^n_{j-\frac{1}{2}} = (1 - u^n_j)\triangle x, \quad j = 1, \cdots, n_x,$$

(6.2a)
$$\begin{aligned} u^{n+1}_1 - u^{n+1}_0 = &\left(-\frac{\triangle\tau}{\triangle x}\beta^{n^-}_{\frac{3}{2}}\right)(u^n_2 - u^n_1) \\ &+ \left(1 + \triangle\tau(1 - u^n_1 - u^n_0) - \frac{\triangle\tau}{\triangle x}\beta^{n^+}_{\frac{1}{2}}\right)(u^n_1 - u^n_0) \\ &+ (1 + \triangle\tau(1 - u^n_0))u^n_0 - u^{n+1}_0, \end{aligned}$$

for $j = 1, \cdots, n_x - 1$,

(6.2b)
$$\begin{aligned} u^{n+1}_{j+1} - u^{n+1}_j = &\left(-\frac{\triangle\tau}{\triangle x}\beta^{n^-}_{j+\frac{3}{2}}\right)(u^n_{j+2} - u^n_{j+1}) \\ &+ \left(1 + \triangle\tau(1 - u^n_{j+1} - u^n_j) - \frac{\triangle\tau}{\triangle x}\beta^{n^+}_{j+\frac{1}{2}} + \frac{\triangle\tau}{\triangle x}\beta^{n^-}_{j+\frac{1}{2}}\right)(u^n_{j+1} - u^n_j) \\ &+ \left(\frac{\triangle\tau}{\triangle x}\beta^{n^+}_{j-\frac{1}{2}}\right)(u^n_j - u^n_{j-1}), \end{aligned}$$

and

(6.2c)
$$\begin{aligned} u^{n+1}_{n_x+1} - u^{n+1}_{n_x} = &u^{n+1}_{n_x+1} - \left(1 + \triangle\tau(1 - (u^n_{n_x+1})\right)u^n_{n_x+1} \\ &+ \left(1 + \triangle\tau(1 - u^n_{n_x+1} - u^n_{n_x}) + \frac{\triangle\tau}{\triangle x}\beta^{n^-}_{n_x+\frac{1}{2}}\right)(u^n_{n_x+1} - u^n_{n_x}) \\ &+ \left(\frac{\triangle\tau}{\triangle x}\beta^{n^+}_{n_x-\frac{1}{2}}\right)(u^n_{n_x} - u^n_{n_x-1}). \ \square \end{aligned}$$

LEMMA 6.2. *Suppose that the hypotheses (2.1) and (2.2) and the* CFL *condition (4.1) are satisfied. Then, for Δx satisfying the conditions (4.3), we have*

$$\max_{n_{\inf} \le j \le n^{\sup}} |u^n_{j+1} - u^n_j| \le \| (u_i)_x \|_{L^\infty(0,1)} e^{3T} \triangle x,$$

where

$$n_{\inf} = \begin{cases} 0, & \text{if } \beta(\cdot, 0) > 0, \\ 1, & \text{if } \beta(\cdot, 0) < 0, \end{cases}$$
$$n^{\sup} = \begin{cases} n_x, & \text{if } \beta(\cdot, 1) < 0, \\ n_x - 1, & \text{if } \beta(\cdot, 0) > 0. \end{cases}$$

Proof. Let us consider first the case in which $\beta_0 > 0$ and $\beta_1 < 0$. Set $\xi^n = \max_{0 \le j \le n_x} |u^n_{j+1} - u^n_j|$. Then, by the CFL condition (4.1), (2.2a), (2.2b), and the identities (6.2) of Lemma 6.1, we have

$$|u^{n+1}_1 - u^{n+1}_0| \le \left(1 + \triangle\tau(1 - u^n_1 - u^n_0) - \frac{\triangle\tau}{\triangle x}\beta^{n^+}_{\frac{1}{2}}\right)\xi^n,$$
$$|u^{n+1}_{j+1} - u^{n+1}_j| \le \left(1 + \triangle\tau(1 - u^n_j - u^n_{j+1}) - \frac{\triangle\tau}{\triangle x}(\beta^{n^-}_{j+\frac{3}{2}} + \beta^{n^+}_{j+\frac{1}{2}} - \beta^{n^-}_{j+\frac{1}{2}} - \beta^{n^+}_{j-\frac{3}{2}})\right)\xi^n,$$

for $j = 1, \cdots, n_x - 1$, and

$$|u^{n+1}_{n_x+1} - u^{n+1}_{n_x}| \leq \left(1 + \triangle\tau(1 - u^n_{n_x+1} - u^n_{n_x}) + \frac{\triangle\tau}{\triangle x}\beta^{n^-}_{n_x+\frac{1}{2}}\right)\xi^n.$$

Notice that here we are using the condition (4.3b) to ensure that $\beta^n_{3/2} > 0$ and that $\beta^n_{n_x+1/2} < 0$ by (6.1). Thus, using (6.1), we get

$$\xi^{n+1} \leq (1 + 3\triangle\tau)\xi^n.$$

The result follows easily from this inequality.

To prove the result in the case in which $\beta_0 < 0$ and $\beta_1 > 0$, we proceed as above with $\xi^n = \max_{1\leq j\leq n_x-1}|u^n_{j+1} - u^n_j|$. In this case, thanks to the hypothesis (4.3b), we have that $\beta^n_{3/2} < 0$ and that $\beta^n_{n_x+1/2} > 0$ by (6.1). Hence, it is not necessary to consider the quantities $(u^n_1 - u^n_0)$ and $(u^n_{n_x+1} - u^n_{n_x})$, as in the previous case. The other two cases can be proven in a similar way. This completes the proof of Lemma 6.2. □

The following result follows easily from Lemma 6.1 and simple, but tedious algebraic manipulations.

LEMMA 6.3. *We have, for $n = 0, \cdots, n_T - 1$,*

$$\begin{aligned}
u^{n+1}_2 &- 2u^{n+1}_1 + u^{n+1}_0 \\
&= \left(-\frac{\triangle\tau}{\triangle x}\beta^{n^-}_{\frac{5}{2}}\right)(u^n_3 - 2u^n_2 + u^n_1) \\
&\quad + \frac{\triangle\tau}{\triangle x}\left(-(u^n_2 - u^n_1)\,\Delta x - (\beta^{n^-}_{\frac{5}{2}} - 2\beta^{n^-}_{\frac{3}{2}} + \beta^{n^-}_{\frac{1}{2}})\right)(u^n_2 - u^n_1) \\
&\quad - \triangle\tau(u^n_1 - u^n_0)(u^n_2 - u^n_1) \\
&\quad + \left(1 + \triangle\tau(1 - u^n_1 - u^n_0) - \frac{\triangle\tau}{\triangle x}(\beta^{n^+}_{\frac{3}{2}} - \beta^{n^-}_{\frac{1}{2}})\right)(u^n_2 - 2u^n_1 + u^n_0) \\
&\quad + \frac{\triangle\tau}{\triangle x}\left(\beta^n_{\frac{1}{2}} + (\beta^{n^+}_{\frac{1}{2}} - \beta^{n^+}_{\frac{3}{2}})\right)(u^n_1 - u^n_0),
\end{aligned}$$

for $j = 2, \cdots, n_x - 1$,

$$\begin{aligned}
u^{n+1}_{j+1} &- 2u^{n+1}_j + u^{n+1}_{j-1} \\
&= \left(-\frac{\triangle\tau}{\triangle x}\beta^{n^-}_{j+\frac{3}{2}}\right)(u^n_{j+2} - 2u^n_{j+1} + u^n_j) \\
&\quad + \frac{\triangle\tau}{\triangle x}\left(-(u^n_{j+1} - u^n_j)\,\Delta x - (\beta^{n^-}_{j+\frac{3}{2}} - 2\beta^{n^-}_{j+\frac{1}{2}} + \beta^{n^-}_{j-\frac{1}{2}})\right)(u^n_{j+1} - u^n_j) \\
&\quad - \triangle\tau(u^n_j - \hat{u}^n_j)(u^n_{j+1} - u^n_j) \\
&\quad + \left(1 + \triangle\tau(1 - u^n_j - \hat{u}^n_j) - \frac{\triangle\tau}{\triangle x}(\beta^{n^+}_{j+\frac{1}{2}} - \beta^{n^-}_{j-\frac{1}{2}})\right)(u^n_{j+1} - 2u^n_j + u^n_{j-1}) \\
&\quad - \triangle\tau(\hat{u}^n_j - u^n_j)(u^n_j - u^n_{j-1}) \\
&\quad + \frac{\triangle\tau}{\triangle x}\left(-(u^n_j - u^n_{j-1})\,\Delta x - (\beta^{n^+}_{j+\frac{1}{2}} - 2\beta^{n^+}_{j-\frac{1}{2}} + \beta^{n^+}_{j-\frac{3}{2}})\right)(u^n_j - u^n_{j-1}) \\
&\quad + \left(\frac{\triangle\tau}{\triangle x}\beta^{n^+}_{j-\frac{3}{2}}\right)(u^n_j - 2u^n_{j-1} + u^n_{j-2}),
\end{aligned}$$

where $\hat{u}_j^n$ *is an arbitrary real number, and*

$$
\begin{aligned}
&u^{n+1}_{n_x+1} - 2u^{n+1}_{n_x} + u^{n+1}_{n_x-1} \\
&\quad = \frac{\triangle\tau}{\triangle x}\left(\beta^n_{n_x+\frac12} + (\beta^{n^-}_{n_x+\frac12} - \beta^{n^-}_{n_x-\frac12})\right)(u^n_{n_x+1} - u^n_{n_x}) \\
&\quad + \left(1 + \triangle\tau(1 - u^n_{n_x} - u^n_{n_x+1}) - \frac{\triangle\tau}{\triangle x}(\beta^{n^+}_{n_x+\frac12} - \beta^{n^-}_{n_x-\frac12})\right)(u^n_{n_x+1} - 2u^n_{n_x} + u^n_{n_x-1}) \\
&\quad - \triangle\tau(u^n_{n_x+1} - u^n_{n_x})(u^n_{n_x} - u^n_{n_x-1}) \\
&\quad + \frac{\triangle\tau}{\triangle x}\left(-(u^n_{n_x} - u^n_{n_x-1})\,\Delta x - (\beta^{n^+}_{n_x+\frac12} - 2\beta^{n^+}_{n_x-\frac12} + \beta^{n^+}_{n_x-\frac32})\right)(u^n_{n_x} - u^n_{n_x-1}) \\
&\quad + \left(\frac{\triangle\tau}{\triangle x}\beta^{n^+}_{n_x-\frac32}\right)(u^n_{n_x} - 2u^n_{n_x-1} + u^n_{n_x-2}). \quad \square
\end{aligned}
$$

Lemma 6.4. *Suppose that the hypotheses* (2.1) *and* (2.2) *and the* CFL *condition* (4.1) *are satisfied. Then, for* Δx *satisfying conditions* (4.3), *we have*

$$
\sum_{j=n_{\inf}+1}^{n^{\sup}-1} |u^n_{j+1} - 2u^n_j + u^n_{j-1}| \le C_{10}\triangle x, \quad n = 0, \cdots, n_T,
$$

where $C_{10} = e^{2T}\left\{|\,(u_i)_{xx}\,|_{BV(0,1)} + 3\,e^{3T}\,T\,C_1\,\|\,(u_i)_x\,\|_{L^\infty(0,1)} + 2\,C_3 + 2\,C_4\right\}.$

Proof. Let us consider first the case in which $\beta_0 > 0$ and $\beta_1 < 0$ (thus, $n_{\inf}+1 = 1$ and $n^{\sup} - 1 = n_x$). We take $\hat{u}^n_j = u^n_j$ in Lemma 6.3. After simple, but lengthy algebraic manipulations, we obtain

$$
\begin{aligned}
\sum_{j=1}^{n_x} |u^{n+1}_{j+1} - 2u^{n+1}_j + u^{n+1}_{j-1}| \le &(1 + 2\Delta\tau)\sum_{j=1}^{n_x} |u^n_{j+1} - 2u^n_j + u^n_{j-1}| \\
&+ 3\sup_{1\le j\le n_x} |\,u^n_{j+1} - u^n_j\,|\,|\,u^n_h\,|_{BV(0,1)}\Delta\tau \\
&+ \left(\frac{\triangle\tau}{\triangle x}\beta^{n^+}_{1/2} + \Delta\tau\,|\,1 - u^n_1\,|\right)|\,u^n_1 - u^n_0\,| \\
&+ \left(-\frac{\triangle\tau}{\triangle x}\beta^{n^-}_{n_x+1/2} + \Delta\tau\,|\,1 - u^n_{n_x}\,|\right)|\,u^n_{n_x+1} - u^n_{n_x}\,|.
\end{aligned}
$$

¿From the above inequality, it is easy to obtain, for $0 \le m \le n_T$,

$$
\begin{aligned}
&\sum_{j=1}^{n_x} |u^m_{j+1} - 2u^m_j + u^m_{j-1}| \\
&\qquad \le e^{2T}\sum_{j=1}^{n_x} |u^0_{j+1} - 2u^0_j + u^0_{j-1}| \\
&\qquad + 3\,e^{2T}\sum_{n=0}^{n_T-1}\sup_{1\le j\le n_x} |\,u^n_{j+1} - u^n_j\,|\,|\,u^n_h\,|_{BV(0,1)}\Delta\tau \\
&\qquad + e^{2T}\sum_{n=0}^{n_T-1}\left(\frac{\triangle\tau}{\triangle x}\beta^{n^+}_{1/2} + \Delta\tau\,|\,1 - u^n_1\,|\right)|\,u^n_1 - u^n_0\,| \\
&\qquad + e^{2T}\sum_{n=0}^{n_T-1}\left(-\frac{\triangle\tau}{\triangle x}\beta^{n^-}_{n_x+1/2} + \Delta\tau\,|\,1 - u^n_{n_x}\,|\right)|\,u^n_{n_x+1} - u^n_{n_x}\,|.
\end{aligned}
$$

By (3.1d), (2.2d), Lemma 6.2, properties (4.4c), (4.4e), (4.4a), and (4.4f), and condition (4.3b), we see that

$$\begin{aligned}\sum_{j=1}^{n_x} |u_{j+1}^m - 2u_j^m + u_{j-1}^m| \leq e^{2T} \, |(u_i)_{xx}|_{BV(0,1)} \Delta x \\ + 3\, e^{5T} T\, C_1 \, \|(u_i)_x\|_{L^\infty(0,1)} \Delta x \\ + e^{2T}\, C_3 (1 + 2\Delta x \, |1 - u_1^n| / \beta_0)\, \Delta x \\ + e^{2T}\, C_4 (1 + 2\Delta x \, |1 - u_{n_x}^n| / \beta_1)\, \Delta x \\ \leq e^{2T} \left\{ |(u_i)_{xx}|_{BV(0,1)} + 3\, e^{3T} T\, C_1 \, \|(u_i)_x\|_{L^\infty(0,1)} + 2\, C_3 + 2\, C_4 \right\} \Delta x.\end{aligned}$$

Now, let us consider the case in which $\beta_0 < 0$ and $\beta_1 > 0$ (thus, $n_{\text{inf}} + 1 = 2$ and $n^{\text{sup}} - 1 = n_x - 1$). In this case, we take

$$\hat{u}_j^n = \begin{cases} u_1^n, & \text{for } j = 2, \\ u_j^n, & \text{for } j = 3, \cdots, n_x - 2, \\ u_{n_x}^n, & \text{for } j = n_x - 1, \end{cases}$$

in Lemma 6.3. With this choice and by condition (4.3b), we have

$$\begin{aligned} & u_3^{n+1} - 2u_2^{n+1} + u_1^{n+1} \\ &= \left(-\frac{\triangle\tau}{\triangle x} \beta_{\frac{7}{2}}^{n^-} \right) (u_4^n - 2u_3^n + u_2^n) \\ &+ \frac{\triangle\tau}{\triangle x} \left(-(u_3^n - u_2^n)\, \Delta x - (\beta_{\frac{7}{2}}^{n^-} - 2\beta_{\frac{5}{2}}^{n^-} + \beta_{\frac{3}{2}}^{n^-}) \right) (u_3^n - u_2^n) \\ &- \triangle\tau (u_2^n - u_1^n)(u_3^n - u_2^n) \\ &+ \left(1 + \triangle\tau (1 - u_2^n - u_1^n) - \frac{\triangle\tau}{\triangle x} (\beta_{\frac{5}{2}}^{n^+} - \beta_{\frac{3}{2}}^{n^-}) \right) (u_3^n - 2u_2^n + u_1^n), \end{aligned}$$

and

$$\begin{aligned} & u_{n_x}^{n+1} - 2u_{n_x-1}^{n+1} + u_{n_x-2}^{n+1} \\ &= \left(1 + \triangle\tau (1 - u_{n_x-1}^n - u_{n_x}^n) - \frac{\triangle\tau}{\triangle x} (\beta_{n_x-\frac{1}{2}}^{n^+} - \beta_{n_x-\frac{3}{2}}^{n^-}) \right) (u_{n_x}^n - 2u_{n_x-1}^n + u_{n_x-2}^n) \\ &- \triangle\tau (u_{n_x}^n - u_{n_x-1}^n)(u_{n_x-1}^n - u_{n_x-2}^n) \\ &+ \frac{\triangle\tau}{\triangle x} \left(-(u_{n_x-1}^n - u_{n_x-2}^n)\, \Delta x - (\beta_{n_x-\frac{1}{2}}^{n^+} - 2\beta_{n_x-\frac{3}{2}}^{n^+} + \beta_{n_x-\frac{5}{2}}^{n^+}) \right) (u_{n_x-1}^n - u_{n_x-2}^n) \\ &+ \left(\frac{\triangle\tau}{\triangle x} \beta_{n_x-\frac{5}{2}}^{n^+} \right) (u_{n_x-1}^n - 2u_{n_x-2}^n + u_{n_x-3}^n). \end{aligned}$$

We can thus proceed as in the previous case. The other two cases can be proven in a similar fashion. This completes the proof. ∎

We are now ready to prove the properties (4.4g) and (4.4h). Let us prove (4.4g). Since

$$\begin{aligned}\beta_{j+1/2}^{n^+}(u_{j+1}^n - u_j^n) - \beta_{j-1/2}^{n^+}(u_j^n - u_{j-1}^n) =& \beta_{j+1/2}^{n^+}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \\ &+ (\beta_{j+1/2}^{n^+} - \beta_{j-1/2}^{n^+})(u_j^n - u_{j-1}^n),\end{aligned}$$

we have

$$\sum_{j=1}^{n_x} | \beta^{n+}_{j+1/2}(u^n_{j+1} - u^n_j) - \beta^{n+}_{j-1/2}(u^n_j - u^n_{j-1})| \le \sum_{j=1}^{n_x} \beta^{n+}_{j+1/2} \, | u^n_{j+1} - 2u^n_j + u^n_{j-1} |$$
$$+ \sum_{j=1}^{n_x} | \beta^{n+}_{j+1/2} - \beta^{n+}_{j-1/2} | \, | u^n_j - u^n_{j-1} |$$
$$\le (m^\star C_{10} + \max\{1, u^\star - 1\} \, C_1) \, \Delta x,$$

by Lemma 6.4, (4.4a) and (4.4c). This completes the proof of property (4.4g). The property (4.4h) can be proven in a similar way.

c. Proof of (4.4i). To prove property (4.4i), we need the following preliminary result that follows from Lemma 6.1.

Lemma 6.5. *We have, for $n = 0, \cdots, n_T - 1$, the following equalities:*

$$(u^{n+1}_1 - u^{n+1}_0) - (u^n_1 - u^n_0) = \left(-\frac{\triangle\tau}{\triangle x}\beta^{n-}_{\frac{3}{2}}\right) (u^n_2 - 2u^n_1 + u^n_0)$$
$$+ \triangle\tau(1 - u^n_1)\, u^n_1 - \frac{\triangle\tau}{\triangle x}(\beta^{n+}_{\frac{1}{2}} + \beta^{n-}_{\frac{3}{2}})\,(u^n_1 - u^n_0),$$
$$(u^{n+1}_{j+1} - u^{n+1}_j) - (u^n_{j+1} - u^n_j) = \left(-\frac{\triangle\tau}{\triangle x}\beta^{n-}_{j+\frac{3}{2}}\right)(u^n_{j+2} - 2u^n_{j+1} + u^n_j)$$
$$+ \left(\triangle\tau(1 - u^n_{j+1} - u^n_j) - \frac{\triangle\tau}{\triangle x}(\beta^{n+}_{j+\frac{1}{2}} - \beta^{n+}_{j-\frac{1}{2}}) + (\beta^{n-}_{j+\frac{3}{2}} - \beta^{n-}_{j+\frac{1}{2}})\right)(u^n_{j+1} - u^n_j)$$
$$+ \left(-\frac{\triangle\tau}{\triangle x}\beta^{n+}_{j-\frac{1}{2}}\right)(u^n_{j+1} - 2u^n_j + u^n_{j-1}),$$

for $j = 1, \cdots, n_x - 1$, and

$$(u^{n+1}_{n_x+1} - u^{n+1}_{n_x}) - (u^n_{n_x+1} - u^n_{n_x}) = \frac{\triangle\tau}{\triangle x}\left(\beta^{n-}_{n_x+\frac{1}{2}} + \beta^{n+}_{n_x-\frac{1}{2}}\right)(u^n_{n_x+1} - u^n_{n_x})$$
$$- \triangle\tau(1 - u^n_{n_x})\, u^n_{n_x} + \left(-\frac{\triangle\tau}{\triangle x}\beta^{n+}_{n_x-\frac{1}{2}}\right)(u^n_{n_x+1} - 2u^n_{n_x} + u^n_{n_x-1}).$$

We can now prove the property (4.4i). By Lemma 6.5, Lemma 6.4, (4.4a), (4.4b), (4.4c), and Lemma 6.2, we have

$$\begin{aligned}
&\sum_{j=0}^{n_x} |\,(u_{j+1}^{n+1} - u_j^{n+1}) - (u_{j+1}^n - u_j^n)\,| \\
&\quad \le \sum_{j=1}^{n_x} \frac{\triangle\tau}{\triangle x} \beta_{j+\frac{1}{2}}^{n\,-}\, |\, u_{j+1}^n - 2u_j^n + u_{j-1}^n\,| \\
&\quad + \sum_{j=1}^{n_x} \frac{\triangle\tau}{\triangle x} \beta_{j-\frac{1}{2}}^{n\,+}\, |\, u_{j+1}^n - 2u_j^n + u_{j-1}^n\,| \\
&\quad + \triangle\tau |1 - u_1^n|\, u_1^n + \frac{\triangle\tau}{\triangle x}\left(\beta_{\frac{1}{2}}^{n\,+} - \beta_{\frac{3}{2}}^{n\,-}\right) |u_1^n - u_0^n| \\
&\quad + \sum_{j=1}^{n_x-1} |\triangle\tau(1 - u_{j+1}^n - u_j^n) - \frac{\triangle\tau}{\triangle x}(\beta_{j+\frac{1}{2}}^{n\,+} - \beta_{j-\frac{1}{2}}^{n\,+}) + (\beta_{j+\frac{3}{2}}^{n\,-} - \beta_{j+\frac{1}{2}}^{n\,-})|\, |u_{j+1}^n - u_j^n| \\
&\quad + \frac{\triangle\tau}{\triangle x}(-\beta_{n_x+\frac{1}{2}}^{n\,-} + \beta_{n_x-\frac{1}{2}}^{n\,+})\, |u_{n_x+1}^n - u_{n_x}^n| + \triangle\tau|1 - u_{n_x}^n|\, u_{n_x}^n \\
&\quad \le (2C_5 + C_{11} + 2\|\,(u_i)_x\,\|_{L^\infty(0,1)}\, e^{3T})\triangle\tau \\
&\quad + \frac{\triangle\tau}{\triangle x}\beta_{\frac{1}{2}}^{n\,+}\, |u_1^n - u_0^n| \\
&\quad + \frac{\triangle\tau}{\triangle x}(-\beta_{n_x+\frac{1}{2}}^{n\,-})\, |u_{n_x+1}^n - u_{n_x}^n|,
\end{aligned}$$

where C_{11} depends on $u^\star$ and C_1. Finally, by (4.4e) and (4.4f), we get

$$\begin{aligned}
\sum_{n=0}^{n_T-1}\sum_{j=0}^{n_x} |\,(u_{j+1}^{n+1} - u_j^{n+1}) - (u_{j+1}^n - u_j^n)\,| \le &(2C_5 + C_{11} + 2\|\,(u_i)_x\,\|_{L^\infty(0,1)}\, e^{3T})\, T \\
&+ 2(C_3 + C_4)\,\Delta x.
\end{aligned}$$

This completes the proof of property (4.4i).

7. Proof of Lemma 5.4. In this section we prove Lemma 5.4. The first equality of the lemma follows directly from the definition of the entropy form $E^{\epsilon_0,\epsilon}(\tilde{u}_h, u; \beta_h)$, (5.3), the definition of $\tilde{u}_h$, (5.4), and a simple integration by parts.

To obtain the inequality, we observe that, by (5.3),

$$E^{\epsilon_0,\epsilon}(u_h, u; \beta_h) \le \int_0^T\!\!\int_0^1 |\,\mathrm{residue}(\tau, x)\,|\, dx\, d\tau.$$

For $\tau^n < \tau < \tau^{n+1}$ and for $x_{j-1/2} < x < x_j$, we have, using Taylor expansions, (5.4), and (3.2),

$$
\begin{aligned}
&\text{residue}(\tau,x)\\
&= (\tilde{u}_h)_\tau(\tau,x) + \beta_h(\tau^n,x)\,(\tilde{u}_h)_x(\tau,x) + (\beta_h)_x(\tau^n,x)\,\tilde{u}_h(\tau,x)\\
&= \Big\{(\tilde{u}_h)_\tau(\tau^n,x_j) + \beta_h(\tau^n,x_{j-1/2})\,(\tilde{u}_h)_x(\tau^n,x_j-0) + (\beta_h)_x(\tau^n,x)\,\tilde{u}_h(\tau^n,x_j)\Big\}\\
&+ \Big\{\beta_h(\tau^n,x) - \beta_h(\tau^n,x_{j-1/2})\Big\}\,(\tilde{u}_h)_x(\tau^n,x_j-0)\\
&+ \Big\{(\tau-\tau^n)\,\beta_h(\tau^n,x) + (x-x_j) + (\tau-\tau^n)(x-x_j)\beta_h(\tau^n,x)/2\Big\}\,(\tilde{u}_h)_{\tau x}(\tau^n,x_j-0)\\
&+ \Big\{(\tau-\tau^n)\,(\tilde{u}_h)_\tau(\tau^n,x_j) + (x-x_j)\,(\tilde{u}_h)_x(\tau^n,x_j-0)\Big\}\,\beta_h(\tau^n,x)\\
&= \Big\{-\beta^{n^-}_{j+1/2}(u^n_{j+1}-u^n_j) + \beta^{n^-}_{j-1/2}(u^n_j-u^n_{j-1})\Big\}\\
&+ \Big\{\beta_h(\tau^n,x) - \beta_h(\tau^n,x_{j-1/2})\Big\}\,(\tilde{u}_h)_x(\tau^n,x_j-0)\\
&+ \Big\{(\tau-\tau^n)\,\beta_h(\tau^n,x) + (x-x_j) + (\tau-\tau^n)(x-x_j)\beta_h(\tau^n,x)/2\Big\}\,(\tilde{u}_h)_{\tau x}(\tau^n,x_j-0)\\
&+ \Big\{(\tau-\tau^n)\,(\tilde{u}_h)_\tau(\tau^n,x_j) + (x-x_j)\,(\tilde{u}_h)_x(\tau^n,x_j-0)\Big\}\,\beta_h(\tau^n,x).
\end{aligned}
$$

Similarly, for $\tau^n < \tau < \tau^{n+1}$ and $x_j < x < x_{j+1/2}$, we have,

$$
\begin{aligned}
&\text{residue}(\tau,x)\\
&= \Big\{\beta^{n^+}_{j+1/2}(u^n_{j+1}-u^n_j) - \beta^{n^+}_{j-1/2}(u^n_j-u^n_{j-1})\Big\}\\
&+ \Big\{\beta_h(\tau^n,x) - \beta_h(\tau^n,x_{j+1/2})\Big\}\,(\tilde{u}_h)_x(\tau^n,x_j+0)\\
&+ \Big\{(\tau-\tau^n)\,\beta_h(\tau^n,x) + (x-x_j) + (\tau-\tau^n)(x-x_j)\beta_h(\tau^n,x)/2\Big\}\,(\tilde{u}_h)_{\tau x}(\tau^n,x_j+0)\\
&+ \Big\{(\tau-\tau^n)\,(\tilde{u}_h)_\tau(\tau^n,x_j) + (x-x_j)\,(\tilde{u}_h)_x(\tau^n,x_j+0)\Big\}\,\beta_h(\tau^n,x).
\end{aligned}
$$

It is thus easy to see that, by Theorem 4.2,

$$
\begin{aligned}
&E^{\epsilon_0,\epsilon}(u_h,u;\beta_h)\\
&\qquad \le \int_0^T\!\!\int_0^1 |\,\text{residue}(\tau,x)\,|\,dx\,d\tau\\
&\qquad \le \Big\{\max\{1,u^\star-1\}\,T\,(C_1+C_2) + 2C_5 + (2+\max\{1,u^\star-1\}/2)\,C_6\Big\}\,\Delta x,
\end{aligned}
$$

since $\|\,(\tilde{u}_h)_x\,\|_{L^\infty(0,T;L^1(0,1))} = |\,u_h\,|_{L^\infty(0,T;BV(0,1))}$ and $\|\,(\tilde{u}_h)_{\tau x}\,\|_{L^1(0,T;L^1(0,1))}$ is equal to $\sum_{n=0}^{n_T-1}\sum_{j=0}^{n_x}|\,(u^{n+1}_{j+1}-u^{n+1}_j)-(u^n_{j+1}-u^n_j)|$. This completes the proof of Lemma 5.4.

8. Extensions to the full unipolar model. In this section we shall extend the results in the last few sections to the full model

$$u_\tau + (u\,\beta)_x = \lambda^2\, u_{xx}, \quad \tau > 0,\; x \in (0,1), \tag{8.1a}$$
$$u(\tau,0) = u_0(\tau), \quad \tau \geq 0, \tag{8.1b}$$
$$u(\tau,1) = u_1(\tau), \quad \tau \geq 0, \tag{8.1c}$$
$$u(0,x) = u_i(x), \quad x \in (0,1), \tag{8.1d}$$

where λ is the normed Debye length and β is still given by (1.2). Since λ ranges from 10^{-3} to 10^{-5}, system (8.1) is convection-dominated. As mentioned in the introduction, the results derived here can apply to other similar problems arising from different fields.

With the same notation as before, the approximate solution u_h to problem (8.1) is taken to be in the space $W_{\Delta\tau} \otimes W_{\Delta x}$ such that

$$(u_j^{n+1} - u_j^n)/\Delta\tau + (f^n_{j+\frac{1}{2}} - f^n_{j-\frac{1}{2}})/\Delta x - \lambda^2 (q^n_{j+\frac{1}{2}} - q^n_{j-\frac{1}{2}})/\Delta x = 0, \tag{8.2a}$$

where the function $q_h \in W_{\Delta\tau} \otimes V_{\Delta x}$ is the solution of

$$(\, q_h(\tau^n), v_{\Delta x}\,) = (\, u_h(\tau^n), (v_{\Delta x})_x\,) + u_{1,\Delta\tau}(\tau^n)\, v_{\Delta x}(1) - u_{0,\Delta\tau}(\tau^n)\, v_{\Delta x}(0), \quad \forall v_{\Delta x} \in V_{\Delta x}, \tag{8.2b}$$

after the mass matrix has been mass-lumped. This simple way to compute q_h can be easily extended to multidimensional cases. In our case, the expression for the degrees of freedom of q_h is taken as follows:

$$q^n_{j+1/2} = \begin{cases} (u_1^n - u^n_{0,\Delta\tau})/(\Delta x/2), & \text{for } j = 0, \\ (u^n_{j+1} - u^n_j)/\Delta x, & \text{for } j = 1, \cdots, n_x - 1, \\ (u^n_{1,\Delta\tau} - u^n_{n_x})/(\Delta x/2), & \text{for } j = n_x, \end{cases} \tag{8.2c}$$

The approximate solution $(\beta_h, \phi_h) \in W_{\Delta\tau} \otimes V_{\Delta x} \times W_{\Delta\tau} \otimes W_{\Delta x}$ is still defined by (3.3) and the numerical initial and boundary conditions are defined in the same way as before. We have the following simple stability result.

THEOREM 8.1. (Stability). *Let u_h and (β_h, ϕ_h) be defined by* (8.2) *and* (3.3), *respectively. Assume that the hypotheses on the data* (2.2) *and the following* CFL *condition are satisfied:*

$$\Delta\tau^n \leq \min\Big\{ \frac{1}{u^\star + 2\lambda^2/(\Delta x)^2}, \frac{\Delta x}{(2u^\star - 1)\Delta x + m^\star + 2\lambda^2/\Delta x} \Big\}, \tag{8.3}$$

where $m^\star$ is defined as in Theorem 4.2. Then

$$u_h(\tau, x) \in [0, u^\star], \quad (\tau, x) \in [0,T] \times [0,1], \tag{8.4a}$$
$$\|\, \beta_h \,\|_{L^\infty(0,T;L^\infty(0,1))} \leq m^\star. \tag{8.4b}$$

Notice that since the values of λ are very small, it is reasonable to take $\Delta x \gg \lambda^2$ in practice. In this case, the CFL-condition (8.6) becomes essentially the CFL condition (4.1) for the zero diffusion case. Thus, discretizing the second-order term in (8.1a) in an explicit way does not increase significantly the complexity of the numerical method.

In the present case, we consider initial data u_i and $\phi_1(0)$ such that

$$\beta(0,0) \geq \beta_0 > 0, \tag{8.5a}$$

$$\beta(0,1) \leq \beta_1 < 0. \tag{8.5b}$$

Since, by (1.2), (3.3), (2.2d), (2.2g), and (3.1),

$$\beta_h(0,0) \geq \beta(0,0) - |\, u_i \,|_{BV(0,1)}\, \Delta x - \|\, (\phi_1)_\tau \,\|_{L^\infty(0,\Delta\tau)}\, \Delta\tau,$$
$$\beta_h(0,1) \leq \beta(0,1) + |\, u_i \,|_{BV(0,1)}\, \Delta x + \|\, (\phi_1)_\tau \,\|_{L^\infty(0,\Delta\tau)}\, \Delta\tau,$$

it is easy to see that for Δx and $\Delta\tau$ such that

$$|\, u_i \,|_{BV(0,1)}\, \Delta x + \|\, (\phi_1)_\tau \,\|_{L^\infty(0,\Delta\tau)}\, \Delta\tau \leq \min\{\beta_0, -\beta_1\}/2, \tag{8.6}$$

we have

$$\beta_h(0,0) \geq \beta_0/2 > 0, \tag{8.5a'}$$

$$\beta_h(0,1) \leq \beta_1/2 < 0. \tag{8.5b'}$$

Moreover, since there is a constant C_{12} such that

$$||\beta_h^{n+1} - \beta_h^n||_{L^\infty(0,1)} \leq C_{12}\Delta\tau,$$

(see Lemma 3.11 in [1]), we have that there is a time $T^\star \geq \max\{\beta_0, -\beta_1\}/(2\, C_{12})$ such that, for $T \leq T^\star$,

$$\beta^n_{\frac{1}{2}} \geq 0, \text{ and } \beta^n_{n_x+\frac{1}{2}} \leq 0, \;\; n = 0, \cdots, n_T - 1.$$

In some cases, $T^\star = \infty$, for example when $u^\star = 1$ and $\phi_1 = 0$. The hypotheses (8.5) on the initial data u_i and $\phi_1(0)$ guarantee the absence of boundary layers of the solution u of (8.1) and (1.2). The treatment of the case in which boundary layers are present requires a much more complicated analysis.

We now state the following results on the numerical scheme (8.2).

Theorem 8.2. (Compactness). *Assume, in addition to the assumptions of Theorem* 8.1*, that the hypothesis* (8.5) *holds. Then, for* $T \leq T^\star$ *and* Δx *and* $\Delta\tau$ *satisfying* (8.6)*, we have*

$$\max_{0\leq j\leq n_x} |\, u^n_{j+1} - u^n_j \,| \leq C_3\, \Delta x, \qquad \textit{for } n = 0, \cdots, n_T, \tag{8.7a}$$

$$\max_{0\leq n\leq n_T-1} |\, u^{n+1}_j - u^n_j \,| \leq C_3\, (\Delta x + \Delta\tau), \qquad \textit{for } j = 0, \cdots, n_x+1, \tag{8.7b}$$

$$\sum_{j=1}^{n_x} |\, u^n_{j+1} - 2u^n_j + u^n_{j-1} \,| \leq C_{10}\, \Delta x, \qquad \textit{for } n = 0, \cdots, n_T. \tag{8.7c}$$

The above results allows us to conclude that the method converges to the exact solution of (8.1) and (1.2) and that the continuous version of the estimates (8.4) and (8.7) also hold for the exact solution; that is,

$$u(\tau,x) \in [0, u^\star], \quad (\tau,x) \in [0,T]\times[0,1],$$
$$\|\, \beta \,\|_{L^\infty(0,T;L^\infty(0,1))} \leq m^\star,$$
$$\|\, u_x \,\|_{L^\infty(0,T;L^\infty(0,1))} \leq C_3,$$
$$\|\, u_\tau \,\|_{L^\infty(0,T;L^1(0,1))} \leq C_3,$$
$$\|\, u_{xx} \,\|_{L^\infty(0,T;L^1(0,1))} \leq C_{10}.$$

With these results, we can easily obtain the following error estimates.

THEOREM 8.3. (Error estimates). *Suppose that the hypotheses on the data* (2.2) *and* (8.5) *and the* CFL *condition* (8.3) *are satisfied and that* (u, β) *and* (u_h, β_h) *are the respective solutions of the systems given by* (8.1) *and* (1.2) *and by* (8.2) *and* (3.3). *Then, there is a constant* C *dependent only of the data and* T *such that, for* $T \leq T^\star$ *and for* Δx *and* $\Delta\tau$ *satisfying* (8.6),

$$\begin{aligned} \|u - u_h\|_{L^\infty(0,T;L^1(0,1))} &\leq C(\Delta x + \lambda^2), \\ \|\beta - \beta_h\|_{L^\infty(0,T;L^\infty(0,1))} &\leq C(\Delta x + \lambda^2). \end{aligned}$$

The proof of Theorems 8.1, 8.2, and 8.3 can be carried out in the same manner as in the zero-diffusion case; we omit the details. We remark that, as $\lambda = 0$, the error estimates above reduce to the results given in Theorem 5.5.

9. Extensions and concluding remarks. The requirement that $u_0 \equiv 0$ and $u_1 \equiv 1$ has been used in the proof of (4.4e) and (4.4f) and thus in the proof of Lemma 6.4. However, the results are still valid if $u_0 \equiv 1$ and $u_1 \equiv 0$. Moreover, our results remain valid for other sets of boundary conditions that ensure the smoothness of the electron concentration. For example, when the initial condition satisfies the hypothesis

$$u_i, u_{ix} \in \mathrm{BV}(0,1) \text{ and } \int_0^1 u_i = 1,$$

and the boundary conditions (1.2c) and (1.2d) are replaced by the Neumann boundary condition

$$\phi_x(x=0) = \phi_x(x=1) = 0.$$

In this case, the analysis in the previous sections is in fact simpler since the boundary terms involved with $\beta_h(\cdot, 0)$ and $\beta_h(\cdot, 1)$ drop out. Also, the results hold for the case where we have the Neumann boundary condition in place of (8.1b-c):

$$u_x(x=0) = u_x(x=1) = 0.$$

In the present case, we do not need to make any assumption on ϕ_1 like (8.5).

Let us finish this paper by pointing out that the main difficulty in the analysis of the numerical method for the full unipolar drift diffusion model is associated with the presence of the boundaries; the case in which the domain is not the interval $(0, 1)$ but the real line can be easily handled. Moreover, without an hypothesis like (8.5) that allow us to have some control on the sign of the approximate (negative) electric field at the boundaries, it is not possible to use the technique used in the zero-diffusion case. For example, this technique allows us to prove Lemma 6.2 by means of a very local-in-time estimate, namely, $\xi^{n+1} \leq (1 + 3\triangle\tau)\xi^n$, which is simply not true when λ is not equal to zero. To handle this case a new technique must be found which does not have the local character of the technique used for the zero diffusion case.

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