QUASI-MONTE CARLO STUDIES OF SPATIAL AVERAGES OF QUADRATIC MAPS

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ABSTRACT. Using Quasi-Monte Carlo integration, we found numerically that the spatial averages of chaotic quadratic maps either fluctuate periodically or converge to constants.

1. Introduction

A series of papers [3] [4] [5] [7] motivated us to study the dynamics of spatial averages of dynamical systems, especially, those with chaotic behaviors. The problem arises naturally when one considers the collective behavior of a large lattice system with subsystems being identical and chaotic. The main interest of the study is to investigate the dynamics of the thermodynamic limit of some global quantities such as spatial averages as the lattice size tends to infinity. In numerical simulations, the problem is often approached by taking a large lattice size. However such approach has major difficulties when the dimension of the lattice is even moderately high. In a previous paper [2], we first proved the existence of the thermodynamic limit under very general assumptions. As an easy consequence of Ergodic Theorem, we then obtained an explicit formula for the thermodynamic limit of spatial averages. This enables us to study the dynamics of spatial averages independent of the lattice size. We showed that the dynamics of spatial averages is closely related to the existence of an asymptotic invariant measure (SRB-measure) for the local subsystem when the interactions among subsystems are weak. If such measure exists, the spatial average converges to a constant.

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One interesting question that remains open is whether there exists a dynamical system whose spatial average can fluctuate chaotically in time. Even though there is an example in [2] showing that the spatial average can have non-periodic fluctuation, such example does not appear naturally. It was constructed solely to illustrate possible behaviors of spatial averages and it has infinitely many non-differentiable points. Since it is known that for uncountably many parameters of λ, the quadratic family

$$f_\lambda(x) = \lambda x(1-x)$$

does not have an asymptotic invariant measure [9], one would like to investigate the dynamics of spatial averages for this entire family, i.e.,

$$A(n,\lambda) = \int_0^1 f_\lambda^n(x)dx, \tag{1}$$

where $f_\lambda^n = f_\lambda \circ \cdots \circ f_\lambda$ is the nth iterate of f_λ. Some simple results along this direction were reported in [2]. These simple results concern only the case when $f_\lambda(x)$ has an attracting periodic orbit or f_λ has an asymptotic measure.

In this article, we report our numerical results on the complete picture of the dynamics of spatial averages $A(n,\lambda)$ for the quadratic family $2 \le \lambda \le 4$. Our results confirm that there is a cascade of period two bifurcation at the beginning of the parameter interval. However, this bifurcation does not lead to any chaotic behavior. instead, it leads to a period two orbit. Our numerical results do not show any inverse cascade of bifurcation, nor any chaotic behavior in the entire parameter interval.

An interesting observation from our numerical computation is that the spatiotemporal average of the quadratic family seems to exit and depends on the parameter λ continuously. We formulate this observation as a conjecture. It also remains unknown what can happen when we consider a weighted spatial average

$$\widetilde{A}(n,\lambda) = \int_0^1 \varphi(f_\lambda^n(x))dx$$

for various functions $\varphi(x)$ over the interval $[0,1]$.

2. Main Results

We first describe the general problem.

Let $\mathbb{Z}^d$ be a d-dimensional integer lattice and M be a smooth Riemannian manifold possibly with boundary. Let Φ be a continuous map from $\mathcal{M} = \otimes_{i\in\mathbb{Z}^d} M_i$, the direct product of identical copies of M over $\mathbb{Z}^d$, into itself with respect to the product topology. For any continuous function φ on M, by the multidimensional ergodic theorem [8] the following limit exists for almost every point $\bar{x}$ in $\mathcal{M}$ with respect to the direct product of Lebesgue measures $\otimes dx$ induced by Riemannian metric on M:

$$A(n) = \lim_{V\to\mathbb{Z}^d} \frac{1}{|V|} \sum_{i\in V} \varphi(\Phi_i^n(\bar{x})) = \int_{\mathcal{M}} \varphi P_0[\Phi^n(\underline{\mathrm{x}})] \otimes dx,$$

where Φ_i^n denotes the coordinate of Φ^n, the nth iterate of Φ, at the lattice site $i \in \mathbb{Z}^d$, i.e., $\Phi = (\Phi_i)_{i\in\mathbb{Z}^d}$; P_0 denotes the projection from $\mathcal{M}$ to M_0 [2].

The function φ can be considered as a local observable and therefore, $A(n)$ is the spatial average of φ over the entire lattice $\mathbb{Z}^d$. The quantity $A(n)$ has different names in different contexts. It is called *collective behavior* in [5] and *population activity* in [6].

The particular model we study in this paper does not have spatial interaction, i.e., the map Φ is a direct product of a map $f_\lambda(x) = \lambda x(1-x)$ from $M = [0,1]$ into itself: $\Phi = (f_{\lambda i})_{i\in\mathbb{Z}^d}$. In this simple case, $A(n)$ can be written in the following form:

$$A(n,\lambda) = \int_M \varphi(f_\lambda^n(x))dx, \tag{2}$$

where dx denotes the Lebesgue measure and $2 \le \lambda \le 4$ is a fixed parameter.

We report our numerical results below when $\varphi(x) = x$ with illustrations at the end of this article. Since the function $f_\lambda^n(x)$ is very oscillatory when n is large, we use Monte Carlo and Quasi-Monte Carlo integrations in evaluation of the integral (2).

Main Results:

(1) Asymptotically, for every $\lambda: \ 2 \leq \lambda \leq 4$, $A(\lambda, n)$ *is either periodic or constant. When* λ *increases,* $A(n, \lambda)$ *also undergoes a periodic doubling bifurcation. However, this bifurcation does not lead to any chaotic behavior. Instead, it leads to a periodic two fluctuation. In the regions when* $f_\lambda(x)$ *is very chaotic, the spatial average approaches constant quickly.*

Figure 1 plots the values of $A(n, \lambda)$ for 500 points of λ in $[3.5, 4]$ and $41 \leq n \leq 90$. For each fixed λ, different values of $A(n, \lambda)$ are plotted on the same vertical line. Figures 2, 3, and 4 shows more details at different area including the so-called tongue-like structure [10]. One can see clearly that the 'tongues' correspond to the windows in the graph of the orbits of the maps.

(2) (Conjecture) (a) The time average of $A(\lambda, n)$*:*

$$STA(\lambda) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} A(\lambda, n),$$

called spatiotemporal average of $f_\lambda(x)$ *exists for every* $\lambda \in [2, 4]$.

(b)$STA(\lambda)$ *is a continuous function in* λ *and has self-similar properties.*

Figures 5-8 provide supporting evidences of our conjectures.

(3) There exists $\lambda \in [2, 4]$ *(e.g.,* $\lambda = 3.9616$*) such that* $f_\lambda(x)$ *does not have attracting periodic orbits, but the corresponding spatial average* $A(\lambda, n)$ *fluctuates periodically as* n *tends to infinity. The mechanism for causing such fluctuation is close to the one described in* [2]*: points of the orbit are mostly concentrated in several disjoint sets.*

Figure 9 shows the dynamics of the spatial average when $\lambda = 3.9616$. It is clearly asymptotically periodic. Figures 10, 11 , and 12 show the typical behavior of the orbits of $f_\lambda(x)$ for $\lambda \approx 3.96$. Figures 13 and 14 show the dynamics of spatial average $A(n, \lambda)$ when $\lambda = 3.9938$ with different number of points chosen in Quasi-Monte Carlo integration. They indicate that the fluctuation of the average is mainly caused by the numerical error. The true picture of $A(n, 3.9938)$ can only be obtained with higher precision computation.

Remarks. (1) These results are only numerical observations. Theoretically, $A(\lambda, n)$ is still possibly chaotic. But, the variation of its values is so small that it is usually covered by the numerical noise and very difficult to be detected by routine numerical schemes. We have used quadruple precisions in all computations.

(2) Note that the existence of an asymptotic invariant measure is equivalent to the existence of the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_0^1 [f_\lambda^n(x)]^k dx$$

for every positive integer k since polynomials are dense in $C[0,1]$. The claim of item (2) is much weaker than the existence of an asymptotic invariant measure. We believe that the existence of spatiotemporal average holds for a rather general class of smooth dynamical systems and it depends on parameters continuously.

(3) Our numerical results also imply that in coupled map lattice cases where the lattice size is much greater than the spatial correlation length, the spatial average is either asymptotically periodic or convergent to a constant.

3. Error Estimation

In this section, we give justifications for our numerical methods.

3.1. **Error in Monte Carlo integration.** Assume that $f(x)$ is a measurable function on the unit interval $[0,1]$ and $\{x_i\}$ is a sequence of independently uniformly distributed random variables in $[0,1]$. The error in Monte Carlo integration of $f(x)$,

$$\epsilon = |\int_0^1 f(x) - \frac{1}{N} \sum_1^N f(x_i)|$$

can be estimated using the center limit theorem. The most convenient formula is the so-called “ the rule of three sigmas”: with a confidence level of 0.997, the error ϵ satisfies the inequality

$$\epsilon \leq \frac{3\sigma}{\sqrt{N}}, \tag{3}$$

where σ^2 is the variance of $f(x)$, i.e.,

$$\sigma^2(f) = \int_0^1 f^2(x)dx - (\int_0^1 f(x)dx)^2.$$

The confidence level can be significantly raised if the coefficient in front of σ is just slightly increased.

When $f_\lambda(x) = \lambda x(1-x)$, we have $\sigma^2 \leq 2$. This estimation holds also for all iterates of $f_\lambda(x)$. So we have

$$\epsilon \leq \frac{3\sqrt{2}}{\sqrt{N}}$$

with a confidence level of 0.997. In numerical calculation of $\int_0^1 f_4^n(x)dx$, the result is much better than the estimation above. We have $\epsilon \approx \frac{1}{\sqrt{N}}$. The random number generator used in our simulation is from the SGI's Fortran compiler.

3.2. **Error in Quasi-Monte Carlo integration.** Since the error in Monte Carlo method is random number generator dependent, we implemented our numerical integration with Quasi-Monte Carlo method. In our case, the domain of the integral is of dimension one, so we simply use the following formula to approximate the integral:

$$\int_0^1 f_\lambda^n(x)dx \approx \frac{1}{p^k}\sum_{i=1}^{p^k} f_\lambda^n(\frac{i}{p^k}),$$

where p is any prime number and k is a positive integer. The traditional error estimation involving the derivatives of $f_\lambda^n(x)$ will not give any meaningful bound on the error since the derivatives of $f_\lambda^n(x)$ increases exponentially fast in terms of n. We use Fourier series expansion to obtain a formal representation of the error.

Proposition 1. *Assume that*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi nx + b_n 2\pi nx, \; x \in [0,1].$$

Then

$$\frac{1}{p^k}\sum_{i=1}^{p^k} f(\frac{i}{p^k}) - \int_0^1 f(x)dx = \sum_{m=1}^{\infty} b_{mp^k}. \tag{4}$$

The formula (4) can be easily verified by integrating both sides of the Fourier expansion of $f(x)$. There are several advantages of using this scheme of numerical integration: it is random generator independent; on many machines, it is faster than Monte Carlo method; and one can make different choices of the prime number p to make assure that a good estimation of the integral is obtained. Theoretically, the method should work especially well when $f(x)$ can be smoothly extended into a periodic function because b_m goes to zero very fast when m tends to infinity [11]. When $f(x)$ is only continuous, the convergence rate depends on how well $f(x)$ can be uniformly approximated by trigonometric polynomials [11].

We compared the errors (Figures 15 and 16) from both Monte Carlo and Quasi-Monte Carlo integration of

$$\int_0^1 f_4^n(x)dx = \frac{2^{2n-1}}{2^{2n}-1}$$

where the exact value of the integral is available. Quasi-Monte Carlo method shows smaller error when n is not large and approximately the same magnitude of error when n is large.

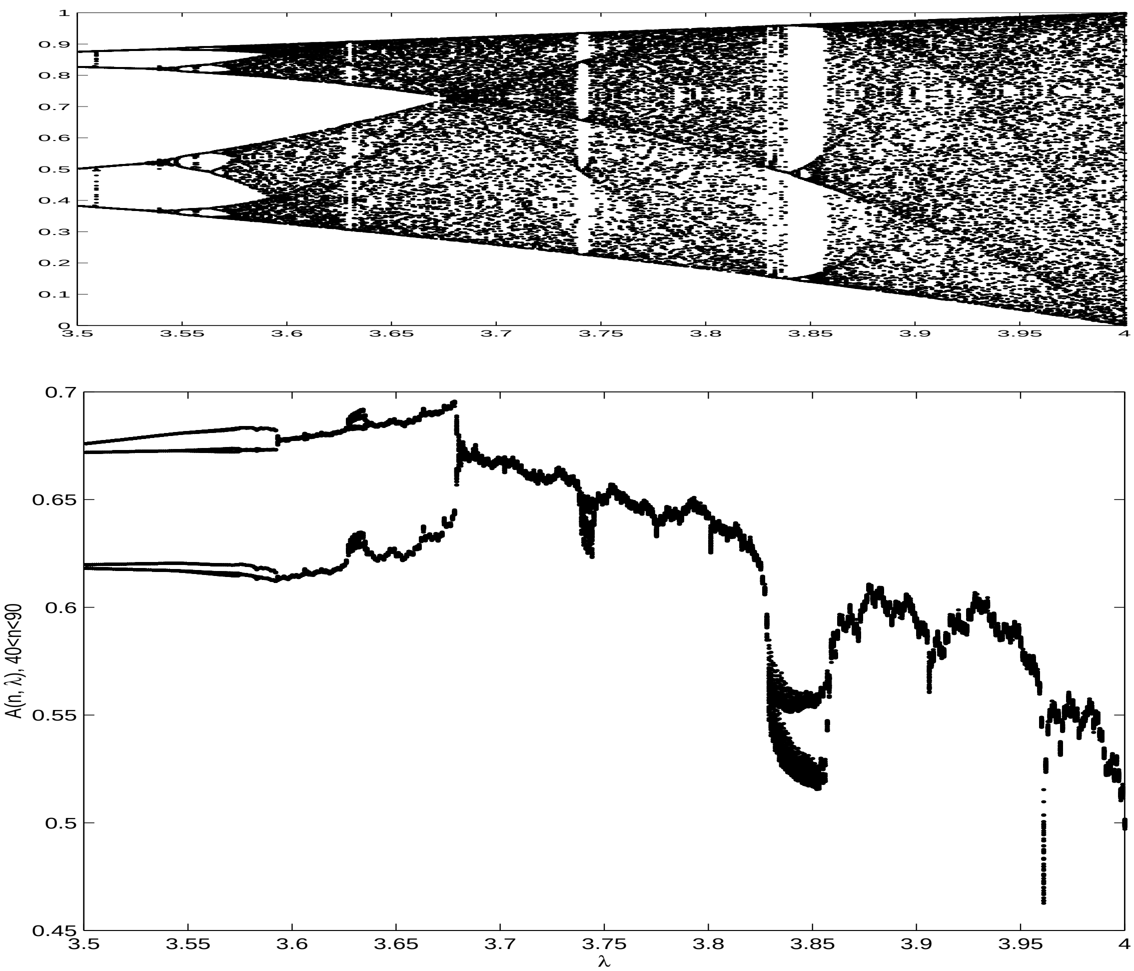

FIGURE 1. Spatial Averages (a)

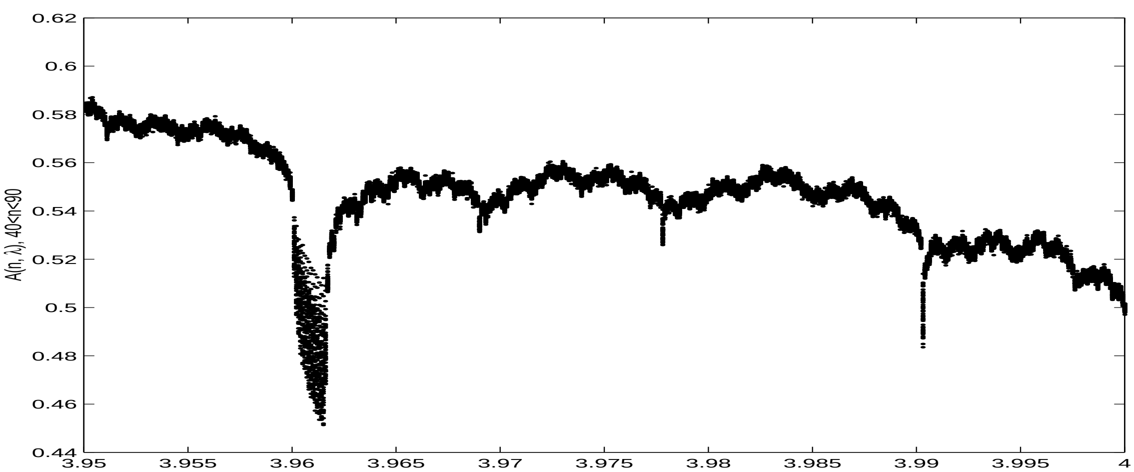

FIGURE 2. Spatial Averages (b)

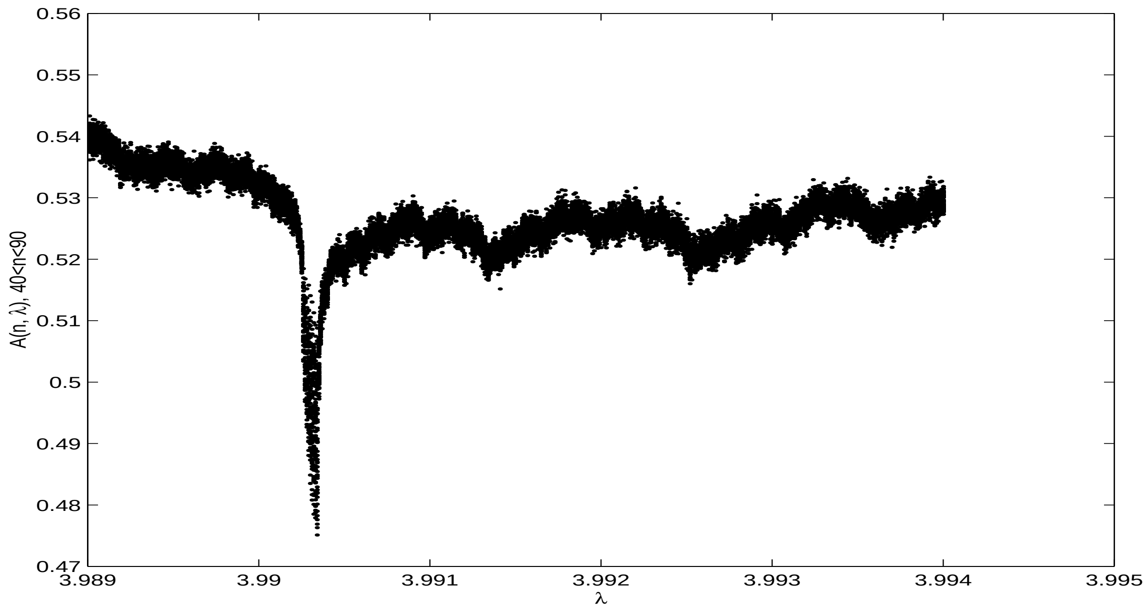

FIGURE 3. Spatial Averages (c)

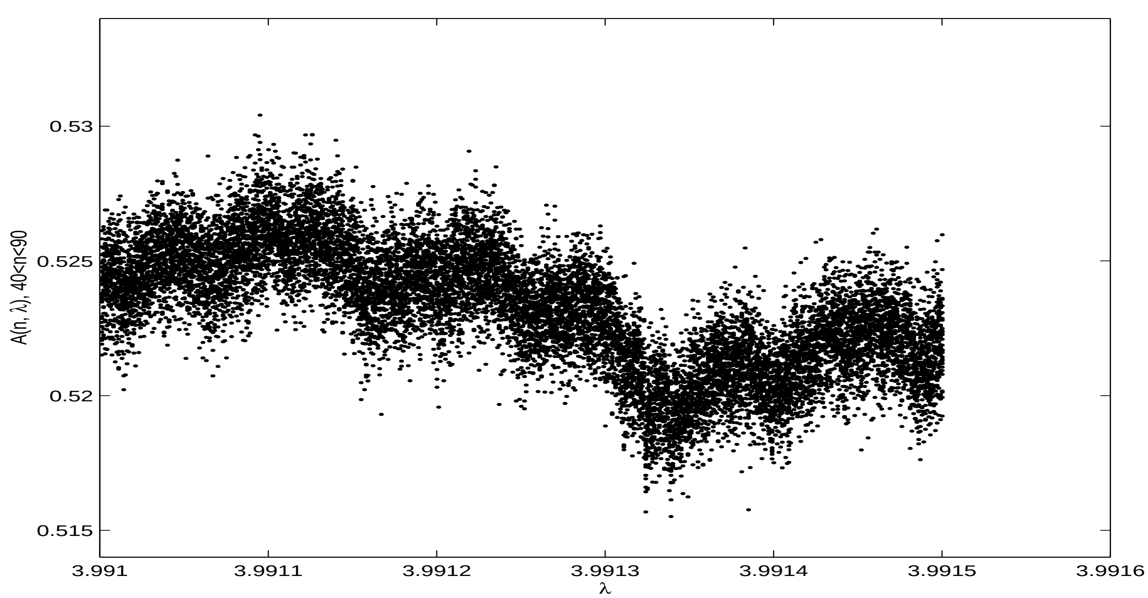

FIGURE 4. Spatial Averages (d)

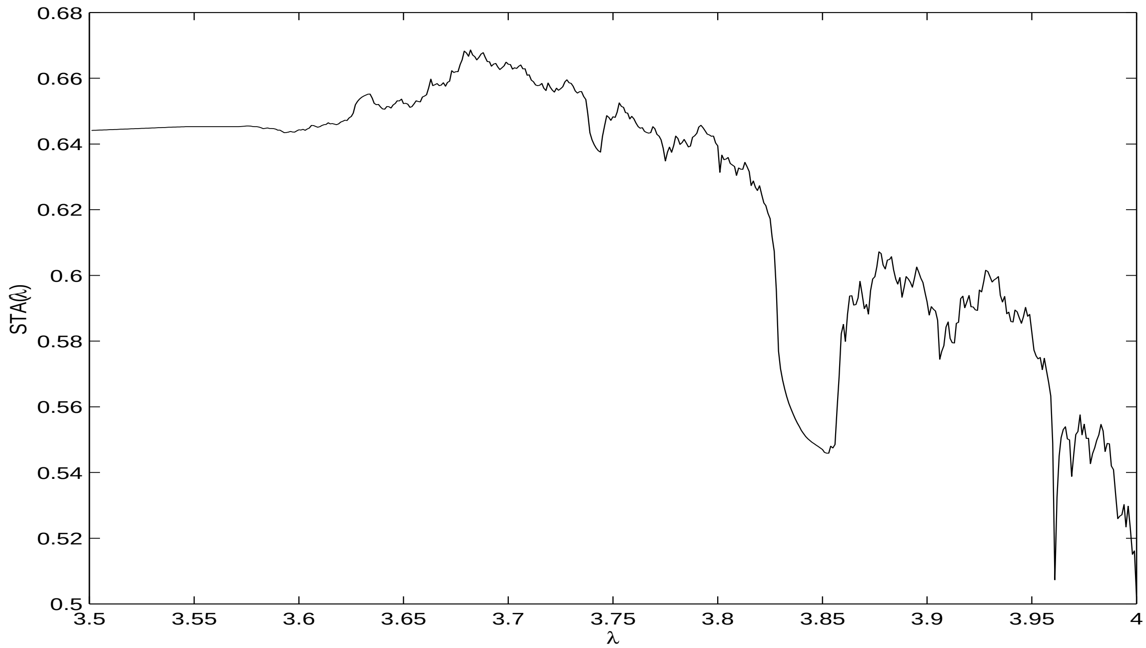

FIGURE 5. Spatiotemporal Average (a)

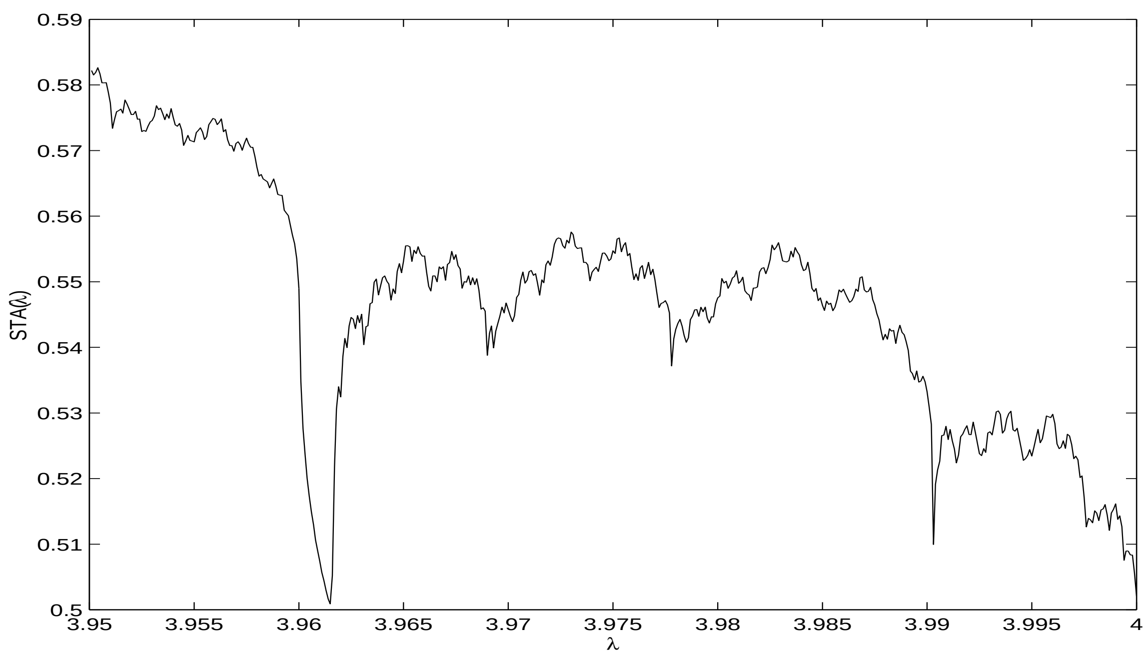

FIGURE 6. Spatiotemporal Average (b)

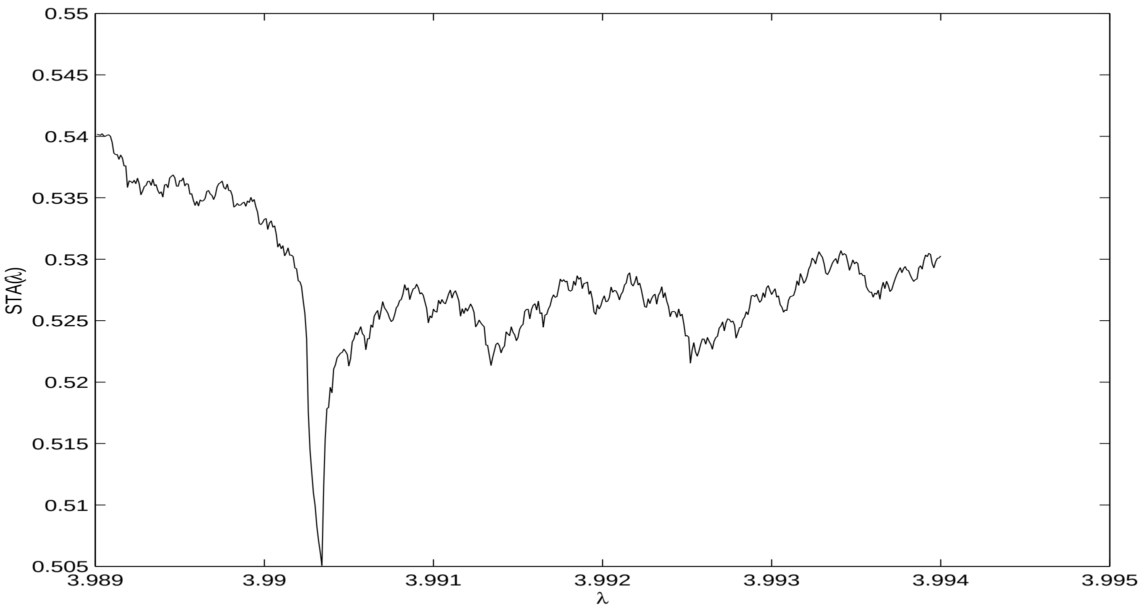

FIGURE 7. Spatiotemporal Average (c)

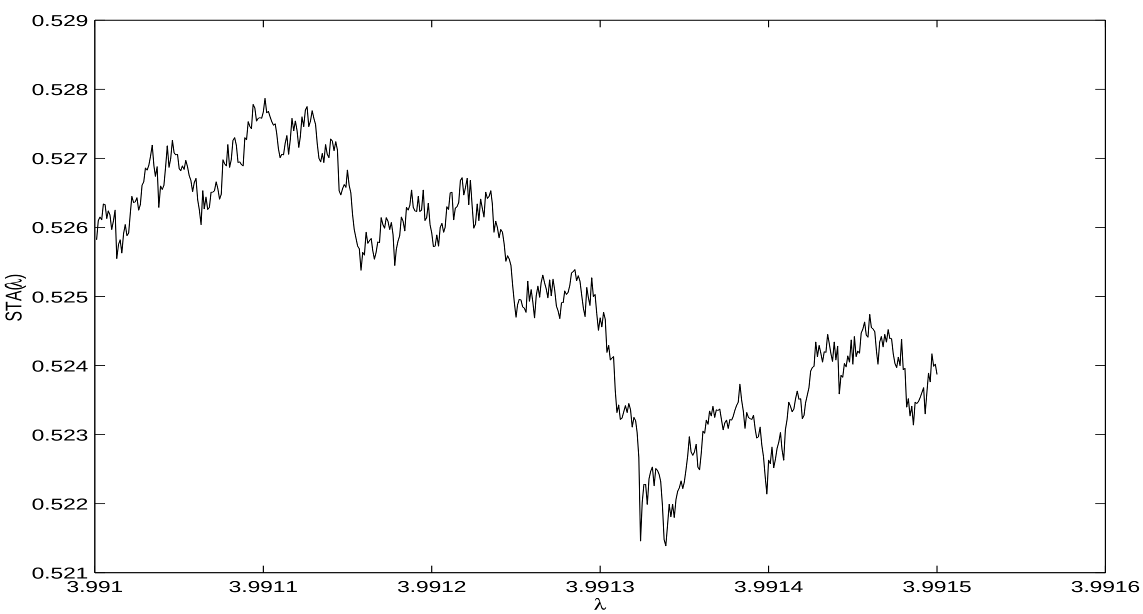

FIGURE 8. Spatiotemporal Average (d)

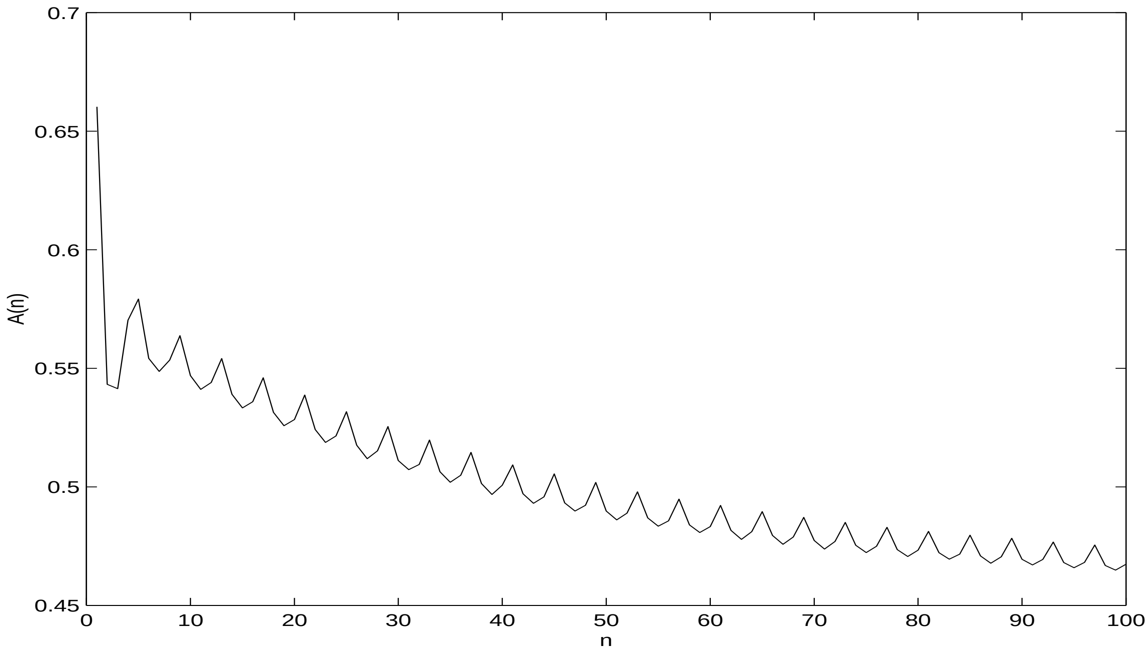

FIGURE 9. Dynamics of Spatial Average $\lambda = 3.9616$

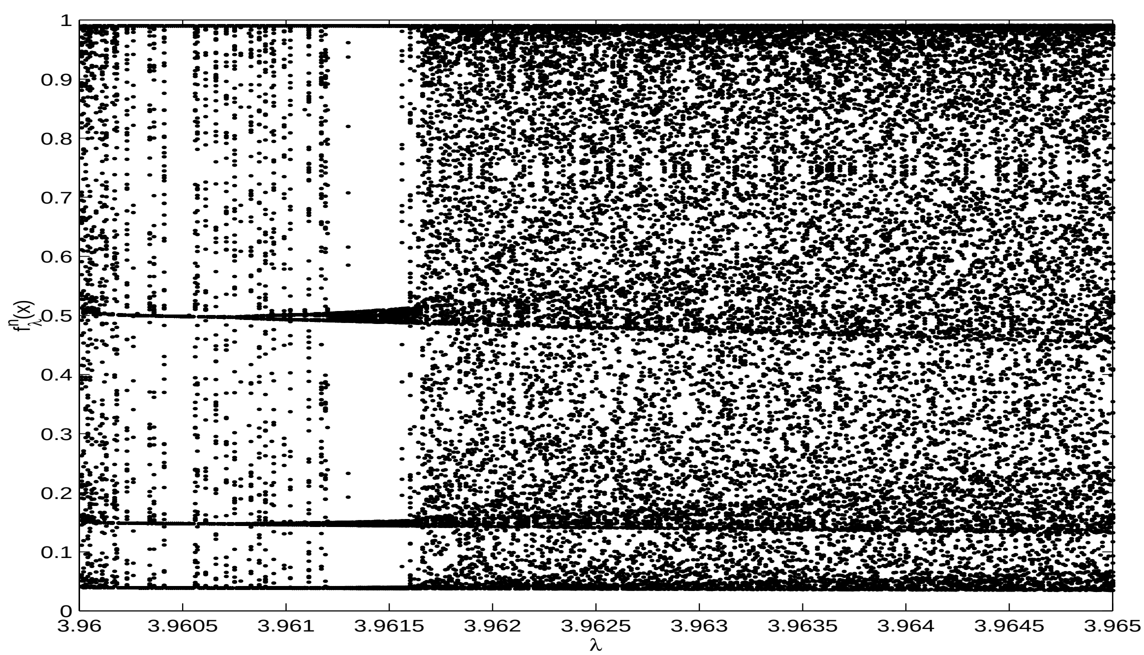

FIGURE 10. Orbits of Quadratic Maps

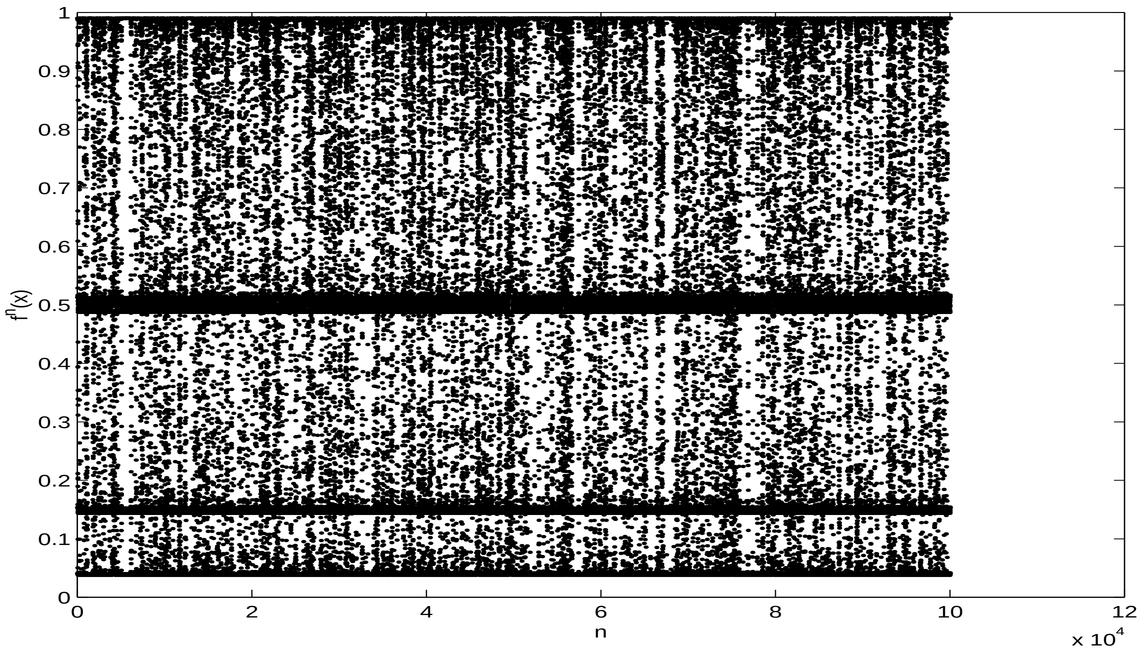

FIGURE 11. A typical Orbit of $f(x) = 3.9616x(1-x)$ (a)

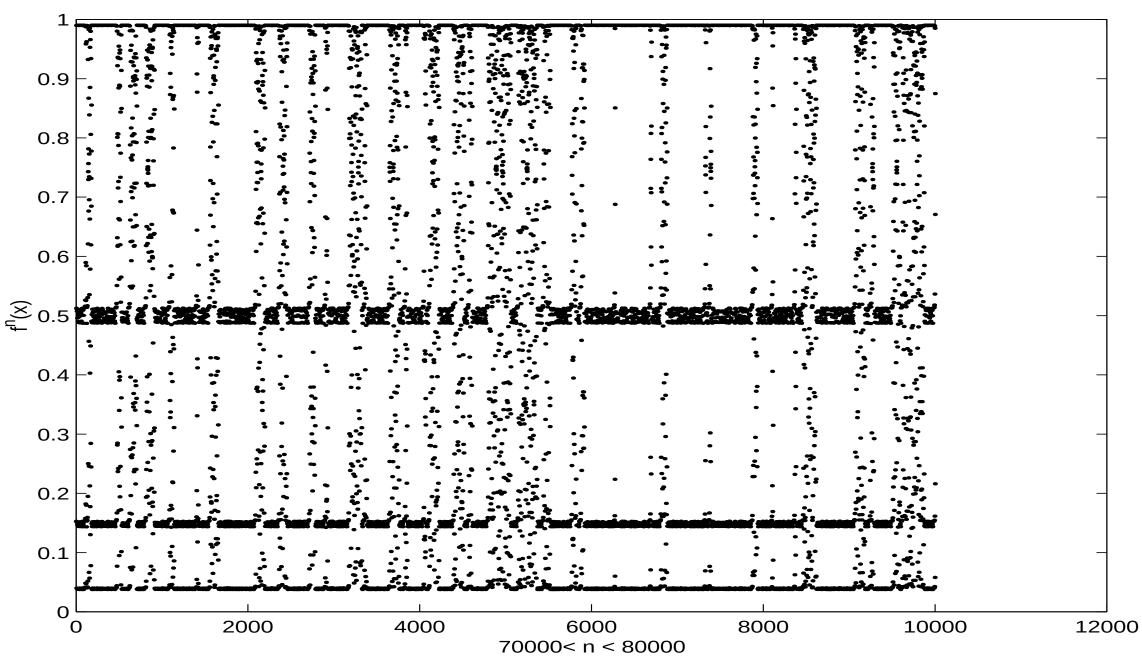

FIGURE 12. A typical Orbit of $f(x) = 3.9616x(1-x)$ (b)

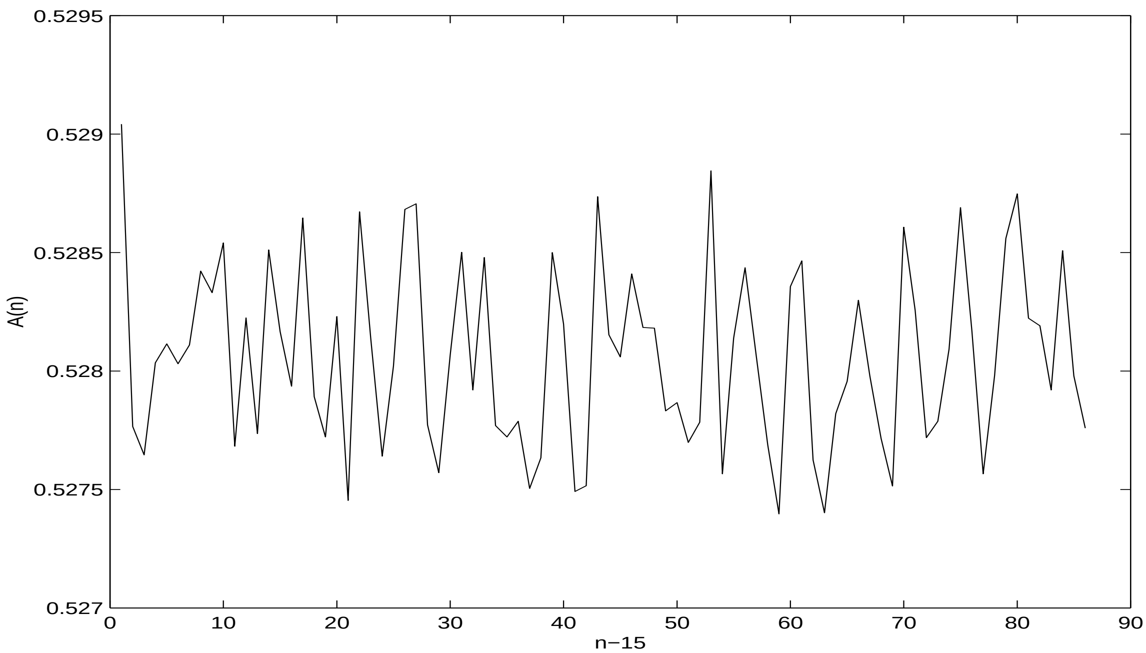

FIGURE 13. Dynamics of Spatial Average $\lambda = 3.9938, N = 10^6$

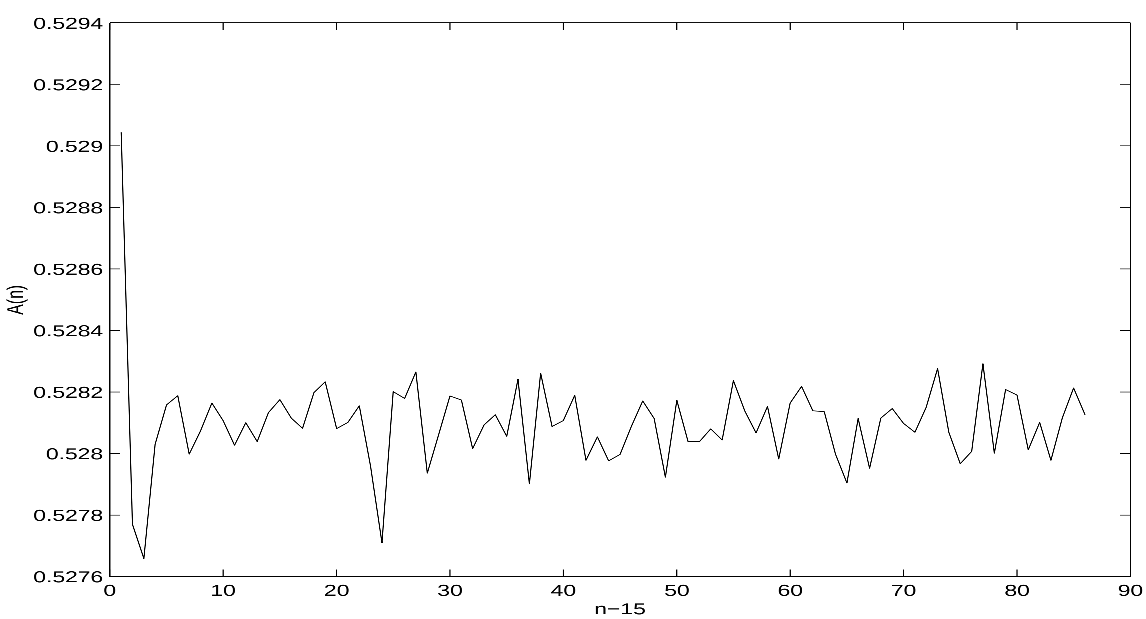

FIGURE 14. Dynamics of Spatial Average $\lambda = 3.9938, N = 10^7$

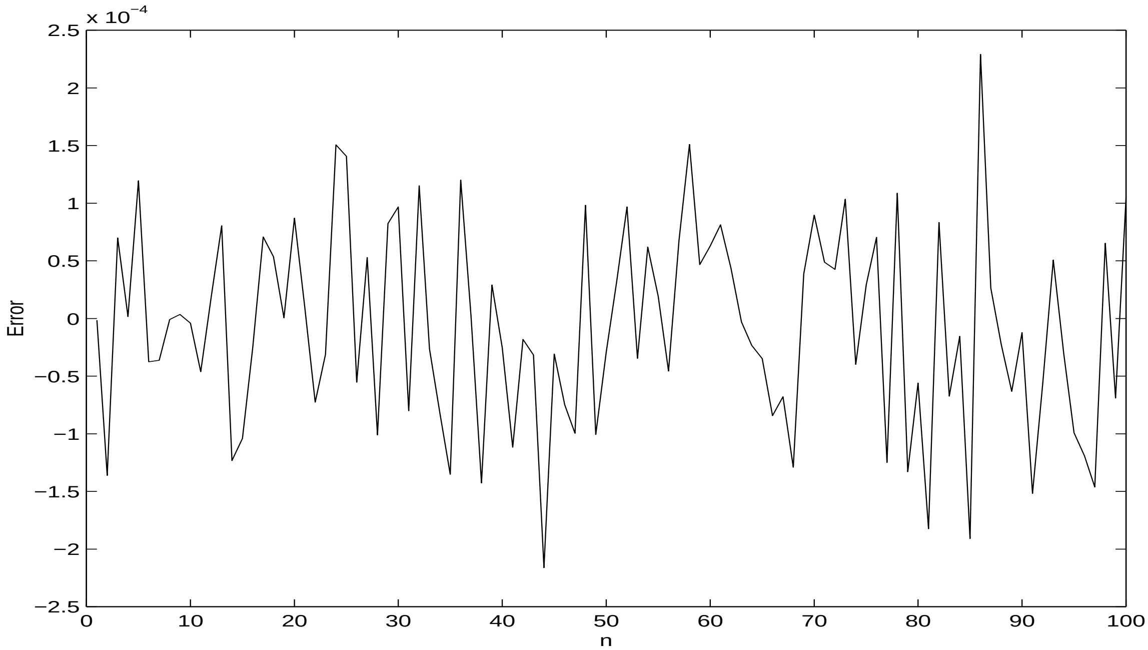

FIGURE 15. Error in Monte Carlo Integration of $f_4^n(x)$

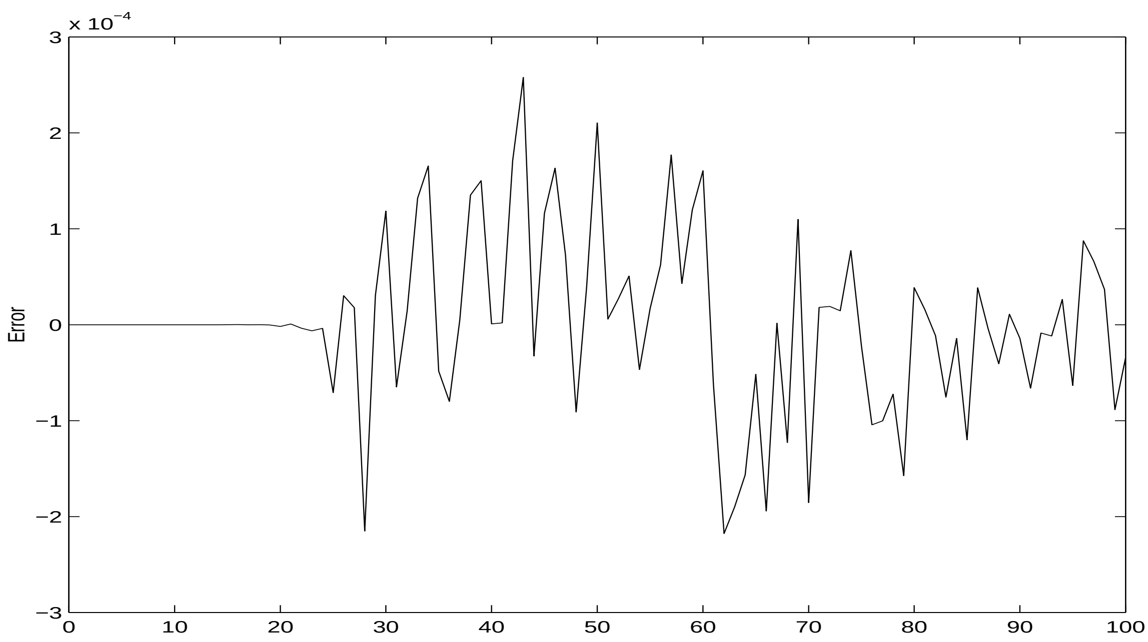

FIGURE 16. Error in Quasi-Monte Carlo Integration of $f_4^n(x)$

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