The Analysis of a Coating Flow with Evaporation

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Abstract. This work is concerned with a plane steady-state coating flow problem including evaporation effects. The motion is governed by a free boundary value problem for a coupled system of Navier-Stokes and Stefan equations. One of the a priori unknown free surfaces is noncompact and the other one has an a priori unknown contact point at the rigid wall. The main objective of the paper is to prove that for sufficiently small data (flux, wall speed and wall temperatures) the free boundary value problem is uniquely solvable in appropriate weighted Hölder spaces. The proof is realized in several steps and it is based on methods known for the Navier-Stokes equations.

Key Words. Coating flow, evaporation, free boundary problems, Navier- Stokes equations, Stefan problem, dynamic contact point.

AMS(MOS) subject classification. Primary 35R35, 35Q35. Secondary 35Q30, 76D05.

1 The mathematical model

This paper is devoted to a plane steady-state nonisothermal flow problem with two free boundaries describing a coating process with evaporation. A heavy viscous incompressible fluid is coated onto a horizontally moving rigid wall (cf. Fig.1). One of the a priori unknown free surfaces is noncompact and the other one ends at an a priori unknown contact point on the rigid wall. The flow domain is unbounded and evaporation is taken into account. This is important in many technological and scientific applications; interesting examples can be found in the field of materials processing, particularly in coating and solidification processes with evaporation [2], [4], [10], [11], [16] or in crystal growth processes [12], [15], [17]. We assume that the viscosity of the liquid is constant, although in general case it may depend on temperature, too.

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Let us consider the mathematical model for this coating process. A heavy viscous incompressible liquid fills the infinite region $V \in \mathbf{R}^2$ bounded by the straight line $\Sigma_1 = \{x \in \mathbf{R}^2; x_2 = 0\}$ and the half-lines

$$\Sigma_2 = \{x \in \mathbf{R}^2;\, x_1 \leq 0,\, x_2 = h_1 - x_1 \tan\alpha\}$$

$$\Sigma_3 = \{x \in \mathbf{R}^2;\, x_1 \leq 0,\, x_2 = h_2 - x_1 \tan\alpha,\, h_2 > h_1 > 0\},$$

where α is a real number with $0 \leq \alpha < \pi/2$ (cf. Fig.1). The region V is then the union of the first quadrant of $\mathbf{R}^2$ with the half-strip between Σ_2 and Σ_3.

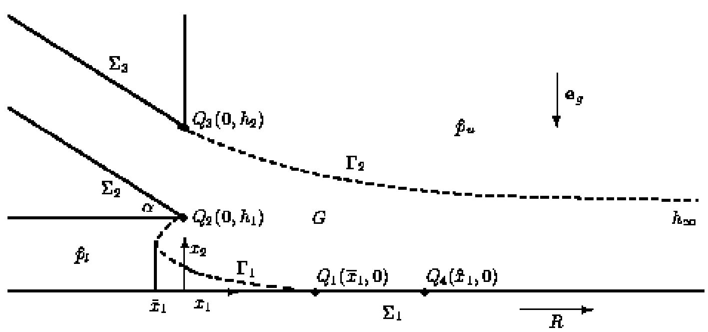

Fig.1: Flow domain of a coating flow

The force of gravity is directed along the vector $e_g = (0,-1)^T$. We suppose that the flow of the liquid is generated by the stream $F^0(\alpha)$ and by the motion of the lower rigid wall Σ_1 with constant velocity R in x_1-direction. The values R and $F^0(\alpha)$ are assumed to be positive. In agreement with many experimental studies we further suppose that the free surfaces separate from the rigid walls at the "sharp" corner points Q_2 and Q_3. The lower free surface Γ_1 "touches" the *moving* rigid wall Σ_1 at the a priori unknown point $Q_1(\overline{x}_1, 0)$ - the so called dynamic contact point. We assume that Γ_1 is described as the graph of a function ψ_1 with respect to $x_2 \in [0, h_1]$. This assumption makes sense physically and it is a key to handling the lower free surface.

The following notations are used: $\Gamma_m (m = 1,2)$- free surfaces with the representations $x_1 = \psi_1(x_2)$ and $x_2 = \varphi_2(x_1)$; $G = \{x \in V;\, x_2 < \varphi_2(x_1)$ for

$x_1 > 0$ and $x_1 > \psi_1(x_2)$ for $x_2 < h_1\}$ - flow domain of the fluid; $\delta(t) = \{x \in G;\, x_1 = t\}$; n and τ are unit vectors directed along the exterior normal and along the tangent to ∂G, resp.; $\nabla = (\partial/\partial x_1, \partial/\partial x_2)^T$, $\nabla p = \text{grad } p$, $\nabla \cdot v = \text{div } v$ and ∇^2 is the Laplace operator; $S(v)$ is the tensor of deformation velocities with elements $S_{ij} = \partial v_i/\partial x_j + \partial v_j/\partial x_i$. Finally, $a \cdot b$ is the inner product of the vectors a, b in $\mathbf{R}^2$.

The dimensionless viscosity and density are assumed to be equal to 1. The symbols $\lambda, \tilde{g}, \sigma(\theta)$ denote the dimensionless thermal conductivity, the acceleration of gravity and the surface tension function, resp.

Thus, the mathematical problem consists of the determination of the unbounded domain G occupied by the fluid, i.e. the determination of the functions ψ_1 and φ_2 inclusive of the dynamic contact point $Q_1(\overline{x}_1, 0)$, the velocity $v(x) = (v_1(x), v_2(x))^T$, the pressure $\tilde{p}(x)$ and the temperature $\theta(x)$ which satisfy in G the equations

$$-\nabla^2 v + (v \cdot \nabla) v + \nabla p = 0, \qquad \nabla \cdot v = 0, \tag{1.1}$$

$$-\lambda \nabla^2 \theta + (v \cdot \nabla)\, \theta = 0 \tag{1.2}$$

and the following *boundary conditions* (=BCs)

$$v|_{\Sigma_j} = 0 \, (j = 2,3), \quad v \cdot n|_{\Gamma_m} = \left.\frac{\partial \theta}{\partial n}\right|_{\Gamma_m}, \; \tau \cdot S(v) n|_{\Gamma_m} = 0 \quad (m = 1,2),$$

$$\tilde{B}\tau \cdot S(v)n + [v - (R,0)^T] \cdot \tau = 0, \quad v \cdot n = 0 \quad (x \in \Sigma_1;\, \overline{x}_1 \le x_1 \le \hat{x}_1),$$

$$v|_{\Sigma_1} = (R,0)^T \qquad (x \in \Sigma_1;\, x_1 \ge \hat{x}_1 > 0), \tag{1.3}$$

$$\theta|_{\Sigma_j} = \overline{\theta}_j \quad (j = 1, \dots, 3), \qquad \theta|_{\Gamma_m} = \hat{\theta}_m \quad (m = 1,2), \tag{1.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_2} \frac{\psi_1'(x_2)}{[\,1 + (\psi_1'(x_2))^2\,]^{1/2}} + \beta(\theta) x_2 = \frac{1}{\sigma(\theta)} [\,\overline{p} + p - n \cdot S(v) n\,],$$

$$\frac{\mathrm{d}}{\mathrm{d}x_1} \frac{\varphi_2'(x_1)}{[\,1 + (\varphi_2'(x_1))^2\,]^{1/2}} - \beta(\theta)\varphi_2(x_1) = \frac{1}{\sigma(\theta)} (-p + n \cdot S(v) n),$$

$$\varphi_2(0) = h_2, \qquad \psi_1(h_1) = 0, \qquad \psi_1'(0) = \cot \gamma_1 = -A. \tag{1.5}$$

The fact that the motion is caused by nonzero flux is mathematically formulated in the form

$$\int_{\delta(t)} v_1(t, x_2)\, \mathrm{d}x_2 = F^0(\alpha). \qquad (t \in \mathbf{R}) \tag{1.6}$$

In (1.1),(1.5) the transformation $p(x) = \hat{p}(x) + \hat{g}\, x_2 - \hat{p}_u$, where $\hat{p}$ denotes the original physical pressure, is realized and the symbols $\beta(\theta) = \hat{g}/\sigma(\theta)$, $\overline{p} = \hat{p}_u - \hat{p}_l$ are also used. In the last *equation* (=eq.) $\hat{p}_l$, $\hat{p}_u$ denote the (constant) atmospheric pressures outside Γ_1 and Γ_2, resp.

It was shown (cf. e.g. [3], [12], [14], [21]) for a large number of liquids that the surface tension can be regarded as a linear function of the temperature, i.e.

$$\sigma(\theta) = a - b\,\theta, \tag{1.7}$$

with constant positive coefficients a, b. The restriction

$$|p(x)| \leq \text{ const. as } x_1 \longrightarrow +\infty, \tag{1.8}$$

resulting from physical considerations completes the free *boundary value problem* (=BVP) (1.1)-(1.6). For a more detailed discussion of BVP (1.1)-(1.6) we refer to the literature [5], [13], [20]. Let us emphasize that problem (1.1)-(1.6) represents some coupling of the well-known free BVP for the Navier-Stokes eqs. and the Stefan problem (cf. [5], [13]). For the Stefan problem we refer - e.g. - to the book [6] of A. Friedman.

Another essential feature of BVP (1.1)-(1.6) is the presence of a so called dynamic contact point. There are several papers presenting analytical studies of different free BVPs including dynamic contact points or lines (cf. [7], [8], [20], [25] and the references given therein). The nature of dynamic contact lines and angles as well as the nessity of slip boundary conditions like $(1.3)_2$ [i.e. the second line of (1.3)] were explained in [20]. Here we only assume that γ_1, $\hat{x}_1$, $\tilde{B}$ are prescribed with $\pi/2 < \gamma_1 < \pi$, (i.e. $0 < A < +\infty$), $\hat{x}_1 > 0$ and $\tilde{B} =$ const. > 0. Note that the restriction $\gamma_1 < \pi$ (i.e.$\gamma_1 \neq \pi$ and consequently $A < +\infty$) are mathematically not nessessary. It was only assumed in order to simplify the function spaces and the notations. In [25] the results of [20] were extended to the case $\gamma_1 = \pi$ and $\tilde{B} = 0$. The main result of the present paper (cf. Theorem 6.1) can also be extended to the case $\gamma_1 = \pi$, $\tilde{B} = 0$ in the same manner as in [25].

The present study aims to prove the unique solvability of BVP (1.1)-(1.6) in weighted Hölder spaces for small data $F^0(\alpha), R, |\overline{\theta}_j|$ using functional-analytic methods. Similar stationary problems in bounded domains have been analytically investigated by many authors [14], [26]. The corresponding isothermal problem (i.e. BVP (1.1),(1.3),(1.5),(1.6) with $b = 0$) was solved analytically and numerically in former papers [18], [20]. There is a great number of studies presenting numerical procedures for nonisothermal free BVPs (cf. [3], [9], [14]). Numerical results of such problems including temperature-dependent viscosities and dissipation terms are also given by the author [22]. The solvability of the full free BVP (1.1)-(1.6) will be shown in several steps. Because the present study is an extension of former results [20] to the temperature-depending case we concentrate special attention to the eqs. and BCs for the temperature.

2 Function spaces

Let B be an arbitrary domain in $\mathbf{R}^2$ and $N \subset \overline{B}$ a manifold of dimension $\overline{n} < 2$. The symbol $\rho_N(x)$ denotes the distance $\mathrm{dist}(x, N) := \inf_{y \in N} |x - y|$. Let $\beta = (\beta_1, \beta_2)$ be a multiindex in this section with

$$|\beta| = \beta_1 + \beta_2 \text{ and } D^\beta u = \frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \, (\beta_i \in \mathbb{N} \cup \{0\}) \, .$$

The symbol $[r]$ will denote the integer part of r.
$C^r(B)(r > 0$, non-integer) denotes the Hölder space of functions defined in a domain $B \subset \mathbf{R}^2$ with a finite norm

$$|u|_B^{(r)} = \sum_{|\beta| < r} \sup_{x \in B} |D^\beta u| + \sum_{|\beta| = [r]} \sup_{x,y \in B} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^{r - [r]}}.$$

Let $\dot{C}_s^r(B, N)$ be the weighted Hölder space of functions defined in $B \backslash N$ and having a finite norm

$$|u|_{\dot{C}_s^r(B,N)} = \sum_{|\beta| < r} \sup_{x \in B \backslash N} \rho_N^{|\beta| - s}(x) |D^\beta u(x)|$$

$$+ \sum_{|\beta| = [r]} \sup_{x \in B \backslash N} \rho_N^{r - s}(x) \sup_{|x - y| < \frac{1}{2} \rho_N(x)} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^{r - [r]}}.$$

$C_s^r(B, N)(r > s > 0; r, s$ non-integer) denotes the space of functions with a finite norm

$$|u|_{C_s^r(B,N)} := |u|_B^{(s)} + \sum_{s < |\beta| < r} \sup_{x \in B \backslash N} \rho_N^{|\beta| - s}(x) |D^\beta u(x)|$$

$$+\sum_{|\beta|=[r]}\sup_{x\in B\setminus N}\rho_N^{r-s}(x)\sup_{|x-y|<\frac{1}{2}\rho_N(x)}\frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^{r-[r]}}.$$

Clearly, $\dot{C}_s^r(B,N)$ is a subspace of $C_s^r(B,N)$ consisting of functions vanishing on N together with their derivatives of order up to $[s]$. For $s<0$ assume $C_s^r(B,N):=\dot{C}_s^r(B,N)$.

For the definition of a generalized solution to the linear auxiliary problem we need also the following function spaces. Note that the domain G is unbounded. Define the sets $\Sigma_{11}:=\{x\in\Sigma_1;\,\overline{x}_1<x_1<\hat{x}_1\}$ and $\Sigma_{12}:=\{x\in\Sigma_1;\,x_1>\hat{x}_1\}$.

By $C_0^\infty(G,\Gamma)$ we mean the set of all infinitely differentiable vector functions v vanishing for $|x|\gg 1$ and satisfying the following BCs

$$v|_{\Sigma_j}=0\,(j=2,3),\quad v|_{\Sigma_{12}}=0,\quad v\cdot n=0\,(x\in\Gamma_1\cup\Gamma_2\cup\Sigma_{11}).$$

Further we have:

$$J_0^\infty(G,\Gamma):=\{v\in C_0^\infty(G,\Gamma),\ \operatorname{div}v=0\}.$$

The Sobolev spaces $D(G)$ and $H(G)$ are the completions of the sets $C_0^\infty(G,\Gamma)$ and $J_0^\infty(G,\Gamma)$ with respect to the Dirichlet norm

$$\|u_x\|_G^2:=\|u_x\|_{L_2(G)}^2=\int_G\sum_{i,j=1}^2\left(\frac{\partial u_i}{\partial x_j}\right)^2\mathrm{d}x.$$

$C_{\theta,0}^\infty(G,\Gamma)$ is the set of all infinitely differentiable scalar fields $\theta(x)$ vanishing for $|x|\gg 1$ and satisfying the BCs

$$\theta|_{\Sigma_j}=0\,(j=1,2,3),\qquad \theta|_{\Gamma_m}=0\,(m=1,2).$$

The Sobolev space $D_\theta(G)$ is then the completion of the set $C_{\theta,0}^\infty(G,\Gamma)$ with respect to the Dirichlet norm

$$\|\theta_x\|_G^2:=\|\theta_x\|_{L_2(G)}^2=\int_G\sum_{i=1}^2\left(\frac{\partial\theta}{\partial x_i}\right)^2\mathrm{d}x.$$

Finally we define the weighted Hölder spaces to which the generalized solutions to the problem (1.1)-(1.6) belong. Let x^* be the value $x^*:=\max(|\tilde{x}_1|,\hat{x}_1)$ where $\tilde{x}_1$ denotes the value $\tilde{x}_1:=\min_{x_2\in[0,h_1]}\psi_1(x_2)$. We use the following notations: $Q^* := Q_1 \cup Q_2 \cup Q_3 \cup Q_4$, $J_1 :=]0, h_1[$, $Y_1 := \{0, h_1\}$, $G^0 := \{x \in G; |x_1| < x^* + 2\}$, $G^+ := \{x \in G; x_1 > x^* + 1\}$, $G^- := \{x \in G; x_1 < -x^* - 1\}$, $\hat{Q} = Q_1 \cup Q_2$, $J_2^0 := \{x_1 \in \mathbf{R}, 0 < x_1 < x^* + 2\}$, $J_2^+ := \{x_1 \in \mathbf{R}, x_1 > x^* + 1\}$.

For an arbitrary real number $z > 0$ define the space

$$C^r_{s,z}(G) = \{u(x), u|_{G^0} \in C^r_s(G^0, Q^*), \exp(zx_1)u(x)|_{G^+} \in C^r(G^+),$$

$$\exp(-zx_1)u(x)|_{G^-} \in C^r(G^-)\}$$

with the norm:

$$\| u \|^{r,z}_{G,s} := |u|_{C^r_s(G^0,Q^*)} + |\exp(zx_1)u|^{(r)}_{G^+} + |\exp(-zx_1)u|^{(r)}_{G^-}.$$

For functions $f(x_1)$ defined in $\mathbf{R}^1_+$ we introduce the space $C^r_{s,z}(\mathbf{R}^1_+)$ with the norm

$$\| f \|^{r,z}_{\mathbf{R}^1_+,s} = |f|_{C^r_s(J^0_2,0)} + |f(x_1)\exp(zx_1)|^{(r)}_{J^+_2}.$$

The norm $|\tilde{g}|_{C^r_s(J_1,Y_1)}$ is defined analogously.

3 Linear auxiliary problem with homogeneous boundary conditions

The linear auxiliary problem in the domain G with *fixed* boundaries consists of the linearized Eqs.(1.1),(1.2) with given right-hand sides f, r and g, resp., and the homogeneous BCs (1.3),(1.4), where $v \cdot n$ was set at zero. Thus, we obtain

$$-\nabla^2 v + \nabla p = f, \qquad \nabla \cdot v = r, \qquad (x \in G), \tag{3.1}$$

$$-\nabla^2 \theta = g, \qquad (x \in G), \tag{3.2}$$

$$v = 0 \quad (x \in \Sigma_{12} \cup \Sigma_2 \cup \Sigma_3), \qquad v \cdot n = 0 \quad (x \in \Gamma_1 \cup \Gamma_2 \cup \Sigma_{11}),$$

$$\tau \cdot S(v)n|_{\Gamma_m} = 0 \ (m = 1, 2), \quad \tilde{B}\tau \cdot S(v)n + v \cdot \tau = 0 \ (x \in \Sigma_{11}), \tag{3.3}$$

$$\theta|_{\Sigma_j} = 0 \quad (j = 1, \ldots, 3), \qquad \theta|_{\Gamma_m} = 0 \quad (m = 1, 2). \tag{3.4}$$

This problem can be decomposed into a BVP (3.1),(3.3) for v, p and a second BVP (3.2),(3.4) for θ. The solution to BVP (3.1),(3.3) was completely presented by the author [20]. Let us give the concept of a weak solution to BVP (3.2),(3.4).

Definition 3.1. *By a weak solution to BVP (3.2),(3.4) we understand a scalar field $\theta \in D_\theta(G)$ satisfying the integral identity*

$$E_\theta(\theta, \xi) := \int_G (\nabla\theta \cdot \nabla\xi)\,\mathrm{d}x = \int_G g\xi\,\mathrm{d}x \tag{3.5}$$

for all scalar fields $\xi \in D_\theta(G)$.

The existence of such a solution can be shown in the same manner as in [21], where a modified unbounded domain for two liquids was considered. This procedure is well known for the Laplace equation in bounded domains. Let ψ_1, φ_2 be the representation functions described in Section 1. We define the dynamic contact angle γ_1 and both static contact angles γ_2, γ_3 at the contact points Q_1, Q_2, Q_3, resp., by the relations

$$\gamma_1 = \frac{\pi}{2} + \arctan[-\psi_1'(0)], \qquad \gamma_2 = \frac{3\pi}{2} - \alpha - \arctan[-\psi_1'(h_1)],$$

$$\gamma_3 = \pi + \alpha + \arctan[-\varphi_2'(0)].$$

Analysing some model problems [19], [23] for θ in a neighbourhood of Q_1, Q_2, Q_3, Q_4, it could be shown that the weak solution θ belongs to $C_s^{s+2}(G^0)$ with s satisfying the condition $0 < s < s_t := \min_{j\in\{1,2,3\}}[1/3,\ \pi/(2\gamma_j)]$. Assume now that the fixed boundaries $\Gamma_m\,(m = 1, 2)$ are of class C_s^{3+s}. Finally, we need the set $G(\mu, t) := \{x \in G, \mu - t < x_1 < \mu + t\}$.

Theorem 3.1 *There is a positive real number z_t such that for any $g \in C^s_{s-2,z}(G)$ with $s \in]0, s_t[$ and $z \in]0, z_t[$ the generalized solution $\theta \in D_\theta(G)$ to BVP (3.2),(3.4) satisfies the inequality*

$$\int_{G(\mu,1)} |\nabla\theta|^2 \mathrm{d}x \le c_0 e^{-2z|\mu|} \left(\| g \|^{s,z}_{G,s-2} \right)^2, \tag{3.6}$$

where $|\mu| > x^* + 4$ holds and c_0 does not depend on μ.

Proof: Multiplying Eqs.(3.2) by θ and integrating by parts the result in the domain $G(\mu, t)$ one obtains

$$E_\theta(\theta, \theta)|_{G(\mu,t)} = \int_{G(\mu,t)} g\theta\mathrm{d}x + \int_{\delta(\mu+t)} \lambda\theta\frac{\partial\theta}{\partial x_1}\mathrm{d}x_2 - \int_{\delta(\mu-t)} \lambda\theta\frac{\partial\theta}{\partial x_1}\mathrm{d}x_2. \tag{3.7}$$

Integrating Eq.(3.7) with respect to t over the interval $]\eta - 1, \eta[$ we get

$$Z(\mu, \eta) \le c_1 \left\{ \left| \int_{\eta-1}^{\eta} \left(\int_{G(\mu,t)} g\theta\mathrm{d}x \right) \mathrm{d}t \right| + \left| \int_{\Omega^{\mp}(\mu,\eta)} \lambda\theta\frac{\partial\theta}{\partial x_1}\mathrm{d}x_2 \right| \right\}, \tag{3.8}$$

where the notations $Z(\mu,\eta) := \int_{\eta-1}^{\eta}(\int_{G(\mu,t)}\theta_x^2 dx)dt$, $\Omega^+(\mu,\eta) := \{x \in G, \mu+\eta-1 < x_1 < \mu+\eta\}$, $\Omega^-(\mu,\eta) := \{x \in G, \mu-\eta < x_1 < \mu-\eta+1\}$ are used. Now we estimate the right-hand side of (3.8). Taking into account Friedrichs' inequality it is easy to prove that

$$\left|\int_{\Omega^{\mp}(\mu,\eta)} \lambda\theta\frac{\partial\theta}{\partial x_1}dx_2\right| \leq c_2 \parallel \theta_x \parallel^2_{\Omega^{\mp}(\mu,\eta)}, \tag{3.9}$$

$$\left|\int_{\Omega^{\mp}(\mu,\eta)}\left(\int_{G(\mu,t)} g\theta dx\right)dt\right| \leq \parallel g \parallel_{G(\mu,\eta)} \left(\int_{\eta-1}^{\eta} \parallel \theta \parallel^2_{G(\mu,t)} dt\right)^{1/2}$$

$$\leq c_3 \parallel g \parallel_{G(\mu,\eta)} \left(\int_{\eta-1}^{\eta} \parallel \theta_x \parallel^2_{G(\mu,t)} dt\right)^{1/2} \leq c_4 \parallel g \parallel^2_{G(\mu,\eta)} + \frac{1}{4c_1}Z(\mu,\eta), \tag{3.10}$$

where $c_4 := c_1c_3^2/4$. By virtue of inequalities (3.9),(3.10) it follows from (3.8) that

$$Z(\mu,\eta) \leq c_1c_4 \parallel g \parallel^2_{G(\mu,\eta)} + \frac{1}{4}Z(\mu,\eta) + c_5 \parallel \theta_x \parallel^2_{\Omega^{\mp}(\mu,\eta)}$$

$$\leq c_6 \parallel g \parallel^2_{G(\mu,\eta)} + c_7 \parallel \theta_x \parallel^2_{\Omega^{\mp}(\mu,\eta)} . \tag{3.11}$$

Since $dZ(\mu,\eta)/d\eta = \parallel \theta_x \parallel^2_{\Omega^+(\mu,\eta)} + \parallel \theta_x \parallel^2_{\Omega^-(\mu,\eta)}$ the estimate (3.11) can be represented in the form $Z(\mu,\eta) \leq c_7 dZ(\mu,\eta)/d\eta + c_6 \parallel g \parallel^2_{G(\mu,\eta)}$. Multiplying the last inequality by $\exp(-(\eta-2)/c_7)$ and integrating with respect to η over the interval $[2,\mu/2]$ we get

$$Z(\mu,2) \leq Z\left(\mu,\frac{\mu}{2}\right)\exp\left(\frac{-\mu/2-2}{c_7}\right) + \frac{c_6}{c_7}\int_2^{\mu/2} \parallel g \parallel^2_{G(\mu,\eta)} \exp\left(-\frac{\eta-2}{c_7}\right)d\eta. \tag{3.12}$$

From the estimate $\parallel \theta_x \parallel_G \leq \bar{c}_0 \parallel g \parallel_G$ which results from the existence of a generalized solution to BVP (3.2,(3.4) (cf.[21]) it follows that

$$Z\left(\mu,\frac{\mu}{2}\right) = \int_{\mu/2-1}^{\mu/2}\left(\int_{G(\mu,t)}\theta_x^2 dx\right)dt \leq \parallel \theta_x \parallel^2_G \leq c_8 \parallel g \parallel^2_G \leq c_9(\parallel g \parallel^{s,x}_{G,s-2})^2. \tag{3.13}$$

Inequality (3.13) can be obtained with the help of the norms in the spaces $C^r_{s,x}(G)$ and $L_2(G)$. Now we estimate the second term on the right-hand side of (3.12). For $\mu > x^* + 4$ we have

$$\parallel g \parallel^2_{G(\mu,\eta)} \leq c_{10} \sup_{x\in G(\mu,\eta)} [\exp(zx_1)|g(x)|]^2 \int_{\mu-\eta}^{\mu+\eta} \exp(-2zx_1)dx_1$$

$$\leq c_{11} \exp(-2z\mu) \sinh(2z\eta)(\| g \|^{s,z}_{G,s-2})^2.$$

Therefore, for $z \in]0, 1/(2c_7)[$ and $\mu > x^* + 4$, it follows that

$$\int_2^{\mu/2} \| g \|^2_{G(\mu,\eta)} \exp\left(-\frac{\eta-2}{c_7}\right) \mathrm{d}\eta \leq c_{12} \exp(-2z\mu)(\| g \|^{s,z}_{G,s-2})^2 \times$$

$$\times \int_2^{\mu/2} \sinh(2z\mu) \exp\left(-\frac{\eta}{c_7}\right) \mathrm{d}\eta \leq c_{13} \exp(-2z\mu)(\| g \|^{s,z}_{G,s-2})^2. \qquad (3.14)$$

Since $\int_{G(\mu,1)} |\nabla\theta|^2 \mathrm{d}x \leq Z(\mu, 2)$, the relations (3.12)-(3.14) bring us to the inequality (3.6) for arbitrary $\mu > x^* + 4$ and $z \in]0, z_t[$ where $z_t := 1/(4c_7)$ holds. The same considerations prove the inequality (3.6) in the case $\mu < -x^* - 4$. This proves the theorem. ■

Theorem 3.2 *For arbitrary $g \in C^s_{s-2,z}(G)$ with $s \in]0, s_t[$ and $z \in [0, z_t[$ the BVP (3.2),(3.4) has a unique solution $\theta \in C^{s+2}_{s,z}(G)$ and the estimate*

$$\| \theta \|^{s+2,z}_{G,s} \leq c_{14} \| g \|^{s,z}_{G,s-2}, \qquad (3.15)$$

holds where c_{14} does not depend on g.

Note that Eq.(3.2) is elliptic in the sense of Douglis-Nirenberg [1],[24] and that the BCs (3.4) fulfil the complementarity conditions [24]. Therefore, one is able to prove Theorem 3.2 in a well-known manner [19], too. Joining the solution θ to BVP (3.2),(3.4) given here with the solution v, p to BVP (3.1),(3.3) (cf.[19]), we obtain the solvability of the full linear problem (3.1)-(3.4).

4 Linear auxiliary problem with nonhomogeneous boundary conditions

In this section we study the following BVP with nonhomogeneous BCs

$$-\nabla^2 v + \nabla p = f, \qquad \nabla \cdot v = r, \qquad (x \in G), \qquad (4.1)$$

$$-\nabla^2 \theta = g, \qquad (x \in G), \qquad (4.2)$$

$$v|_{\Sigma_j} = 0 \ (j=2,3), \quad v \cdot n|_{\Sigma_{11}} = 0, \quad \tilde{B}\tau \cdot S(v)n + v \cdot \tau = a_1 \ (x \in \Sigma_{11}),$$

$$v \cdot \tau|_{\Sigma_{12}} = a_2, \quad v \cdot n|_{\Gamma_m} = b_m, \quad \tau \cdot S(v)n|_{\Gamma_m} = d_m \ (m=1,2), \tag{4.3}$$

$$\theta|_{\Sigma_j} = 0 \quad (j=1,2,3), \qquad \theta|_{\Gamma_m} = \hat{\theta}_m \quad (m=1,2). \tag{4.4}$$

The boundary values must fulfil the compatibility conditions

$$\int_G r\mathrm{d}x = \int_{\Gamma_1} b_1 \mathrm{d}s - \int_{\Gamma_2} b_2 \mathrm{d}s, \ b_1(\hat{x}_1, 0) = 0, \ b_m(0, h_m) = 0 \ (m=1,2), \tag{4.5}$$

$$\hat{\theta}_1(0, h_1) = 0, \qquad \hat{\theta}_1(\overline{x}_1, 0) = 0, \qquad \hat{\theta}_2(0, h_2) = 0. \tag{4.6}$$

The problem (4.1)-(4.4) can also be decomposed into a BVP (4.1),(4.3) for v, p and a second BVP (4.2),(4.4) for the temperature θ. The solution to BVP (4.1),(4.3) was given in [20], too. Let us present the corresponding theorem. Remark that the numbers s_*, z_* denote some constants resulting from model problems in angular domains (cf. [20]) for the isothermal BVP (3.1),(3.3).

Theorem 4.1 *Let h_∞ be a positive constant and $(\varphi_2 - h_\infty) \in C^{3+s}_{1+s,z}(\mathbf{R}^1_+)$, $\psi_1 \in C^{3+s}_{1+s}(J_1, Y_1)$. For arbitrary $s \in]0, s_*[$, $z \in]0, z_*[$, $f \in C^s_{s-2,z}(G)$, $r \in C^{s+1}_{s-1,z}(G)$, $a_1 \in C^{s+2}_s(\Sigma_{11}, Q_1)$, $a_2 \in C^{s+2}_{s,z}(\Sigma_{12}, Q_4)$, $b_1 \in C^{s+2}_s(\Gamma_1, \hat{Q})$, $b_2 \in C^{s+2}_{s,z}(\Gamma_2)$, $d_1 \in C^{s+1}_{s-1}(\Gamma_1, \hat{Q})$, $d_2 \in C^{s+1}_{s-1,z}(\Gamma_2)$ such that conditions (4.5) are fulfilled, BVP (4.1),(4.3) is uniquely solvable with $v \in C^{s+2}_{s,z}(G)$, $\nabla p \in C^s_{s-2,z}(G)$. Moreover, the inequality*

$$\|v\|^{s+2,z}_{G,s} + \|\nabla p\|^{s,z}_{G,s-2} \le c_{15} \Big(\|f\|^{s,z}_{G,s-2} + \|r\|^{s+1,z}_{G,s-1} + |a_1|_{C^{s+2}_s(\Sigma_{11}, Q_1 \cup Q_4)}$$

$$+ \|a_2\|^{s+2,z}_{\Sigma_{12},s} + |b_1|_{C^{s+2}_s(\Gamma_1,\hat{Q})} + \|b_2\|^{s+2,z}_{\Gamma_2,s} + |d_1|_{C^{s+1}_{s-1}(\Gamma_1,\hat{Q})} + \|d_2\|^{s+1,z}_{\Gamma_2,s-1} \Big) \tag{4.7}$$

applies.

Next we are studying BVP (4.2),(4.4) under the compatibility conditions (4.6).

Theorem 4.2 *Let h_∞ be a positive constant and $(\varphi_2 - h_\infty) \in C^{3+s}_{1+s,z}(\mathbf{R}^1_+)$, $\psi_1 \in C^{3+s}_{1+s}(J_1, Y_1)$. For arbitrary $s \in]0, s_t[$, $z \in]0, z_t[$, $g \in C^s_{s-2,z}(G)$, $\hat{\theta}_1 \in C^{s+2}_s(\Gamma_1, \hat{Q})$, $\hat{\theta}_2 \in C^{s+2}_{s,z}(\Gamma_2)$, such that conditions (4.6) are fulfilled, BVP (4.2),(4.4) has a unique solution $\theta \in C^{s+2}_{s,z}(G)$. Moreover, the estimate*

$$\|\theta\|^{s+2,z}_{G,s} \le c_{16} \Big(\|g\|^{s,z}_{G,s-2} + |\hat{\theta}_1|_{C^{s+2}_s(\Gamma_1,\hat{Q})} + \|\hat{\theta}_2\|^{s+2,z}_{\Gamma_2,s} \Big) \tag{4.8}$$

holds.

Proof: Firstly one constructs a scalar field $\xi \in C^{s+2}_{s,z}(G)$ satisfying the BCs (4.4) and an estimate of type (4.8) where the norm of g on the right-hand side is omitted. Such a field can be obtained as in [20]. Then we set $\theta := \xi + \chi$ where χ is a solution to

$$-\nabla^2 \chi = g + \nabla^2 \xi$$

with homogeneous BCs (3.4). Due to $(g + \nabla^2 \xi) \in C^{s}_{s-2,z}(G)$ the existence of χ follows from Theorem 3.2. ■

Joining the results of Theorems 4.1 and 4.2 one gets the unique solvability of BVP (4.1)-(4.4) and the corresponding estimate for its solution.

5 Nonlinear auxiliary problem

Consider BVP (1.1)-(1.4),(1.6) in G with *fixed* boundary. Let again $h_\infty > 0$ be a given constant and $(\varphi_2 - h_\infty) \in C^{3+s}_{1+s,z}(\mathbf{R}^1_+)$, $\psi_1 \in C^{3+s}_{1+s}(J_1, Y_1)$. The space parameters s and z fulfil the conditions

$$0 < s < s_0 := \min[s_*, s_t], \qquad 0 < z < z_0 := \min[z_*, z_t], \tag{5.1}$$

where s_t, z_t are taken from Theorem 3.1 and s_*, z_* are taken from Theorem 4.1. Let the temperatures at the walls $\overline{\theta}_j$ $(j = 1,2,3)$ be constant and denote by $\overline{\theta}_{max}$ the maximum of their moduli $|\overline{\theta}_j|$ $(j = 1,2,3)$. Furthermore, suppose that $\hat{\theta}_1 \in C^{s+2}_{s}(\Gamma_1, \hat{Q})$, $(\hat{\theta}_2 - \overline{\theta}_1) \in C^{s+2}_{s,z}(\Gamma_2)$ and $|\hat{\theta}_m(x)|_{\Gamma_m} \leq \overline{\theta}_{max}$ $(m = 1,2)$. Finally, assume $\hat{\theta}_2 \equiv \overline{\theta}_1$ for $x_1 \geq \hat{x}_1$. The last assumption is essentiell for the applied solution method and it guarantees a constant layer thickness as $x_1 \to +\infty$.

By $(u^{(-)}, p^{(-)}, \theta^{(-)})$ we understand a solution to BVP (1.1)-(1.4),(1.6) in the channel G^- (here the BCs on $\Sigma_1, \Gamma_1, \Gamma_2$ are excluded). Finally, $(u^{(+)}, p^{(+)}, \theta^{(+)})$ denotes a solution to BVP (1.1)-(1.4),(1.6) in $\hat{G}^+$, where $\hat{G}^+$ is a domain approximating the curve Γ_2 by the half-line $x_2 = h_\infty$ as $x_1 \geq \hat{x}_1$ (here the BCs on $\Sigma_j (j = 2,3)$ are excluded). These solutions can explicitly be calculated. They have the form

$$u_1^{(-)}(x_1, x_2) = \frac{6F^0(\alpha)}{(h_2 - h_1)^3}(x_2 - h_1 + x_1 \tan\alpha)(h_2 - x_2 - x_1 \tan\alpha),$$

$$u_2^{(-)}(x_1,x_2) = -\frac{6F^0(\alpha)\tan\alpha}{(h_2-h_1)^3}(x_2-h_1+x_1\tan\alpha)(h_2-x_2-x_1\tan\alpha),$$

$$p^{(-)}(x_1,x_2) = -\frac{12F^0(\alpha)}{(h_2-h_1)^3\cos^2\alpha}[x_1-(x_2-h_1)\tan\alpha],$$

$$\theta^{(-)}(x_1,x_2) = \overline{\theta}_2 + \frac{x_2-h_1+x_1\tan\alpha}{h_2-h_1}(\overline{\theta}_3-\overline{\theta}_2), \qquad (5.2)$$

$$u_1^{(+)}(x_1,x_2) = R + \frac{3}{(h_\infty)^2}\left(R-\frac{F^0(\alpha)}{h_\infty}\right)x_2\left(\frac{x_2}{2}-h_\infty\right), \qquad (x\in\hat{G}^+)$$

$$u_2^{(+)}\equiv 0,\ \theta^{(+)}\equiv\overline{\theta}_1,\ p^{(+)} = \frac{3}{(h_\infty)^2}\left(R-\frac{F^0(\alpha)}{h_2^*}\right)x_1, \qquad (x\in\hat{G}^+). \quad (5.3)$$

Now the solution $(u^{(+)},p^{(+)},\theta^{(+)})$ hitherto only defined on $\hat{G}^+$ can be extended or restricted to $G^+\neq\hat{G}^+$ with the help of formula (5.3). In this case the notation is preserved. Let $M(F,R,\overline{\theta}) := \max(F^0(\alpha),R,\overline{\theta}_{max})$ and let in the remaining part of this paper g be a smooth real function vanishing for $t\leq\hat{x}_1+1$ and being equal to 1 for $t\geq\hat{x}_1+2$.

Theorem 5.1 *For a sufficiently small number $M(F,R,\overline{\theta})$ and for s,z satisfying condition (5.1) the BVP (1.1)-(1.6) has a unique solution (v,p,θ) permitting the representation*

$$v = g(-x_1)u^{(-)} + g(x_1)u^{(+)} + w, \qquad p = g(-x_1)p^{(-)} + g(x_1)p^{(+)} + q,$$

$$\theta = g(-x_1)\theta^{(-)} + g(x_1)\theta^{(+)} + \theta_0, \qquad (5.4)$$

where $u^{(+)},u^{(-)},p^{(+)},p^{(-)},\theta^{(+)}$ and $\theta^{(-)}$ are given by formulae (5.2),(5.3) and $w,\theta_0\in C^{s+2}_{s,z}(G), \nabla q\in C^s_{s-2,z}(G)$. Moreover,

$$\| w \|^{s+2,z}_{G,s} + \| \nabla q \|^{s,z}_{G,s-2} + \| \theta_0 \|^{s+2,z}_{G,s} \leq c_{16}(F^0(\alpha),R,\overline{\theta}_{max}) \qquad (5.5)$$

holds and $c_{16}\longrightarrow 0$ for $M(F,R,\overline{\theta})\longrightarrow 0$.

Proof: Introduce the notations $u := v-w$, $\tilde{p} := p-q$ and $\tilde{\theta} := \theta-\theta_0$ where v,w,p,q,θ and θ_0 are given in (5.4). Taking into account the expressions (5.2),(5.3) for $u,\tilde{p}$ and $\tilde{\theta}$ we obtain the following system for (w,q,θ_0)

$$-\nabla^2 w + \nabla q = M_1(w), \qquad \nabla\cdot w = r, \qquad (5.6)$$

$$-\nabla^2\theta_0 = M_2(\theta_0,w), \qquad (5.7)$$

$$w|_{\Sigma_j} = 0\ (j=2,3), \quad w\cdot n|_{\Sigma_1} = 0, \quad \tilde{B}\tau\cdot S(w)n + v\cdot\tau = a_1\ (x\in\Sigma_{11}),$$

$$w\cdot\tau|_{\Sigma_{12}} = a_2, \quad w\cdot n|_{\Gamma_m} = b_m, \quad \tau\cdot S(w)n|_{\Gamma_m} = d_m\ (m=1,2), \tag{5.8}$$

$$\theta|_{\Sigma_j} = 0 \quad (j=1,2,3), \qquad \theta|_{\Gamma_m} = \hat{\Theta}_m \quad (m=1,2), \tag{5.9}$$

where $a_1, a_2, b_m, d_m, \hat{\Theta}_m\ (m=1,2), M_1(w), M_2(\theta_0, w)$ and r have the representations

$$a_1 = 0, \quad a_2 = R\,[1-g(x_1)], \quad b_1 = 0, \quad d_1 = 0,$$

$$b_2 = g(x_1)\varphi_2'(x_1)[1+(\varphi_2')^2)]^{-1/2}\, u_1^{(+)}|_{\Gamma_2},$$

$$d_2 = [1+(\varphi_2')^2)]^{-1}\left\{2g'(x_1)\varphi_2'(x_1)u_1^{(+)}|_{\Gamma_2}\right.$$

$$\left. -g(x_1)[1-(\varphi_2')^2]\frac{3}{(h_\infty)^2}\left(r - \frac{F^0(\alpha)}{h_\infty}\right)[\varphi_2(x_1) - h_\infty]\right\},$$

$$M_1(w) = F_1 - (u\cdot\nabla)w - (w\cdot\nabla)u + (w\cdot\nabla)w,$$

$$F_1 = \nabla^2 u - (u\cdot\nabla)u - \nabla\tilde{p}, \quad r = g'(-x_1)u_1^{(-)} - g'(x_1)u_1^{(+)}, \tag{5.10}$$

$$M_2(\theta_0, w) = F_2 - (w\cdot\nabla)\tilde{\theta} - (u\cdot\nabla)\theta_0 + (w\cdot\nabla)\theta_0,$$

$$F_2 = \lambda\nabla^2\tilde{\theta} - (u\cdot\nabla)\tilde{\theta},$$

$$\hat{\Theta}_1 = \hat{\theta}_1, \qquad \hat{\Theta}_2 = \hat{\theta}_2 - g\,\overline{\theta}_1. \tag{5.11}$$

In [20] it was proved that the problem (5.6),(5.8) with right-hand sides (5.10) is uniquely solvable provided that $\max[F^0(\alpha), R]$ is sufficiently small. Furthermore, the inequality

$$\|w\|_{G,s}^{s+2,x} + \|\nabla q\|_{G,s-2}^{s,x} \le c_{17}(F^0(\alpha), R) \tag{5.12}$$

was shown where $c_{17} \longrightarrow 0$ as $\max[F^0(\alpha), R] \longrightarrow 0$. The proof is based on Banach's fixed point theorem and the results of Theorem 4.1.

Let us consider problem (5.7),(5.9) with right-hand sides (5.11). Obviously, $F_2 \equiv 0$ for $|x_1| \le x^* + 1$. Therefore, $F_2 \in C^s_{s-2,x}(G)$ holds. Let $w \in C^{s+2}_{s,x}(G)$ be the unique solution to the problem (5.6),(5.8) with right-hand sides (5.10). Suppose that $\theta_0 \in C^{s+2}_{s,x}(G)$. In this case we can conclude

$$\|M_2(\theta_0, w)\|_{G,s-2}^{s,x} \le c_{18}(F^0(\alpha), R, \overline{\theta}_{max})\left[\|\theta_0\|_{G,s}^{s+2,x} + \|w\|_{G,s}^{s+2,x} + \right.$$

$$\left. c_{19}\|w\|_{G,s}^{s+2,x}\|\theta_0\|_{G,s}^{s+2,x} + c_{18}(F^0(\alpha), R, \overline{\theta}_{max})\right], \tag{5.13}$$

where $c_{18} \longrightarrow 0$ as $M(F,R,\overline{\theta}) \longrightarrow 0$. Simple calculations using the assumptions $(\varphi_2 - h_\infty) \in C^{3+s}_{1+s}(\mathbf{R}^1_+)$ and $\psi_1 \in C^{3+s}_{1+s}(J_1, Y_1)$ show that the functions $\hat{\theta}_m (m=1,2)$ from (5.11) fulfil the conditions of Theorem 4.2 and the compatibility conditions (4.6). Thus problem (5.7),(5.9) can be written in the form

$$\theta_0 = \mathcal{A}M_2(\theta_0, w) =: \mathcal{B}(\theta_0), \tag{5.14}$$

where $\mathcal{A}$ is an operator which maps a scalar field $\overline{g} \in C^s_{s-2,\varkappa}(G)$ into a solution θ of problem (4.2),(4.4). By virtue of Theorem 4.2, the range of this operator is the set of all scalar fields belonging to $C^{s+2}_{s,\varkappa}(G)$, satisfying BCs (5.11) and

$$\begin{aligned} \|\mathcal{A}g\|^{s+2,\varkappa}_{G,s} &\leq \|\mathcal{A}\|[\|g\|^{s,\varkappa}_{G,s-2} + |\hat{\theta}_1|_{C^{s+2}_{\varkappa}(\Gamma_1,Q)} + \|\hat{\theta}_2\|^{s+2,\varkappa}_{\Gamma_2,s}] \\ &\leq \|\mathcal{A}\|[\|g\|^{s,\varkappa}_{G,s-2} + c_{20}(F^0(\alpha), R, \overline{\theta}_{max})], \end{aligned} \tag{5.15}$$

where $\|\mathcal{A}\| < +\infty$ and $c_{20} \longrightarrow 0$ as $M(F,R,\overline{\theta}) \longrightarrow 0$. Repeating the considerations from [19] and using (5.13) it is possible to prove that the operator $\mathcal{B}$ carries the ball $\|\theta_0\|^{s+2,\varkappa}_{G,s} < \varepsilon$ of the space $C^{s+2}_{s,\varkappa}(G)$ into itself and that it is there a contraction operator. In this case we put $\varepsilon := \min[1; (c_{21}\|\mathcal{A}\|)^{-1}]$ and $M(F,R,\overline{\theta}) := \varepsilon / c_{22}$ with constants c_{21}, c_{22} independent of F, R, θ. Furthermore, from (5.13) the estimate

$$\|\theta_0\|^{s+2,\varkappa}_{G,s} \leq c_{23}(\varepsilon)\|\mathcal{A}\|c_{24}(F^0(\alpha), R, \overline{\theta}_{max}) \tag{5.16}$$

follows in this ball and $c_{24} \longrightarrow 0$ as $M(F,r,\overline{\theta}) \longrightarrow 0$. From Banach's fixed point theorem we conclude finally the unique solvability of (5.14) and the inequality

$$\|\theta_0\|^{s+2,\varkappa}_{G,s} \leq c_{25}(F^0(\alpha), R, \overline{\theta}_{max}). \tag{5.17}$$

Joining inequalities (5.17) and (5.12) one gets the estimate (5.5). Thus the theorem is completely proved. ■

Next, let us consider BVP (1.1)-(1.4),(1.6) in a slightly different domain $G' = \{x \in V; x_2 < \widetilde{\varphi_2}(x_1) \quad \text{for} \quad x_1 > 0 \quad \text{and} \quad x_1 < \widetilde{\psi_1}(x_2) \quad \text{for} \quad x_2 < h_1\}$ where $(\widetilde{\varphi_2} - h_\infty) \in C^{3+s}_{1+s,\varkappa}(\mathbf{R}^1_+), \widetilde{\psi_1} \in C^{3+s}_{1+s}(J_1, Y_1)$ are functions close to φ_2 and ψ_1 in the $C^{3+s}_{1+s,\varkappa}(\mathbf{R}^1_+)$-norm and in the $C^{3+s}_{1+s}(J_1, Y_1)$-norm, respectively. Moreover, we require that $\widetilde{\varphi_2}(0) = h_2$ and $\widetilde{\psi_1}(h_1) = 0$ hold. Let the solution be (v', p', θ'). To estimate the difference of solutions of both problems we map the domain G' onto G. Constructing this mapping as in [20] we obtain a nonhomogeneous problem of determining the transformed solution $(\hat{v}, \hat{p}, \hat{\theta})$ in G. Repeating the same conclusions as in [19] and performing some small modifications due to the temperature θ one can prove the following theorem for the differences $U := v - \hat{v}, P := p - \hat{p}, TE := \theta - \hat{\theta}$. Let s and z be the same as above.

Theorem 5.2 *If $M(F, R, \overline{\theta})$ is sufficiently small and*

$$\Lambda := |\widetilde{\psi_1} - \psi_1|_{C^{s+z}_{1+z}(J_1,Y_1)} + \|\widetilde{\varphi_2} - \varphi_2\|^{3+s,z}_{\mathbb{R}^1_+,1+s} \leq 1 \qquad (5.18)$$

then the inequality

$$\| U \|^{s+2,z}_{G,s} + \| \nabla P \|^{s,z}_{G,s-2} + \| TE \|^{s+2,z}_{G,s} \leq c_{26}(F^0(\alpha), R, \overline{\theta}_{max}) \cdot \Lambda \qquad (5.19)$$

holds, in which $c_{26} \longrightarrow 0$ as $M(F, R, \overline{\theta}) \longrightarrow 0$.

6 The complete free surface problem

Now we study the solvability of the complete BVP (1.1)-(1.6) under the conditions (1.7) and (1.8). We consider v, p and θ in Eqs.(1.5) as the solution of the nonlinear auxiliary problem (1.1)-(1.4),(1.6) depending on the functions $\psi_1(x_2), \varphi_2(x_1)$, and we show that for sufficiently small $M(F, R, \overline{\theta})$ the functions ψ_1, φ_2 are determined from (1.5) uniquely. Firstly, it follows from the representation of the solution to BVP (1.1)-(1.6) that it is necessary to choose $h_\infty = F^0(\alpha)/R$ in order to satisfy condition (1.8). This implies

$$\varphi_2(x_1) \longrightarrow \frac{F^0(\alpha)}{R} \text{ as } x_1 \longrightarrow +\infty. \qquad (6.1)$$

In this case the pressure $p(x)$ is bounded for $x_1 > 0$. Note that $\theta \in C^{s+2}_{s,z}(G)$ yields the continuity of θ up to the boundaries of G. Thus the inverse surface tension $\sigma^{-1}(\theta) = (a - b\,\theta)^{-1}$ is also a continuous function provided that $\theta < a/b$ is sufficiently small. For this reason we can conclude that

$$p(x) \longrightarrow p_* = \frac{F^0(\alpha)}{R} \text{ as } x_1 \longrightarrow +\infty \qquad (6.2)$$

hold applying the same operations as in the isothermal case (cf. [20]).

Let be $\omega_2(x_1) := \varphi_2(x_1) - \overline{\varphi}_2(x_1)$, where $\overline{\varphi}_2(x_1)$ denotes a solution to the following BVP

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\frac{\overline{\varphi}_2'(x_1)}{[1+(\overline{\varphi}_2'(x_1))^2]^{1/2}} - \beta(0)\overline{\varphi}_2(x_1) = -\beta(0)\frac{F^0(\alpha)}{R}, \qquad (x_1 \in \mathbf{R}^1_+)$$

$$\overline{\varphi}_2(0) = h_2, \qquad \varphi_2(x_1) \longrightarrow \frac{F^0(\alpha)}{R} \text{ as } x_1 \longrightarrow +\infty. \tag{6.3}$$

Analogously, let be $\omega_1(x_2) := \psi_1(x_2) - \overline{\psi}_1(x_2)$, where $\overline{\psi}_1(x_2)$ denotes a solution to the BVP

$$\frac{\mathrm{d}}{\mathrm{d}x_2}\frac{\overline{\psi}_1'(x_2)}{[1+(\overline{\psi}_1'(x_2))^2]^{1/2}} + \beta(0)x_2 = \beta(0)\frac{F^0(\alpha)}{R} + \frac{\beta\overline{p}}{\hat{g}}, \qquad (x_2 \in]0, h_1[)$$

$$\overline{\psi}_1(h_1) = 0, \qquad \overline{\psi}_1'(0) = -A = \cot\theta_1. \tag{6.4}$$

These BVPs result from (1.5) in the case $v \equiv 0, p = p_* = \text{const.}, \theta = 0$ being a solution to the auxiliary BVP (1.1)-(1.4),(1.6) for the parameters $R = F^0(\alpha) = \overline{\theta}_1 = \overline{\theta}_2 = \overline{\theta}_3 = 0$. It was easy to show [19] that the problem (6.3) has a unique solution , if the condition $|h_2 - h_\infty| < [2/\beta(0)]^{1/2}$ is satisfied. Furthermore, the difference $(\varphi_2(x_1) - h_\infty)$ is equivalent to $\exp(-\sqrt{\beta(0)}x_1)$ as $x_1 \longrightarrow +\infty$.

The BVP (6.4) was studied [20] in detail and the corresponding results will be included in Theorem 6.1. For the unknown functions $\omega_m(x_1)$ we obtain a two-point BVP like BVP (8.8) from [19] subtracting Eqs.(6.3) and (6.4) from Eq.(1.5). A difference to BVP (8.8) consists in the following. We have to substitute β by $\beta(0)$ everywhere and, additionally, we introduce the operators $T_m^{(3)}$ defined as

$$T_m^{(3)}\omega_m := \frac{\delta\theta}{\sigma(\theta)}\omega_m = \frac{\sigma(0) - \sigma(\theta)}{\sigma(\theta)}\omega_m. \tag{6.5}$$

The remaining part of the proof of the main theorem can be realized as a modified repetition of the proof of Theorem 8.1 in [19]. Instead of the operator Eq.(8.10) from [19] we have to study the following one:

$$\omega_m = L_m(T_m^{(1)}\omega_m + T_m^{(2)}\omega_m + T_m^{(3)}\omega_m) =: B_m\omega_m \tag{6.6}$$

with $T_m^{(3)}$ given in (6.5) and the other part taken from [19]. Since $T_m^{(3)}$ is a contraction operator for small θ we can conclude as in [19] that B_m is a contraction operator too in the ball $\| \omega_m \|^{3+s,z}_{\mathbb{R}^1_+,1+s} < \epsilon$. Consequently, we have proved the main result of the present paper.

Theorem 6.1 *There exist positive real numbers* $\overline{s} < s_0$, $\overline{z} \leq \min[z_0, \sqrt{\beta(0)}]$ *and* M_0 *such that for arbitrary* $s \in]0, \overline{s}[$, $z \in]0, \overline{z}[$, $M(F, R, \theta) < M_0$ *and for positive* $h_1, h_2, F_0(\alpha), R$ *satisfying the conditions*

$$\left| h_2 - \frac{F_0(\alpha)}{R} \right| < \sqrt{\frac{2}{\beta(0)}}, \qquad 0 < h_1 < \sqrt{\frac{2(\tilde{A}+1)}{\beta(0)}},$$

$$h_1 \leq \frac{F_0(\alpha)}{R} + \frac{(\hat{p}_u - \hat{p}_l)}{\hat{g}} < \frac{h_1}{2} + \frac{\tilde{A}+1}{\beta(0) h_1} \tag{6.7}$$

the free BVP (1.1)-(1.6) has a unique solution $\{v, p, \theta, \varphi_2, \psi_1\}$ *which can be represented in the form*

$$v = g(-x_1)u^{(-)} + g(x_1)(R, 0)^T + w, \qquad \varphi_2(x_1) = \overline{\varphi}_2(x_1) + \omega_2(x_1),$$

$$p = g(-x_1)p^{(-)} + p_0(x) + p_*, \qquad \psi_1(x_2) = \overline{\psi}_1(x_2) + \omega_1(x_2),$$

$$\theta = g(-x_1)\,\theta^{(-)} + g(x_1)\,\theta_1 + \theta_0(x), \tag{6.8}$$

where $g, (u^{(-)}, p^{(-)}, \theta^{(-)})$ *are taken from Theorem 5.1 and* $\tilde{A} := A(1+A^2)^{-1/2}$ *was set. Moreover,* $\theta_0, w \in C^{s+2}_{s,z}(G), p_0 \in C^{s+1}_{s-1,z}(G^0 \cup G^+), \nabla p_0 \in C^{s}_{s-2,z}(G)$ *and* $\omega_2 \in C^{3+s}_{1+s,z}(\mathbf{R}^1_+), \omega_1 \in C^{3+s}_{1+s}(J_1, Y_1)$ *hold.*

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