

A SINGULAR LIMIT FOR A SYSTEM OF DEGENERATE CAHN–HILLIARD EQUATIONS

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ABSTRACT. A singular limit is considered for a system of Cahn–Hilliard equations with a degenerate mobility matrix near the deep quench limit. Via formal asymptotics, this singular limit is seen to give rise to geometric motion in which the interfaces between the various pure phases move by motion by minus the surface Laplacian of mean curvature. These interfaces may couple at triple junctions whose evolution is prescribed by Young’s law, balance of fluxes, and continuity of the chemical potentials. Short time existence and uniqueness is proven for this limiting geometric motion in the parabolic Hölder space $C_{t,p}^{1+\frac{\alpha}{4}, 4+\alpha}$, $0 < \alpha < 1$, via parametrization of the interfaces.

1. INTRODUCTION

Isothermal phase separation in multicomponent systems can be modeled by a system of Cahn–Hilliard equations. In such a system the diffusional mobility matrix in general can be expected to be dependent on the concentration of the components. In regions in which one component is predominant, the mobility tends to be much smaller than in regions in which there is a mixture of different components. This implies that diffusion in interfacial regions is relatively enhanced. In particular, this phenomenon becomes more pronounced at low temperatures where entropy contributions are smaller. Therefore, it is reasonable to consider systems of Cahn–Hilliard equations in which the mobility matrix depends on the concentrations of the different components (cf. [11, 22, 31]). The mass balance for the various components then gives the following evolution equations for the concentrations u^i , $i = 1, \dots, N$,

$$\begin{aligned} u_t^i &= -\nabla \cdot J^i, \\ J^i &= -\sum_{j=1}^N L^{ij}(\mathbf{u}) \nabla w^j, \\ w^i &= D_i \Psi(\mathbf{u}) - \frac{1}{N} (D\Psi(\mathbf{u}) \cdot \mathbf{e}) - \gamma \Delta u^i, \end{aligned} \tag{1}$$

where $\mathbf{e} = (1, \dots, 1)^t$ and $\mathbf{u} = (u^1, \dots, u^N)^t$. The number N denotes the number of the components and J^i is the flux of the i 'th component, which is taken here to be a linear combination of the thermodynamical forces ∇w^j . The functions w^i are

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modified chemical potentials which are prescribed as functional derivatives of the free energy functional

$$\mathcal{E}(\mathbf{u}) = \int_{\Omega} (\Psi(\mathbf{u}) + \frac{\gamma}{2} |\nabla \mathbf{u}|^2)$$

where the variational derivative has been taken over the set of all functions which satisfy the constraint $\sum_{i=1}^N u^i = 1$. This constraint reflects the fact that the u^i are defined here as volume (or molar) fractions of the components. In this paper we assume the homogeneous part of the free energy Ψ to be of regular solution type; i.e.,

$$\Psi(\mathbf{u}) = \theta \sum_{i=1}^N u^i \ln u^i - \mathbf{u} \cdot \mathbf{A} \mathbf{u},$$

where \mathbf{A} is a $N \times N$ matrix. The gradient energy coefficient $\gamma > 0$ that appears in the gradient part of the free energy is a small parameter and we will see that $\sqrt{\gamma}$ is directly proportional to the thickness of the interfaces. Furthermore, \mathcal{E} is taken to be defined on a bounded domain $\Omega \subset \mathbb{R}^n$ with a sufficiently smooth boundary. As boundary conditions for the system (1), we assume no-flux conditions and natural boundary conditions for the concentrations; i.e., $J^i \cdot \nu = 0$ and $\nabla u^i \cdot \nu = 0$ where ν is the exterior normal to $\partial\Omega$.

Phase separation for multicomponent systems were first studied by Morral and Cahn [23] and De Fontaine [13]. In particular their linear stability analysis suggested that systems of Cahn–Hilliard equations model spinodal decomposition in multicomponent systems. This was later supported by the numerical simulations of Eyre [20] and Blowey, Copetti, and Elliott [5]. We also refer to the work of Barrett and Blowey [3, 4] who analyze a numerical method for a system of Cahn–Hilliard equations with a concentration dependent mobility matrix. They performed numerical simulations with a concentration dependent mobility which is taken to be small in the pure component and as expected they observed that the time scale of the evolution increases as the mobility in the pure component decreases. A derivation of the system (1) from basic thermodynamical principles was given by Elliott and Luckhaus [18] in the case of a constant mobility matrix. They also proved global existence and uniqueness. Elliott and Garcke [17] studied a system of Cahn–Hilliard equations with a concentration dependent mobility matrix which allowed the mobility to vanish for the pure component. They discussed properties of the mobility matrix and showed existence of a weak solution of the resulting system of fourth order degenerate parabolic equations. We also refer to the work of Tombakoglu and Ziya Akcasu [31] who derived the system (1) with a concentration dependent mobility matrix from a mean field theory.

The simplest choice of a concentration dependent mobility matrix which is physically relevant, has as its entries

$$L^{ij}(\mathbf{u}) = u^i (\delta_{ij} - u^j) \quad i, j = 1, \dots, N \quad (2)$$

(cf. [31, 17]). This matrix is symmetric and fulfills the condition $\sum_{j=1}^N L^{ij} = 0$ which is necessary in order to ensure that the constraint $\sum_{i=1}^N u^i = 1$ is preserved during the evolution. Although our results are also true for more general mobility matrices

(cf. [17]), we restrict our discussion to the form (2) in order to keep the presentation simple.

Our goal in this paper is to relate the system (1) to a sharp interface model in the limit in which the interfacial thickness tends to zero as the time is rescaled appropriately. The first result in this direction for a single Cahn–Hilliard equation with a constant mobility was obtained by Pego [26]. By the means of formal asymptotic expansions, Pego [26] showed that in the scaling $\gamma = \varepsilon^2$, $t \rightarrow \varepsilon t$, as $\varepsilon \searrow 0$ solutions of the scalar Cahn–Hilliard equation with constant mobility can be expected to converge to solutions of the Mullins–Sekerka evolution problem. Later Pego’s result was justified rigorously in the radially symmetric case by Stoth [30] and by Alikakos, Bates, and Chen [1] in the case that the limiting motion has a smooth solution. Cahn, Elliott and Novick–Cohen performed a formal asymptotic analysis for a single Cahn–Hilliard equation with a concentration dependent mobility. With the scaling $\gamma = \varepsilon^2$, $t \rightarrow \varepsilon^2 t$, they obtained motion by surface diffusion as the singular limit near the deep quench; i.e. as $\varepsilon, \theta \searrow 0$. This evolution law is defined purely in terms of local geometric quantities in contrast to Mullins–Sekerka evolution where the geometric data of the interfaces are coupled with the evolution of bulk fields. We also mention the work by Elliott and Garcke [16] and Escher, Mayer, and Simonett [19] for existence and stability results for diffusive surface motion laws.

The connection between a multiphase Ginzburg–Landau system and a sharp interface problem was first demonstrated by Baldo [2]. He applied Γ –convergence techniques to show that functions that minimize \mathcal{E} over a set of functions with prescribed mass converge to solutions of a sharp interface problem which consists of finding a partition of Ω into N phases which fulfills a minimal interface condition. Here, each of the phases is constrained to have fixed volume. Via formal asymptotic expansions, Bronsard and Reitich [8] related a vector valued Allen–Cahn equation to mean curvature evolution for interfaces which are coupled by an angle condition, known as Young’s law [33], at triple junctions. They also showed existence of solutions for the limiting evolution problem. The angle condition at the triple junction can be viewed as a balance of mechanical forces and it arises in the formal asymptotic expansions as a solvability condition for a three layered stationary wave. Recently Bronsard, Gui and Schatzman [6] proved the existence of such a wave in the case of a symmetric three–well potential.

Cahn and Novick–Cohen [12] and Novick–Cohen [24] studied an Allen–Cahn/Cahn–Hilliard system modelling simultaneous phase separation and ordering in binary alloys. Assuming a degenerate mobility, in [24] formal asymptotic expansions were applied to show that the singular limit of this system couples motion by mean curvature and quasi–static diffusion with surface diffusion. Triple junctions have also been derived recently by formal asymptotics in the context of a phase field model for solidification of a eutectic alloy in the stationary case by Wheeler, McFadden and Boettinger [32]. In parallel with our work, Bronsard, Garcke and Stoth [6] studied an asymptotic limit of a system of Cahn–Hilliard equations with a constant mobility matrix. They obtained an evolution law that couples Laplace’s equation in bulk

regions, the Gibbs–Thomson law on interfaces, Young’s law at triple junctions to a dynamic law of Stefan type. For the limiting evolution they demonstrated a conditional existence result using an implicit time discretization and methods from geometric measure theory.

In this paper we show that with the scaling $\gamma = \varepsilon^2$, $t \rightarrow \varepsilon^2 t$, the singular limit of (1) in the deep quench limit yields motion by minus the Laplacian of the mean curvature for each of the interfaces. At triple junctions Young’s law, a condition that follows from the continuity of chemical potentials and a no-flux condition hold. At the intersection of an interface with an external boundary a Neumann type angle condition and a no-flux condition are obtained. Furthermore, we derive a fourth order parabolic boundary value problem such that solutions of this initial boundary value problem parametrize curves which solve this limiting geometric evolution problem. We prove local in time existence for this initial boundary value problem. Taking into account equivalence classes of solutions obtained by reparametrization of the evolutionary curves and allowing for variability in fixing the tangential velocity uniqueness for the geometric problem is also proved.

2. FORMAL ASYMPTOTIC EXPANSIONS

In what follows we consider the scaling $t \rightarrow \varepsilon^2 t$ and $\gamma = \varepsilon^2$ (cf. Cahn, Elliott, Novick–Cohen [10]) and write (1) as

$$\varepsilon^3 u_t^i = -\varepsilon \nabla \cdot J^i, \quad (3)$$

$$J^i = -\sum_{j=1}^N L^{ij}(\mathbf{u}) \nabla w^j, \quad (4)$$

$$w^i = D_i \Psi(\mathbf{u}) - \frac{1}{N} (D \Psi(\mathbf{u}) \cdot \mathbf{e}) - \varepsilon^2 \Delta w^j, \quad (5)$$

where Ψ is chosen to be of regular solution type. We assume that $\mathbf{A} = (a^{ij})_{i,j=1,\dots,N}$ is positive definite when restricted to $\{\mathbf{u} \mid \sum_{i=1}^N u^i = 1\}$. The coefficients a^{ij} are pair interaction coefficients of components i and j and since it is reasonable to assume that a phase does not interact with itself, a^{ii} is taken here to be zero (cf. De Fontaine [13]). This implies that the minima of Ψ are of equal height in the limit $\theta \searrow 0$. The assumptions on \mathbf{A} give rise to a N -well structure for Ψ when $\theta > 0$ is sufficiently small. Moreover, each of the N phases can be identified by the property that one of the N components is predominant. Throughout our discussion, we assume that

$$\theta = \mathcal{O}(\varepsilon).$$

The derivation of the outer and inner solutions will only be sketched. For a more detailed analysis in the binary case we refer to Cahn, Elliott, and Novick–Cohen [10] and Novick–Cohen [24].

2.1. The outer expansion. Let us denote by $\mathbf{a}(k)$, $k = 1, \dots, N$, the N local minima of Ψ on the Gibbs simplex $\mathcal{G} = \{\mathbf{u} \mid \sum_{i=1}^N u^i = 1, u^i \geq 0\}$. This implies that the $\mathbf{a}(k)$

are solutions of the system of algebraic equations

$$D_i \Psi(\mathbf{u}) - \frac{1}{N} (D \Psi(\mathbf{u}) \cdot \mathbf{e}) = 0 \quad i = 1, \dots, N.$$

Since the $\mathbf{a}(k)$ converge at the corners of the Gibbs triangle exponentially fast as $\varepsilon \searrow 0$, we can label the $\mathbf{a}(k)$ so that

$$\begin{aligned} |a^k(k) - 1| &\leq C_1 e^{-\frac{C_2}{\varepsilon}}, \\ |a^i(k)| &\leq C_1 e^{-\frac{C_2}{\varepsilon}} \quad \text{for } i \neq k, \end{aligned}$$

where C_1, C_2 are constants which are independent of ε for ε sufficiently small and are therefore independent of θ .

Let us consider now a region of the outer solution which is dominated by component k ; i.e., $\mathbf{u} \approx \mathbf{a}(k)$. We make here the self-consistent ansatz that \mathbf{u} is of the form

$$\begin{aligned} u^i &= a^i(k) + (\varepsilon u_1^i + \varepsilon^2 u_2^i + \dots) e^{-\frac{C}{\varepsilon}} + H.O.T. && \text{for } i = 1, \dots, N, \\ J^i &= (J_0^i + \varepsilon J_1^i + \varepsilon^2 J_2^i + \dots) e^{-\frac{C}{\varepsilon}} + H.O.T. && \text{for } i = 1, \dots, N, \\ w^i &= (\varepsilon w_1^i + \varepsilon^2 w_2^i + \dots) + H.O.T. && \text{for } i = 1, \dots, N, \end{aligned}$$

where by H.O.T. we denote higher order terms.

2.2. The inner expansion. We now analyze the behaviour of the solution in an interfacial region separating two phases under the assumption that we are finitely away from both triple junctions and external boundaries. The two phases which bound this interfacial region are denoted here as k_1 and k_2 . We employ the "classical" variables (ρ, s) (cf. Caginalp and Fife [9] and Rubinstein, Sternberg, and Keller [27]) for the scaled distance function from the interface $\rho = d/\varepsilon$ where d is the signed distance from the interface and s is an arc-length variable along the interface. We use the convention that the unit normal defining the sign of ρ points into the domain denoted as k_2 . Any given function $v(x, t)$ written in terms of the new variables as $\underline{v}(t, \rho, s)$ satisfies

$$\nabla v = \frac{1}{\varepsilon} \frac{\partial}{\partial \rho} \underline{v} \nabla d + \frac{\partial}{\partial s} \underline{v} \nabla s, \quad (6)$$

$$v_t = \frac{\partial}{\partial t} \underline{v} - \frac{1}{\varepsilon} \frac{\partial}{\partial \rho} \underline{v} V + \frac{\partial}{\partial s} \underline{v} s_t. \quad (7)$$

Here $V = -d_t$ denotes the velocity of the interface. We now assume that there exist asymptotic expansions for $u^i, J^i, w^i, i = 1, \dots, N$, given in terms of these variables; i.e.,

$$\begin{aligned} \underline{u}^i &= \underline{u}_0^i + \varepsilon \underline{u}_1^i + \dots, && i = 1, \dots, N, \\ \underline{J}^i &= \underline{J}_0^i + \varepsilon \underline{J}_1^i + \dots, && i = 1, \dots, N, \\ \underline{w}^i &= \varepsilon \underline{w}_1^i + \varepsilon^2 \underline{w}_2^i + \dots, && i = 1, \dots, N \end{aligned}$$

such that $\sum_{i=1}^N \underline{u}_0^i = 1$ and $\sum_{i=1}^N \underline{u}_k^i = 0, k = 1, 2, \dots$.

At $\mathcal{O}(1)$, for $i = 1, \dots, N$

$$0 = -(n \cdot \underline{J}_0^i)_{,\rho}, \quad (8)$$

$$n \cdot \underline{J}_0^i = - \sum_{j=1}^N L^{ij}(\underline{\mathbf{u}}_0) \underline{w}_{1,\rho}^j, \quad (9)$$

$$0 = D_i \Psi(\underline{\mathbf{u}}_0) - \frac{1}{N} (D \Psi(\underline{\mathbf{u}}_0) \cdot \mathbf{e}) - \mathbf{u}_{0,\rho\rho}^i \quad (10)$$

where n denotes the unit normal to the interface and the notation $v_{,\rho}$ indicates the derivative of a function v with respect to the variable ρ . Integrating (8) yields the existence of a function g^i which is independent of ρ such that

$$n \cdot \underline{J}_0^i = g^i(s, t), \quad (11)$$

and matching of the normal components of the fluxes with the fluxes in the outer region shows that

$$g^i(s, t) = T.S.T., \quad (12)$$

where T.S.T. denote terms which are transcendentally small in ε (cf. [24]). Equation (10) with boundary conditions dictated by the outer expansions, has a stationary wave connecting the minima of Ψ as its solution (cf. Sternberg [29]). We shall assume that the stationary wave solution has only two non-trivial components which we denote by k_1, k_2 ; i.e.,

$$\underline{u}_0^i = T.S.T. \quad \text{for } i \neq k_1, k_2. \quad (13)$$

This implies that

$$L^{ij}(\underline{\mathbf{u}}_0) = T.S.T. \quad \text{for } i, j \notin \{k_1, k_2\}. \quad (14)$$

Neglecting transcendentally small terms

$$L^{k_1 k_2}(\underline{\mathbf{u}}_0) = -L^{k_1 k_1}(\underline{\mathbf{u}}_0) = -L^{k_2 k_2}(\underline{\mathbf{u}}_0) = L^{k_2 k_1}(\underline{\mathbf{u}}_0) \quad (15)$$

and these terms are different from zero. Thus there are effectively only four non-trivial entries in $L^{ij}(\underline{\mathbf{u}}_0)$, $i, j = 1, \dots, N$. From here on we employ the notation $L_0^{ij} := L^{ij}(\underline{\mathbf{u}}_0)$, $i, j = 1, \dots, N$.

From (9), (11), (14), and (15) it follows that there exists a function $\tilde{g}(s, t)$ which is transcendentally small, such that on the interface between k_1 and k_2 ,

$$\begin{aligned} n \cdot \underline{J}_0^{k_1} &= - [L_0^{k_1 k_2} \underline{w}_{1,\rho}^{k_2} + L_0^{k_1 k_1} \underline{w}_{1,\rho}^{k_1} + \tilde{g}(s, t)] = \\ &= -L_0^{k_1 k_2} [\underline{w}_{1,\rho}^{k_2} - \underline{w}_{1,\rho}^{k_1}] + \tilde{g}(s, t) = g^{k_1}(s, t). \end{aligned} \quad (16)$$

From (16), since $L_0^{k_1 k_2} \neq 0$ it follows that

$$[\underline{w}_1^{k_2} - \underline{w}_1^{k_1}]_{,\rho} = \frac{\tilde{g} - g^{k_1}}{L_0^{k_1 k_2}}. \quad (17)$$

By our assumptions on $L_0^{k_1 k_2}$ and on the normal fluxes in the outer regions,

$$\frac{\tilde{g} - g^{k_1}}{L_0^{k_1 k_2}} = \mathcal{O}(1).$$

Hence, matching the chemical potentials in the inner region with the chemical potentials in the outer region yields

$$\tilde{g} - g^{k_1} = \mathcal{O}\left(e^{-\frac{c}{\varepsilon}}\right) \quad (18)$$

and

$$\underline{w}_1^{k_1} - \underline{w}_1^{k_2} = h(s, t) \quad (19)$$

where $h(s, t)$ is not necessarily T.S.T..

At $\mathcal{O}(\varepsilon)$:

$$0 = (n \cdot \underline{J}_1^i)_{,\rho} + (\tau \cdot \underline{J}_0^i)_{,s}, \quad (20)$$

$$\underline{w}_1^i = \sum_{j=1}^N D_{ij} \Psi \cdot \underline{u}_1^j - \frac{1}{N} (\underline{\mathbf{u}}_1 \cdot D^2 \Psi \mathbf{e}) - \underline{u}_{1,\rho\rho}^i - \kappa_0^{k_1, k_2} \underline{u}_{0,\rho}^i, \quad (21)$$

where the sign convention is used that the curvature $\kappa_0^{k_1, k_2}$ is computed with respect to the normal pointing into the k_2 phase and $\kappa_0^{k_1, k_2}$ denotes the first term in a regular perturbation expansion in ε for κ^{k_1, k_2} . Since employing the Frenet formulas $\tau_s \cdot \underline{J}_0^i = \kappa^{k_1, k_2} n \cdot \underline{J}_0^i$, we conclude from (11) that $\tau_s \cdot \underline{J}_0^i = T.S.T.$ and therefore (20) simplifies to

$$0 = (n \cdot \underline{J}_1^i)_{,\rho} + (\tau \cdot \underline{J}_{0,s}^i) + T.S.T..$$

By matching with the outer solution, it follows that $n \cdot \underline{J}_1^i = T.S.T.$. Multiplying (21) by $\underline{u}_{0,\rho}^i$, integrating over $(-\infty, \infty)$ with respect to the variable ρ , and summing over i gives

$$\begin{aligned} \int_{-\infty}^{\infty} \sum_{i=1}^N \underline{w}_1^i \underline{u}_{0,\rho}^i &= \int_{-\infty}^{\infty} \sum_{i,j=1}^N D_{ij} \Psi \underline{u}_1^j \underline{u}_{0,\rho}^i - \int_{-\infty}^{\infty} \sum_{i=1}^N \frac{1}{N} (D^2 \Psi \underline{\mathbf{u}}_1 \cdot \mathbf{e}) \cdot \underline{u}_{0,\rho}^i \\ &\quad - \int_{-\infty}^{\infty} \sum_{i=1}^N \underline{u}_{1,\rho\rho}^i \underline{u}_{0,\rho}^i - \int_{-\infty}^{\infty} \sum_{i=1}^N \kappa_0^{k_1, k_2} (\underline{u}_{0,\rho}^i)^2. \end{aligned} \quad (22)$$

Integrating by parts and employing (10) yields

$$\int_{-\infty}^{\infty} \sum_{i=1}^N \underline{w}_1^i \underline{u}_{0,\rho}^i = -\kappa_0^{k_1, k_2} \int_{-\infty}^{\infty} \sum_{i=1}^N (\underline{u}_{0,\rho}^i)^2 d\rho + T.S.T.. \quad (23)$$

Now we define $\sigma^{k_1 k_2}$ to be the surface energy of the interface between the k_1 phase and the k_2 phase; i.e.,

$$\sigma^{k_1 k_2} := \int_{-\infty}^{\infty} \sum_{i=1}^N (\underline{u}_{0,\rho}^i)^2 d\rho$$

where \underline{u}_0^i is the connecting orbit between the minima of Ψ corresponding to the k_1 and k_2 phases (cf. Sternberg [29]). From the above discussion, $\underline{u}_0^i = T.S.T.$ for $i \neq k_1, k_2$,

hence:

$$\sigma^{k_1 k_2} := \int_{-\infty}^{\infty} \left\{ (\underline{u}_{0,\rho}^{k_1})^2 + (\underline{u}_{0,\rho}^{k_2})^2 \right\} d\rho + T.S.T.. \quad (24)$$

Since the k_1 and k_2 phases are assumed to be distinct, $\sigma^{k_1 k_2} > 0$. By construction $\sum_{i=1}^N u_0^i = 1$. This implies now that

$$u_0^{k_1} + u_0^{k_2} = 1 + T.S.T.. \quad (25)$$

Employing (24) and (25) in (23) yields

$$(\underline{w}_1^{k_1} - \underline{w}_1^{k_2}) \int_{-\infty}^{\infty} \underline{u}_{0,\rho}^{k_1} d\rho + T.S.T. = -\kappa_0^{k_1, k_2} \sigma^{k_1 k_2}, \quad (26)$$

and this implies that to lowest order

$$(\underline{w}_1^{k_1} - \underline{w}_1^{k_2}) [u_0^{k_1}]_{k_1}^{k_2} = -\kappa_0^{k_1, k_2} \sigma^{k_1 k_2}. \quad (27)$$

Here $[v]_{k_1}^{k_2}$ denotes the jump of a function v across the interface between the k_2 and k_1 phases with the sign convention that we subtract the value in the k_1 phase from the value in the k_2 phase.

At $\mathcal{O}(\varepsilon^2)$, for $i = 1, \dots, N$,

$$-V_0 \underline{u}_{0,\rho}^i = - (n \cdot \underline{J}_2)_\rho - (\tau \cdot \underline{J}_1)_s. \quad (28)$$

At this level, the equation of conservation of mass alone is needed. Integrating (28) between $-\infty$ and ∞ , using the fact that $\tau_{,s} \cdot \underline{J}_1^i$ is T.S.T., and matching with the normal fluxes in the outer region

$$-V_0 [u_0^i]_{k_1}^{k_2} = - \int_{-\infty}^{\infty} (\tau \cdot \underline{J}_{1,s}^i) + T.S.T.. \quad (29)$$

Taking $i = k_1$ (or k_2), to lowest order

$$-V_0 [u_0^{k_1}]_{k_1}^{k_2} = \int_{-\infty}^{\infty} L_0^{k_1 k_2} \underline{w}_{1,ss}^{k_2} d\rho + \int_{-\infty}^{\infty} L_0^{k_1 k_1} \underline{w}_{1,ss}^{k_1} d\rho + T.S.T.,$$

and thus

$$V_0 [u_0^{k_1}]_{k_1}^{k_2} = (\underline{w}_{1,ss}^{k_1} - \underline{w}_{1,ss}^{k_2}) \int_{-\infty}^{\infty} L_0^{k_1 k_2} d\rho. \quad (30)$$

Defining

$$l^{k_1 k_2} := \int_{-\infty}^{\infty} L_0^{k_1 k_2} d\rho \quad (31)$$

which is a positive constant, we can rewrite (30) as

$$V_0 = \frac{l^{k_1 k_2}}{[u_0^{k_1}]_{k_1}^{k_2}} (\underline{w}_{1,ss}^{k_1} - \underline{w}_{1,ss}^{k_2}). \quad (32)$$

Employing (27) in (32) and using the fact that $[u_0^{k_1}]_{k_1}^{k_2} = -1$ yields

$$V^{k_1 k_2} = -l^{k_1 k_2} \sigma^{k_1 k_2} \kappa_{,ss}^{k_1 k_2}. \quad (33)$$

Here, the upper indices in $V^{k_1 k_2}$ and $\kappa^{k_1 k_2}$ indicate that the velocity and curvature are evaluated at the interface between k_1 and k_2 . Additionally, the subscript 0 indicating the order of the expansion has been omitted.

2.3. Expansion near the intersection with the external boundary. Let the interface between the k_1 phase and the k_2 phase be denoted by $\Gamma^{k_1 k_2}$. At the intersections of $\Gamma^{k_1 k_2}$ with the external boundary three conditions must hold.

(i) **Attachment:** An isolated interface $\Gamma^{k_1 k_2}$ does not detach from the boundary $\partial\Omega$. The rationale for the attachment constraint arises from the physical interpretation of the partitioning of the phases.

(ii) **A Neumann boundary condition:** Let $m(t)$ denote the point of intersection of the interface $\Gamma^{k_1 k_2}$ with $\partial\Omega$. Following the arguments of Owen, Rubinstein, and Sternberg [25], Cahn and Novick–Cohen [12] and Novick–Cohen [24] we introduce a rectangle R_ε whose sides are proportional to $\varepsilon^{\frac{1}{2}}$. Let $m(t)$ be located at the midpoint of one of the sides of the rectangle and let that side be tangent to $\partial\Omega$ at $m(t)$. To obtain the Neumann boundary condition, we introduce the variables $\eta = \frac{x-m(t)}{\varepsilon} = (\tau, n)$ where (τ, n) are orthogonal coordinates and τ is a variable which is tangent to $\partial\Omega$ at $m(t)$. We now multiply the equations for the modified chemical potentials by $u_{,\tau}^i$, integrate over the rectangle R_ε , and sum over the components $i = 1, \dots, N$. This gives to leading order

$$\begin{aligned} 0 &= \int_{R_\varepsilon} \sum_{i=1}^N u_{0,\tau}^i \left\{ D_i \Psi(\mathbf{u}_0) - \frac{1}{N} ((D\Psi(\mathbf{u}_0)) \cdot \mathbf{e}) - \Delta_\eta u_0^i \right\} \\ &= \int_{R_\varepsilon} \left\{ \frac{\partial}{\partial \tau} (\Psi(\mathbf{u}_0)) - \frac{\partial}{\partial \tau} \left(\sum_{i=1}^N u_0^i \right) \left(\frac{1}{N} (D\Psi(\mathbf{u}_0)) \cdot \mathbf{e} \right) - \sum_{i=1}^N u_{0,\tau}^i \cdot \Delta_\eta u_0^i \right\}. \end{aligned}$$

Since $\sum_{i=1}^N u_0^i = 1$, the middle term in the above equation term vanishes. Hence

$$0 = \int_{R_\varepsilon} \left\{ \frac{\partial}{\partial \tau} (\Psi(\mathbf{u}_0)) - \sum_{i=1}^N u_{0,\tau}^i \Delta_\eta u_0^i \right\}.$$

This identical form appeared also in [24] and the same arguments may now be employed to deduce that the contact angle with the external boundary is $\frac{\pi}{2}$ up to $\mathcal{O}(\varepsilon)$ accuracy.

(iii) **The no flux condition:** Integration of the balance of mass equation for component k_1 over the rectangle R_ε gives

$$\int_{R_\varepsilon} \varepsilon^3 u_t^{k_1} = -\varepsilon \int_{R_\varepsilon} \nabla \cdot J^{k_1}.$$

It is easy to see that

$$I := \varepsilon^3 \int_{R_\varepsilon} u_t^{k_1} = \mathcal{O}(\varepsilon^2 \text{vol}(R_\varepsilon)) = \mathcal{O}(\varepsilon^3).$$

Defining

$$II := \varepsilon \int_{R_\varepsilon} \nabla \cdot J^{k_1},$$

we can argue as in Novick–Cohen [24] to obtain

$$II = \mathcal{O}(\varepsilon \operatorname{vol}(R_\varepsilon)) = \mathcal{O}(\varepsilon^2).$$

In particular $I = o(II)$, and to leading order from the balance of flux condition, Gauß' identity, and matching with the outer solution

$$\begin{aligned} 0 = II &= \int_{\partial R_\varepsilon} \nu \cdot J^{k_1} = \\ &= -\varepsilon \int_{-\infty}^{\infty} \nu \cdot (L^{k_1 k_2} \{w_{,s}^{k_2} - w_{,s}^{k_1}\}) e_n d\tau + \mathcal{O}(\varepsilon^2) = \\ &= -\varepsilon (\nu \cdot e_n) (w_{,s}^{k_2} - w_{,s}^{k_1}) \int_{-\infty}^{\infty} L^{k_1 k_2} d\rho + \mathcal{O}(\varepsilon^2), \end{aligned}$$

where ν denotes the exterior normal to $\partial\Omega$ and e_n denotes the unit vector pointing in direction of the coordinate n . Here we have used the fact that across the interface $\Gamma^{k_1 k_2}$ only quantities involving k_1 and k_2 give contributions to leading order. Since $\int_{-\infty}^{\infty} L^{k_1 k_2} d\rho \neq 0$ and since the contact angle is $\frac{\pi}{2}$ (which implies that $\nu \cdot e_n \neq 0$), we conclude that

$$w_{,s}^{k_2} - w_{,s}^{k_1} = 0.$$

Matching with the inner solution and noting (27) then gives

$$\kappa_{0,s}^{k_1, k_2} = 0.$$

2.4. Conditions at the triple junction. Four different types of conditions must hold at any triple junction. The first condition is a persistence condition; i.e., the condition three interfaces which meet at an isolated triple junction remain together. The second condition is Young's law; i.e., a condition which determines the angles between the interfaces which meet at the triple junction which results from a balance of mechanical forces. The remaining two conditions consist of a balance of flux condition and a continuity condition resulting from imposing continuity of the chemical potentials across the triple junction. Since the derivation of the first two conditions is similar to the analysis which appears in the treatment of other problems which include triple junctions we do not present the details here. Instead we refer to the works of Bronsard and Reitich [8] and Novick–Cohen [24].

Without loss of generality, it is assumed that phases 1, 2 and 3 meet at the triple junction under consideration. Let $i, j, k \in \{1, 2, 3\}$ be mutually different. Let the interface between i and j be denoted by Γ^k . The same convention is employed for σ^k and l^k .

- (i) Persistence: A single isolated triple junction does not pull apart.
- (ii) Young's law: Following the arguments in Bronsard and Reitich [8] and Novick–Cohen [24] and our discussion of the Neumann condition, it is easy to demonstrate

that Young’s law holds at any triple junction (see also Bronsard, Gui and Schatzman [7]). This condition may be written as:

$$\frac{\sigma^1}{\sin \theta^1} = \frac{\sigma^2}{\sin \theta^2} = \frac{\sigma^3}{\sin \theta^3}$$

where θ^k represents the angle of the k phase at the triple junction and $\sigma^k := \sigma^{ij}$ where σ^{ij} is as defined in subsection 2.2.

(iii) A balance of flux condition: To proceed, let us integrate the mass balance equation over a triangle T_ε which is constructed so that its midpoint coincides with the triple junction $m(t)$, its sides are proportional to $\varepsilon^{\frac{1}{2}}$, and each of its sides is intersected by one of the three interfaces. Employing the stretched variable

$$\eta = \frac{x - m(t)}{\varepsilon},$$

and defining

$$I^i := \int_{T_\varepsilon} \varepsilon^3 u_t^i$$

and

$$II^i := \int_{T_\varepsilon} \nabla_\eta \cdot J^i,$$

it is easy to check that $I^i = \mathcal{O}(\varepsilon^3)$ and $II^i = \mathcal{O}(\varepsilon)$. More precisely, we may conclude that $I^i = o(II^i)$ and may therefore be neglected. Thus, using Gauß’ theorem, matching with the outer and inner solutions, and employing the structure of the inner solutions (cf. [12] and [24]), we obtain to leading order:

$$\begin{aligned} 0 &= \int_{T_\varepsilon} \nabla_\eta \cdot J^i = \varepsilon \int_{\partial T_\varepsilon} \nu \cdot J^i ds \\ &= - \sum_{k=1}^3 \varepsilon \int_{(\partial T_\varepsilon)_k} \nu_k \left\{ \sum_{j=1}^N L^{ij} w_{,s}^j \right\} \cdot e_{\zeta_k} ds, \end{aligned} \quad (34)$$

where ds refers to integration with respect to arc-length. Here, $(\partial T_\varepsilon)_k$, $k = 1, 2, 3$, refer to the edge of the triangle T_ε which intersects the k ’th interface, ν_k denotes the exterior normal to T_ε on $(\partial T_\varepsilon)_k$, and e_{ζ_k} is the unit tangent along the k ’th interface pointing away from the triple junction. By matching with the appropriate inner solutions, we see that along each of the 3 interfaces only two of the N chemical potentials vary to leading order. Hence, in total, at any given triple-junction only three chemical potentials denoted here as w^1 , w^2 and w^3 , enter significantly.

Let us choose three mutually distinct numbers $p, q, r \in \{1, 2, 3\}$. Assuming that we are in the non-degenerate case in which $0 < \theta_i < \pi$, $i = 1, 2, 3$, a triangle T_ε may be constructed such that $\nu_p \cdot e_{\zeta_p} = 0$. Since the component p varies only along interfaces q and r , we obtain from (34) taking $i = p$ that to lowest order

$$0 = \varepsilon \int_{(\partial T_\varepsilon)_q} \{(\nu_q \cdot e_{\zeta_q}) L^{pr} (w_{,s}^r - w_{,s}^p)\} + \varepsilon \int_{(\partial T_\varepsilon)_r} \{(\nu_r \cdot e_{\zeta_r}) L^{pq} (w_{,s}^q - w_{,s}^p)\} . \quad (35)$$

Evaluating the integrals in (35), we conclude from (31) that at the triple junction

$$l^{pr} (w_{,s}^r - w_{,s}^p)_q + l^{pq} (w_s^q - w_s^p)_r = 0. \quad (36)$$

Here the notation $(\cdot)_i$ indicates that the expression in brackets is evaluated at the interface i . Matching with the inner solutions and employing equation (27), gives that the above equation may be written as:

$$l^q \sigma^q \kappa_{,s}^{rp} + l^r \sigma^r \kappa_{,s}^{qp} = 0.$$

Defining $\kappa^1 := \kappa^{23}$, $\kappa^2 := \kappa^{31}$ and $\kappa^3 := \kappa^{12}$, and noting that $\kappa^{ij} = -\kappa^{ji}$, we conclude that

$$l^1 \sigma^1 \kappa_{,s}^1 = l^2 \sigma^2 \kappa_{,s}^2 = l^3 \sigma^3 \kappa_{,s}^3.$$

(iv) Continuity of the chemical potentials: It follows from (27), using the notation from above that

$$\begin{aligned} (w^2 - w^3)_1 &= \kappa^1 \sigma^1, \\ (w^3 - w^1)_2 &= \kappa^2 \sigma^2, \\ (w^1 - w^2)_3 &= \kappa^3 \sigma^3. \end{aligned}$$

Under the assumption that each of the three chemical potentials are continuous across the triple junction, it follows that

$$(w^1)_2 = (w^1)_3, \quad (w^2)_1 = (w^2)_3, \quad (w^3)_1 = (w^3)_2,$$

and we may sum the above equations to obtain:

$$\kappa^1 \sigma^1 + \kappa^2 \sigma^2 + \kappa^3 \sigma^3 = 0. \quad (37)$$

3. THE LIMITING MOTION AND GEOMETRIC PROPERTIES OF THE EVOLUTION

Let us formulate the limiting motion for a three phase system. Generalization to more phases may be obtained in a straightforward manner. The interfaces shall be denoted by $\Gamma^i(t)$, $i = 1, 2, 3$. The quantities σ^i, l^i and κ^i will be defined as in subsection 2.4. Furthermore, we set

$$V^1 := V^{23}, \quad V^2 := V^{31}, \quad V^3 := V^{12}.$$

We remark that the velocities V^{ij} are antisymmetric in i and j ; i.e., $V^{ij} = -V^{ji}$.

The free boundary problem for the interfaces may now be stated as

(A) Along each interface Γ^i , $V^i = -l^i \sigma^i \kappa_{,ss}^i$,

(B) At each intersection of Γ^i with the external boundary, the following conditions hold

- (i) an attachment condition,
- (ii) the contact angle is $\frac{\pi}{2}$,
- (iii) a no-flux condition $\kappa_{,s} = 0$.

(C) At each triple junction

- (i) the triple junction does not pull apart,

(ii) the angles of the phases at the triple junction fulfill Young’s law [33]; i.e.:

$$\frac{\sigma^1}{\sin \theta^1} = \frac{\sigma^2}{\sin \theta^2} = \frac{\sigma^3}{\sin \theta^3}, \quad (38)$$

which may also be written as a balance of mechanical forces (cf. [21])

$$\sum_{i=1}^3 \sigma^i \tau^i = 0 \quad (39)$$

where τ^i denotes a unit tangent to the interface Γ^i pointing away from the triple junction.

(iii) a balance of fluxes condition holds:

$$l^1 \sigma^1 \kappa_{,s}^1 = l^2 \sigma^2 \kappa_{,s}^2 = l^3 \sigma^3 \kappa_{,s}^3.$$

(iv) the chemical potentials are continuous across a triple junction; this implies that

$$\kappa^1 \sigma^1 + \kappa^2 \sigma^2 + \kappa^3 \sigma^3 = 0. \quad (40)$$

Let us now state some geometric properties. It is easy to demonstrate that the total area occupied by each of the different phases is preserved. We leave the simple proof to the reader.

A further property which we wish to show is that the energy

$$E(t) = \sum_{i=1}^3 \sigma^i L^i(t)$$

is non-increasing. Here by $L^i(t)$ we denote the length of $\Gamma^i(t)$. In this context we remark that Baldo ([2]) has been proven that the weighted sum of lengths E is the Γ -limit as $\varepsilon \rightarrow 0$ of the free energy \mathcal{E} . For simplicity we state the property for three curves which each intersect the external boundary at one endpoint and which all meet at a single triple junction at their other endpoint. A generalization to more complex networks of interfaces is straightforward.

Claim: The total energy is a non-increasing function of time; moreover,

$$\frac{d}{dt} E(t) = - \sum_{i=1}^3 (\sigma^i)^2 l^i \int_{\Gamma^i(t)} (\kappa_{,s}^i)^2 ds,$$

where ds denotes integration with respect to arc-length.

Proof. By the transport theorem for evolving curves (see for example Gurtin [21] Section I.2):

$$\frac{d}{dt} L^i(t) = - \int_{\Gamma^i(t)} \kappa^i V^i ds - \int_{\partial \Gamma^i(t)} \tau^i \cdot \dot{R}^i \quad (41)$$

where τ^i denotes the unit tangent to $\Gamma^i(t)$ pointing into the interfaces at their endpoints and \dot{R}^i is the velocity of the endpoint. For an endpoint of $\partial \Gamma^i(t)$ lying on the exterior boundary $\partial \Omega$, we conclude from the angle condition that $\tau^i \cdot \dot{R}^i = 0$.

Therefore, only triple points can contribute to the boundary integral. Summing (41) over the three interfaces each weighted with the appropriate interfacial energy, and employing the law (A) of interfacial motion yields

$$\frac{d}{dt}E(t) = \sum_{i=1}^3 \sigma^i \int_{\Gamma^i(t)} \kappa^i l^i \sigma^i \kappa_{,ss}^i ds - \sum_{i=1}^3 \sigma^i \tau^i \cdot \dot{m}(t). \quad (42)$$

Here, $m(t)$ denotes the location of the triple junction. By (39) the last term in (42) vanishes. Integration by parts and using the no-flux condition $\kappa_{,s}^i = 0$ at the external boundary gives

$$\frac{d}{dt}E(t) = - \sum_{i=1}^3 (\sigma^i)^2 l^i \int_{\Gamma^i(t)} (\kappa_{,s}^i)^2 ds + \sum_{i=1}^3 (\sigma^i)^2 l^i (\kappa^i \kappa_{,s}^i)_{m(t)} \quad (43)$$

where the notation $(\cdot)_{m(t)}$ indicates that the term in the bracket has been evaluated at the point $m(t)$. Employing (iii) and (iv) in (C), the last term in (43) can be seen to vanish which proves the claim.

4. A LOCAL EXISTENCE THEOREM FOR THE LIMITING EVOLUTION PROBLEM

In what follows we restrict ourselves to a situation in which three phases are present which meet at exactly one triple junction. The study of more complex geometries can be treated in a similar manner. We describe the three interfaces Γ^i , $i = 1, 2, 3$, by functions

$$\begin{aligned} X^i &: [0, T] \times [0, 1] \rightarrow \mathbb{R}^2, \\ (t, p) &\mapsto X^i(t, p) \end{aligned}$$

such that for $t \in [0, T]$, $X^i(t, \cdot)$ is a parametrization of $\Gamma^i(t)$ whose properties will be specified in the discussion below, and we formulate an initial boundary value problem in terms of the parametrizations $(X^i)_{i=1,2,3}$ whose solution will parametrize curves that solve the evolution problem (A)–(C) derived in section 3. We then prove local existence within the framework of the parabolic Hölder spaces $C_{t,p}^{1+\frac{\alpha}{4}, 4+\alpha}$, $0 < \alpha < 1$ relying on the results of Solonnikov [28].

First we derive a partial differential equation for X^i such that the evolutionary curves parametrized by X^i fulfill the law of motion

$$V^i = -l^i \sigma^i \kappa_{,ss}^i,$$

which may be written equivalently as

$$X_t^i \cdot N^i = -l^i \sigma^i \kappa_{,ss}^i.$$

Here, N^i is the unit normal to $\Gamma^i(t)$. We use the convention that the sign of N^i is chosen so that (T^i, N^i) is positively oriented, where $T^i = \frac{X_{,p}^i}{|X_{,p}^i|}$ is a unit tangent. The Frenet formulas imply that

$$X_{,ss}^i = \kappa^i N^i,$$

where the index s indicates as before differentiation with respect to arc-length; i.e., for a given function $f(p)$ defined on the curve, $f_{,s} = \frac{f_{,p}}{|X_{,p}|}$.

Setting

$$\alpha^i := T^i \cdot \frac{X_{,pp}^i}{|X_{,p}^i|^2}$$

and

$$\beta^i := -3\kappa^i \kappa_{,s}^i + \alpha_{,ss}^i - \alpha^i (\kappa^i)^2, \quad (44)$$

we obtain, omitting the index i for convenience

$$\begin{aligned} \kappa_{,ss} N + \beta T &= \frac{X_{,pppp}}{|X_{,p}|^4} - 5\alpha \frac{X_{,ppp}}{|X_{,p}|^3} - 2\alpha_{,s} \kappa N - \alpha \kappa_{,s} N - 2\alpha_{,s} \frac{X_{,pp}}{|X_{,p}|^2} + \\ &+ \kappa^3 N - 2\alpha^2 \frac{X_{,pp}}{|X_{,p}|^2}. \end{aligned}$$

The last equality implies that a solution of the partial differential equation

$$\begin{aligned} \frac{1}{\sigma^i l^i} X_t^i &= -\frac{X_{,pppp}^i}{|X_{,p}^i|^4} + 5\alpha^i \frac{X_{,ppp}^i}{|X_{,p}^i|^3} + 2\alpha_{,s}^i \kappa^i N^i + \alpha^i \kappa_{,s}^i N^i + \\ &+ 2\alpha_{,s}^i \frac{X_{,pp}^i}{|X_{,p}^i|^2} - (\kappa^i)^3 N^i + 2(\alpha^i)^2 \frac{X_{,pp}^i}{|X_{,p}^i|^2} \end{aligned} \quad (45)$$

fulfills

$$V^i = -\sigma^i l^i \kappa_{,ss}^i,$$

since (45) may also be written as

$$\frac{1}{\sigma^i l^i} X_t^i = -\kappa_{,ss}^i N^i - \beta^i T^i. \quad (46)$$

To keep the presentation simple, we define a \mathbb{R}^2 -valued function G such that (45) is equivalent to

$$X_t^i = -\sigma^i l^i \frac{X_{,pppp}^i}{|X_{,p}^i|^4} + G(X_{,p}^i, X_{,pp}^i, X_{,ppp}^i) \quad i = 1, 2, 3. \quad (\text{PDE})$$

We remark that G is smooth as long as its first variable is bounded away from zero.

There is some degree of freedom in specifying the parametrization of the evolutionary curves (see the next section), and we find it convenient to make the following choice at the endpoints $p = 0, 1$

$$|X_{,p}^i(t, 0)| = |X_{,p}^i(t, 1)| = 1, \quad i = 1, 2, 3.$$

This choice yields the following formulation of the boundary condition at the triple junction (cf. (C) in Section 3):

$$X^1(t, 0) = X^2(t, 0) = X^3(t, 0), \quad (\text{I})$$

$$|X_{,p}^i(t, 0)|^2 = 1, \quad i = 1, 2, 3, \quad (\text{II})$$

$$\sum_{i=1}^3 \sigma^i X_{,p}^i(t, 0) = 0, \quad (\text{III})$$

$$\sum_{i=1}^3 \sigma^i X_{,pp}^i(t, 0) \cdot N^i(t, 0) = 0, \quad (\text{IV})$$

$$\sigma^1 l^1 \kappa_{,s}^1 = \sigma^2 l^2 \kappa_{,s}^2 = \sigma^3 l^3 \kappa_{,s}^3. \quad (\text{V})$$

Assuming that $\partial\Omega$ is described as the 0-level set of a smooth function $b : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that 0 is a regular value, we obtain as boundary conditions at the external boundary that for $i = 1, 2, 3$ (cf. (B) in Section 3)

$$b(X^i(t, 1)) = 0, \quad (\text{VI})$$

$$X_{1,p}^i b_{,x_2}(X^i)(t, 1) - X_{2,p}^i b_{,x_1}(X^i)(t, 1) = 0, \quad (\text{VII})$$

$$\kappa_{,s}^i(t, 1) = 0, \quad (\text{VIII})$$

$$|X_{,p}^i(t, 1)|^2 = 1. \quad (\text{IX})$$

Altogether this yields 12 boundary conditions at $p = 0$ and $p = 1$ for a system of fourth order parabolic equation which contains six equations. This is precisely the number of conditions which one would expect for this type of system.

To show local in time existence we follow the approach of Bronsard and Reitich [8] where a second order system was studied arising in the context of motion by mean curvature. But since our system is of fourth order the analysis here is more complicated. In order to solve the system of equations (PDE) together with the boundary conditions (I)–(IX) and given initial conditions, we choose parametrizations $X_0^i \in C_p^{4+\alpha}([0, 1], \mathbb{R}^2)$, $i = 1, 2, 3$, of the initial curve which satisfy

$$|X_{0,p}^i(0)| = |X_{0,p}^i(1)| = 1 \quad \text{for } i = 1, 2, 3. \quad (47)$$

The existence proof will be carried out by a contraction argument on the set

$$\begin{aligned} \Xi(T, M) = \{ & (X^1, X^2, X^3) \in C_{t,p}^{1+\frac{\alpha}{4}, 4+\alpha}([0, T] \times [0, 1], \mathbb{R}^2)^3 \mid \\ & X^i(0, \cdot) = X_0^i \text{ and } \|X^i\|_{1+\frac{\alpha}{4}, 4+\alpha} \leq M \text{ for } i = 1, 2, 3 \}. \end{aligned}$$

The numbers T and M will be chosen later. Here $C_{t,p}^{1+\frac{\alpha}{4}, 4+\alpha}([0, T] \times [0, 1], \mathbb{R}^2)$ is defined as a parabolic Hölder space (cf. Solonnikov [28], Chapter IV) with the norm $\|\cdot\|_{1+\frac{\alpha}{4}, 4+\alpha}$, and the space $\Xi(T, M)$ is taken to inherit this norm.

Assume $(\bar{X}^1, \bar{X}^2, \bar{X}^3) \in \Xi(T, M)$ is given. Then linearizing equation (PDE) gives

$$X_t^i + \sigma^i l^i \frac{X_{,pppp}^i}{|X_{0,p}^i|^4} = \sigma^i l^i \frac{\bar{X}_{,pppp}^i}{|\bar{X}_{0,p}^i|^4} - \sigma^i l^i \frac{\bar{X}_{,pppp}^i}{|\bar{X}_p^i|^4} + G(\bar{X}_{,p}^i, \bar{X}_{,pp}^i, \bar{X}_{,ppp}^i). \quad (\text{PDE}_L)$$

This equation is to be solved in conjunction with linearized versions of the boundary conditions (I)–(IX) which we shall denote by (I_L) – (IX_L) . The conditions (I) and (III) are already linear. The linearized boundary conditions (II_L) and (IV_L) are

$$X_{0,p}^i(0)X_{,p}^i(t,0) = X_{0,p}^i(0)\bar{X}_{,p}^i(t,0) - \frac{1}{2}(1 - |\bar{X}_{,p}^i(t,0)|^2), \quad (II_L)$$

$$\sum_{i=1}^3 \sigma^i X_{,pp}^i(t,0)N_0^i(0) = \sum_{i=1}^3 \sigma^i \bar{X}_{,pp}^i(t,0)N_0^i(0) - \sum_{i=1}^3 \sigma^i \bar{X}_{,pp}^i(t,0)\bar{N}^i(t,0) \quad (IV_L)$$

where N_0^i is the unit normal to the parametrization of the initial curve X_0^i and \bar{N}^i is the unit normal to \bar{X}^i .

Using the identity

$$\kappa_{,s}^i = \frac{X_{,ppp}^i}{|X_{,p}^i|^3} \cdot N^i - 3\alpha^i \kappa^i,$$

we can linearize the boundary condition (V) as

$$\begin{aligned} \sigma^i l^i X_{,ppp}^i(t,0)N_0^i(0) - \sigma^j l^j X_{,ppp}^j(t,0)N_0^j(0) &= \sigma^i l^i \bar{X}_{,ppp}^i(t,0)N_0^i(0) - \sigma^j l^j \bar{X}_{,ppp}^j(t,0)N_0^j(0) \\ &\quad - \sigma^i l^i \bar{\kappa}_{,s}^i(t,0) + \sigma^j l^j \bar{\kappa}_{,s}^j(t,0) \end{aligned} \quad (V_L)$$

for $(i, j) = (1, 2)$ and $(i, j) = (2, 3)$. The boundary conditions at the point $p = 1$ can be linearized similarly to give linear conditions (VI_L) – (IX_L) .

We now demonstrate that the boundary conditions (I_L) – (IX_L) together with equation (PDE_L) fulfill the complementary conditions in the sense of Solonnikov (cf. [28]). This will allow us to apply the C^α –theory for linear parabolic systems. Introducing the notation

$$(u_1, \dots, u_6) := (X_1^1, X_2^1, X_1^2, X_2^2, X_1^3, X_2^3),$$

the boundary conditions (I_L) – (IX_L) may be written in the form

$$\sum_{j=1}^6 B_{qj} \left(p, \frac{\partial}{\partial p} \right) u_j(t, p) = \Phi_q(t, p) \quad \text{for } q = 1, \dots, 12, \quad \text{and } p = 0, 1,$$

(cf. Solonnikov [28], Bronsard and Reitich [8]). Here, the Φ_q are defined to be the right hand sides of equations (I_L) – (IX_L) . The B_{qj} ’s are polynomials with respect to $\frac{\partial}{\partial p}$, and here each B_{qj} is homogeneous of a degree that is denoted here by δ_q^0 which does not depend on j . The index 0 indicates that this is the degree at the point $p = 0$. We use the notation δ_q^1 to indicate the degree of the polynomial B_{qj} at the point $p = 1$. Therefore, using the notation of Solonnikov [28], we obtain that $B_{qj} = B_{qj}^0$ where B_{qj}^0 is the principal part of B_{qj} . Below we discuss only the boundary condition at the point $p = 0$, as the analysis for $p = 1$ is similar but simpler. Defining

$$\begin{aligned} (\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Z}_1, \mathcal{Z}_2) := \\ (X_{1,p}^1(0,0), X_{2,p}^1(0,0), X_{1,p}^2(0,0), X_{2,p}^2(0,0), X_{1,p}^3(0,0), X_{2,p}^3(0,0)), \end{aligned}$$

we get

$$\begin{aligned}
(B_{qj}^0(0, \mathbf{i}\tau))_{j=1, \dots, 6} &= (1, 0, -1, 0, 0, 0) && \text{for } q = 1, \\
&= (0, 1, 0, -1, 0, 0) && \text{for } q = 2, \\
&= (0, 0, 1, 0, -1, 0) && \text{for } q = 3, \\
&= (0, 0, 0, 1, 0, -1) && \text{for } q = 4, \\
&= (\mathcal{X}_1 \mathbf{i}\tau, \mathcal{X}_2 \mathbf{i}\tau, 0, 0, 0, 0) && \text{for } q = 5, \\
&= (0, 0, \mathcal{Y}_1 \mathbf{i}\tau, \mathcal{Y}_2 \mathbf{i}\tau, 0, 0) && \text{for } q = 6, \\
&= (0, 0, 0, 0, \mathcal{Z}_1 \mathbf{i}\tau, \mathcal{Z}_2 \mathbf{i}\tau) && \text{for } q = 7, \\
&= (\sigma^1 \mathbf{i}\tau, 0, \sigma^2 \mathbf{i}\tau, 0, \sigma^3 \mathbf{i}\tau, 0) && \text{for } q = 8, \\
&= (0, \sigma^1 \mathbf{i}\tau, 0, \sigma^2 \mathbf{i}\tau, 0, \sigma^3 \mathbf{i}\tau) && \text{for } q = 9, \\
&= (\sigma^1 \mathcal{X}_2 \tau^2, -\sigma^1 \mathcal{X}_1 \tau^2, \sigma^2 \mathcal{Y}_2 \tau^2, -\sigma^2 \mathcal{Y}_1 \tau^2, \sigma^3 \mathcal{Z}_2 \tau^2, -\sigma^3 \mathcal{Z}_1 \tau^2) && \text{for } q = 10, \\
&= (\sigma^1 l^1 \mathcal{X}_2 \mathbf{i}\tau^3, -\sigma^1 l^1 \mathcal{X}_1 \mathbf{i}\tau^3, -\sigma^2 l^2 \mathcal{Y}_2 \mathbf{i}\tau^3, \sigma^2 l^2 \mathcal{Y}_1 \mathbf{i}\tau^3, 0, 0) && \text{for } q = 11, \\
&= (0, 0, \sigma^2 l^2 \mathcal{Y}_2 \mathbf{i}\tau^3, -\sigma^2 l^2 \mathcal{Y}_1 \mathbf{i}\tau^3, -\sigma^3 l^3 \mathcal{Z}_2 \mathbf{i}\tau^3, \sigma^3 l^3 \mathcal{Z}_1 \mathbf{i}\tau^3) && \text{for } q = 12
\end{aligned}$$

where $\mathbf{i} = \sqrt{-1}$. The principal part of the equation (PDE_L) is described by the matrix

$$\mathcal{L}^0(t, p, \frac{\partial}{\partial t}, \frac{\partial}{\partial p}) = (l_{kj}^0)_{k,j=1, \dots, 6}$$

where

$$\begin{aligned}
l_{kj}^0 &= 0 && \text{if } k \neq j, \\
l_{kk}^0 &= \left(\sigma^i l^i \frac{1}{|X_{0,p}^i|^4} \left(\frac{\partial}{\partial p} \right)^4 + \frac{\partial}{\partial t} \right) && \text{if } i = \left[\frac{k+1}{2} \right],
\end{aligned}$$

where $[\cdot]$ is the Gauß bracket and indicates the integer part. To formulate the complementary condition we define

$$\begin{aligned}
\mathbf{L} &:= \det \mathcal{L}^0(t, p, r, \mathbf{i}\tau) = \prod_{j=1}^3 \left(\sigma^j l^j \frac{1}{|X_{0,p}^j|^4} \tau^4 + r \right)^2, \\
\hat{\mathcal{L}}^0 &= (\hat{l}_{kj}^0) := \mathbf{L} (\mathcal{L}^0)^{-1};
\end{aligned}$$

i.e., for $p = 0, 1$ we get

$$\begin{aligned}
\hat{l}_{kj}^0 &= 0 && \text{if } k \neq j, \\
\hat{l}_{kk}^0 &= \prod_{j=1}^3 (\sigma^j l^j \tau^4 + r) \prod_{\substack{j=1 \\ j \neq i}}^3 (\sigma^j l^j \tau^4 + r) && \text{if } i = \left[\frac{k+1}{2} \right].
\end{aligned}$$

The complementary condition at the point $p = 0$ as formulated by Solonnikov [28] requires that the rows of the matrix

$$\mathcal{A}(t, p, r, \mathbf{i}\tau) := \mathcal{B}^0 \hat{\mathcal{L}}^0$$

are linearly independent for all $r \in \mathcal{C}$ and $\operatorname{Re} r > 0$, modulo the polynomial

$$\mathbf{M}^+(r, \tau) = \prod_{j=1}^3 (\tau - \tau_1^j(r))^2 (\tau - \tau_2^j(r))^2 .$$

All functions here are to be interpreted as polynomials in τ , and by τ_l^j ($l = 1, 2$ and $j = 1, 2, 3$) we denote those roots of $\mathbf{L}(t, 0, r, \mathbf{i}\tau)$ which have positive imaginary part. To determine the roots τ_l^j we must calculate the roots of $r + \sigma^j l^j \tau^4$. Therefore writing r in polar coordinates; i.e., $r = |r|e^{i\phi}$ with $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we find that for each $j = 1, 2, 3$ there are precisely two roots with positive imaginary part, namely

$$\begin{aligned} \tau_1^j &= \sqrt[4]{|r|} \sqrt[4]{\frac{1}{\sigma^j l^j}} e^{i(\frac{\phi}{4} + \frac{\pi}{4})}, \\ \tau_2^j &= \sqrt[4]{|r|} \sqrt[4]{\frac{1}{\sigma^j l^j}} e^{i(\frac{\phi}{4} + \frac{3\pi}{4})}. \end{aligned}$$

To decide whether or not the complementary condition is satisfied, we must determine whether there exists a nontrivial vector $(\omega_q)_{q=1, \dots, 12}$ such that

$$\sum_{q=1}^{12} \omega_q B_{qj}^0(\tau) \hat{l}_{jj}^0(\tau) \equiv 0 \quad \text{mod} \quad \prod_{l=1}^3 (\tau - \tau_1^l)^2 (\tau - \tau_2^l)^2$$

for $j = 1, \dots, 6$, or equivalently whether there exists a nontrivial vector $(\omega_q)_{q=1, \dots, 12}$ such that

$$\sum_{q=1}^{12} \omega_q B_{qj}^0(\tau) \equiv 0 \quad \text{mod} \quad (\tau - \tau_1^k) (\tau - \tau_2^k) \quad \text{where } k = \left\lfloor \frac{j+1}{2} \right\rfloor \text{ and } j = 1, \dots, 6.$$

Thus we must decide whether the set of twelve linear equations

$$\begin{aligned} \sum_{q=1}^{12} \omega_q B_{qj}^0(\tau_1^k) &= 0, \\ \sum_{q=1}^{12} \omega_q B_{qj}^0(\tau_2^k) &= 0, \end{aligned}$$

where $k = \left\lfloor \frac{j+1}{2} \right\rfloor$ and $j = 1, \dots, 6$, has a nontrivial solution $(\omega_q)_{q=1, \dots, 12}$.

To derive a more practical version of the complementary condition, we calculate the determinant of the matrix which corresponds to the linear system from above and check the conditions at which the determinant vanishes. Using the property that determinants are multilinear, up to a non-zero factor we must calculate the

determinant of the matrix $D =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \lambda^1 \mathcal{X}_1 & 0 & 0 & \sigma^1 \lambda^1 & 0 & \sigma^1 \lambda^1 \lambda^1 \mathcal{X}_2 & \frac{\mathcal{X}_2}{\lambda^1} & 0 \\ 0 & 1 & 0 & 0 & \lambda^1 \mathcal{X}_2 & 0 & 0 & 0 & \sigma^1 \lambda^1 & -\sigma^1 \lambda^1 \lambda^1 \mathcal{X}_1 & -\frac{\mathcal{X}_1}{\lambda^1} & 0 \\ -1 & 0 & 1 & 0 & 0 & \lambda^2 \mathcal{Y}_1 & 0 & \sigma^2 \lambda^2 & 0 & \sigma^2 \lambda^2 \lambda^2 \mathcal{Y}_2 & -\frac{\mathcal{Y}_2}{\lambda^2} & \frac{\mathcal{Y}_2}{\lambda^2} \\ 0 & -1 & 0 & 1 & 0 & \lambda^2 \mathcal{Y}_2 & 0 & 0 & \sigma^2 \lambda^2 & -\sigma^2 \lambda^2 \lambda^2 \mathcal{Y}_1 & \frac{\mathcal{Y}_1}{\lambda^2} & -\frac{\mathcal{Y}_1}{\lambda^2} \\ 0 & 0 & -1 & 0 & 0 & 0 & \lambda^3 \mathcal{Z}_1 & \sigma^3 \lambda^3 & 0 & \sigma^3 \lambda^3 \lambda^3 \mathcal{Z}_2 & 0 & -\frac{\mathcal{Z}_2}{\lambda^3} \\ 0 & 0 & 0 & -1 & 0 & 0 & \lambda^3 \mathcal{Z}_2 & 0 & \sigma^3 \lambda^3 & -\sigma^3 \lambda^3 \lambda^3 \mathcal{Z}_1 & 0 & \frac{\mathcal{Z}_1}{\lambda^3} \\ 1 & 0 & 0 & 0 & \mathbf{i} \lambda^1 \mathcal{X}_1 & 0 & 0 & \mathbf{i} \sigma^1 \lambda^1 & 0 & -\sigma^1 \lambda^1 \lambda^1 \mathcal{X}_2 & -\mathbf{i} \frac{\mathcal{X}_2}{\lambda^1} & 0 \\ 0 & 1 & 0 & 0 & \mathbf{i} \lambda^1 \mathcal{X}_2 & 0 & 0 & 0 & \mathbf{i} \sigma^1 \lambda^1 & \sigma^1 \lambda^1 \lambda^1 \mathcal{X}_1 & \mathbf{i} \frac{\mathcal{X}_1}{\lambda^1} & 0 \\ -1 & 0 & 1 & 0 & 0 & \mathbf{i} \lambda^2 \mathcal{Y}_1 & 0 & \mathbf{i} \sigma^2 \lambda^2 & 0 & -\sigma^2 \lambda^2 \lambda^2 \mathcal{Y}_2 & \mathbf{i} \frac{\mathcal{Y}_2}{\lambda^2} & -\mathbf{i} \frac{\mathcal{Y}_2}{\lambda^2} \\ 0 & -1 & 0 & 1 & 0 & \mathbf{i} \lambda^2 \mathcal{Y}_2 & 0 & 0 & \mathbf{i} \sigma^2 \lambda^2 & \sigma^2 \lambda^2 \lambda^2 \mathcal{Y}_1 & -\mathbf{i} \frac{\mathcal{Y}_1}{\lambda^2} & \mathbf{i} \frac{\mathcal{Y}_1}{\lambda^2} \\ 0 & 0 & -1 & 0 & 0 & 0 & \mathbf{i} \lambda^3 \mathcal{Z}_1 & \mathbf{i} \sigma^3 \lambda^3 & 0 & -\sigma^3 \lambda^3 \lambda^3 \mathcal{Z}_2 & 0 & \mathbf{i} \frac{\mathcal{Z}_2}{\lambda^3} \\ 0 & 0 & 0 & -1 & 0 & 0 & \mathbf{i} \lambda^3 \mathcal{Z}_2 & 0 & \mathbf{i} \sigma^3 \lambda^3 & \sigma^3 \lambda^3 \lambda^3 \mathcal{Z}_1 & 0 & -\mathbf{i} \frac{\mathcal{Z}_1}{\lambda^3} \end{pmatrix},$$

where we have employed the notation $\lambda^j = (\sigma^j l^j)^{-\frac{1}{4}}$. Assuming that the interfaces are labeled sequentially and counterclockwise, it is easily seen that

$$\begin{aligned} \sin \theta^1 &= \mathcal{Y}_1 \mathcal{Z}_2 - \mathcal{Y}_2 \mathcal{Z}_1, \\ \sin \theta^2 &= \mathcal{Z}_1 \mathcal{X}_2 - \mathcal{Z}_2 \mathcal{X}_1, \\ \sin \theta^3 &= \mathcal{X}_1 \mathcal{Y}_2 - \mathcal{X}_2 \mathcal{Y}_1, \end{aligned}$$

where θ^j denotes the angle opposite Γ^j , the j 'th interface, at the triple junction. Employing the above identities and the symbolic calculation routines of MAPLE, we find

$$\det D = -8(1 + \mathbf{i}) \prod_{j=1}^3 \sigma^j \lambda^j \left[\left(\sum_{j=1}^3 \sigma^j (\lambda^j)^3 \right) \left(\sum_{j=1}^3 \frac{1}{\sigma^j \lambda^j} \sin^2 \theta^j \right) + \left(\sum_{j=1}^3 \lambda^j \sin \theta^j \right)^2 \right].$$

Therefore, the determinant can only vanish if

$$\sin \theta^j = 0 \quad \text{for } j = 1, 2 \text{ and } 3.$$

This condition is fulfilled if

$$0 < \theta^j < \pi \quad \text{for } j = 1, 2 \text{ and } 3. \quad (48)$$

Other cases are not possible. If one angle is zero then by Young's law and the positivity of the surface energies σ^j , the two others have to be π and this would imply that the determinant is zero. The case in which one angle is π can also not occur. This can be seen as follows. Assume without loss of generality that $\theta^3 = \pi$. It then follows that $T^1 = -T^2$ and therefore the balance of tensions (39) implies that

$$(\sigma^1 - \sigma^2)T^1 + \sigma^3 T^3 = 0;$$

hence, $\sigma^3 = 0$. This, however, cannot be admitted within the framework of our analysis because it would imply degeneracy of the equation of evolution for X^3 . To check the complementary condition at $p = 1$ is less complicated and no additional conditions arise.

Remark. It is easy to check that angles $\theta_1, \theta_2, \theta_3$ satisfying Young’s law (38) satisfy condition (48) if and only if the positive surface tensions $\sigma^1, \sigma^2, \sigma^3$ fulfill

$$\sigma^i < \sigma^j + \sigma^k \quad (P_{\text{artial}})$$

for all permutations (i, j, k) of $(1, 2, 3)$. This condition is referred to in the literature as a partial wetting condition on the surface tension (cf. de Gennes [14]) and we will henceforth impose this condition on our system.

We shall assume, moreover, that $X_0^i \in C_p^{4+\alpha}([0, 1])$ fulfill the compatibility condition of order 0 for the initial boundary value problem to $((PDE_L), (I_L)-(IX_L))$; this means that the initial condition satisfy $(I_L)-(IX_L)$ as well as further compatibility conditions which formally are derived by differentiating conditions (I_L) and (VI_L) with respect to time (see Solonnikov [28] for details). The compatibility condition for (I_L) implies that at the point $p = 0$

$$(-\sigma^1 l^1 X_{0,pppp}^1 + G(X_{0,p}^1, X_{0,pp}^1, X_{0,ppp}^1)) = (-\sigma^2 l^2 X_{0,pppp}^2 + G(X_{0,p}^2, X_{0,pp}^2, X_{0,ppp}^2)), \quad (CI_1)$$

and

$$(-\sigma^2 l^2 X_{0,pppp}^2 + G(X_{0,p}^2, X_{0,pp}^2, X_{0,ppp}^2)) = (-\sigma^3 l^3 X_{0,pppp}^3 + G(X_{0,p}^3, X_{0,pp}^3, X_{0,ppp}^3)). \quad (CI_2)$$

The compatibility condition for (VI_L) at the point $p = 1$ reduces to

$$(-\sigma^i l^i X_{0,pppp}^i + G(X_{0,p}^i, X_{0,pp}^i, X_{0,ppp}^i)) \cdot X_{0,p}^i = 0 \quad i = 1, 2, 3. \quad (CVI_i)$$

Finally, we assume that the X_0^i , $i = 1, 2, 3$, are proper parametrizations; i.e., that $|X_{0,p}^i(p)|$ is uniformly bounded away from zero for $p \in [0, 1]$ and $i = 1, 2, 3$. If in addition the $(\sigma^i)_{i=1,2,3}$ are positive and fulfill the partial wetting conditions (P_{artial}) we obtain

Lemma 4.1. *Let $\bar{X} \in \Xi(T, M)$, and let $X_0^i \in C_p^{4+\alpha}([0, 1], \mathbb{R}^2)$, $i = 1, 2, 3$, satisfy the compatibility conditions of order 0. Then there exists a unique solution $X^i \in C_{t,p}^{1+\frac{\alpha}{4}, 4+\alpha}$, $i = 1, 2, 3$ of $((PDE_L), (I_L)-(IX_L))$, such that $X^i(0) = X_0^i$. Moreover there exists a constant $C > 0$, such that the inequality*

$$\begin{aligned} |X^i|_{1+\frac{\alpha}{4}, 4+\alpha} \leq C \sum_{j=1}^3 (|\bar{X}_{\cdot,p}^j, \bar{X}_{\cdot,pp}^j, \bar{X}_{\cdot,ppp}^j, \bar{X}_{\cdot,pppp}^j|_{\frac{\alpha}{4}, \alpha} + |X_0^j|_{4+\alpha}) \\ + C \sum_{q=1}^{12} \left(|\Phi_q(\cdot, 0)|_{1+\frac{\alpha-\delta_q^0}{4}} + |\Phi_q(\cdot, 1)|_{1+\frac{\alpha-\delta_q^1}{4}} \right) \end{aligned}$$

is valid.

Proof. The lemma follows from Theorem 4.9 of Solonnikov [28]. \square

To prove local existence for the nonlinear problem it suffices to demonstrate that the operator which maps $(\bar{X}^1, \bar{X}^2, \bar{X}^3)$ to a solution (X^1, X^2, X^3) of $((PDE_L), (I_L)-(IX_L))$ maps the set $\Xi(T, M)$ into itself and is a contraction for T sufficiently small and M sufficiently large. This however is straightforward employing the a priori estimates given in Lemma 4.1 (see Bronsard and Reitich [8] for the details in the second order case). Therefore we can conclude that under the assumption that the $(\sigma^i)_{i=1,2,3}$ are positive and fulfill the partial wetting conditions (P_{partial}) the following local existence result holds.

Theorem 4.2. *Let $X_0^i \in C_p^{4+\alpha}([0, 1], \mathbb{R}^2)$, $i = 1, 2, 3$, satisfy the compatibility conditions of order 0. Then there is a time $T > 0$ and a positive real number M_0 such that for all $M > M_0$ there exists a unique solution $X \in \Xi(T, M)$ of $((PDE), (I)-(IX))$.*

In the next section we see what geometric constraints on the initial curves are implied by the above compatibility conditions.

5. UNIQUENESS AND THE GEOMETRIC EVOLUTION PROBLEM

Although the contraction argument in the preceding section gives existence and uniqueness for the initial boundary value problem $((PDE), (I)-(IX))$, there remains the question of identifying geometrically admissible initial data (curves) and resolving the possibility of non-uniqueness for the geometric evolution problem. We amplify this issue as follows.

A given set of three curves in the plane which we wish to take as initial data or a set of three evolving curves which we envision as a possible solution to the evolution problem (A)–(C) formulated in Section 3 may be described by a family of smooth parametrizations. This is due to possible reparametrization of the curves. Thus any such set of curves can be equated with an equivalence class of parametrizations which may be used to describe it. Therefore, the first question to resolve is to determine which initial data (curves) are geometrically admissible in the sense that they admit a smooth solution in the following sense:

Definition 5.1. *The geometric evolution problem (A)–(C) with initial curves parametrized by functions $X_0^i \in C_p^{4+\alpha}([0, 1], \mathbb{R}^2)$, $i = 1, 2, 3$, has a smooth solution on the time interval $[0, T]$ if and only if there exist parametrizations*

$$Y^i : [0, T] \times [0, 1] \rightarrow \mathbb{R}^2, \quad i = 1, 2, 3,$$

and functions $\xi_0^i : [0, 1] \rightarrow [0, 1]$, $i = 1, 2, 3$, such that

$$i) \quad Y^i \in C_{t,p}^{1+\frac{\alpha}{4}, 4+\alpha}([0, T] \times [0, 1], \mathbb{R}^2), \quad \xi_0^i \in C_p^{4+\alpha}([0, 1]),$$

$$ii) \quad Y_t^i \cdot N^i = -\sigma^i l^i \kappa_{ss}^i,$$

iii) *the curves parametrized by $Y^i(t, \cdot)$, $0 \leq t \leq T$, fulfill*

- *the persistence conditions,*
- *the angle conditions,*
- *the balance of fluxes*

both at the points of intersection with the exterior boundary and at the triple junction as well as the condition of continuity of the chemical potentials at the triple junction; i.e., the conditions (B) and (C) of Section 3 are satisfied.

- iv) $Y^i(0, \xi_0^i(p)) = X_0^i(p)$,
- v) $\xi_0^i(0) = 0$, $\xi_0^i(1) = 1$, and $\frac{d}{dp}(\xi_0^i)(p) > 0$ for all $p \in [0, 1]$.

We shall see that the initial curves need to fulfill in addition to (B) and (C) only a single geometric compatibility condition in order to guarantee that there exists parametrizations of these initial curves which fulfill the compatibility conditions of order 0 which are needed to apply the existence theorem of Section 4.

Another issue we want to address is to verify uniqueness of the flow within the equivalence class of possible reparametrizations. In the discussion which follows, we take into account the variability in the prescription of the tangential velocity along the interfaces – to be more precise: any velocity is permissible which guarantees that the resulting evolution satisfies the geometric conditions (A)–(C) as well as the regularity requirements of Definition 5.1. We remark that the geometric conditions (A)–(C) do not allow for an arbitrary prescription of the tangential velocities, as the attachment condition and the persistence condition partially determine the tangential velocity at the external boundary and at the triple junction (see (52) and (50) below).

Throughout the remainder of this section we assume that the surface tensions $(\sigma^i)_{i=1,2,3}$ are positive and fulfill the partial wetting condition (P_{artial}). In addition, we use a subscript 0 to indicate that a term is evaluated at time zero. The following theorem makes precise our claim that it is possible to obtain a smooth solution of the geometric evolution problem (A)–(C) provided the initial data fulfill a geometric compatibility condition.

Theorem 5.2. *Suppose the initial curves are parametrized by functions $X_0^i \in C_p^{4+\alpha}([0, 1], \mathbb{R}^2)$, $i = 1, 2, 3$, and assume that the initial curves fulfill (B), (C) and the geometric compatibility condition*

$$(\sigma^1)^2 l^1 \kappa_{0,ss}^1 + (\sigma^2)^2 l^2 \kappa_{0,ss}^2 + (\sigma^3)^2 l^3 \kappa_{0,ss}^3 = 0, \quad (G_{\text{eom}})$$

where κ_0^i indicates the curvatures of the initial curves. Then there exists a time $T > 0$ such that the geometric evolution problem (A)–(C) with initial data parametrized by X_0^i , $i = 1, 2, 3$, has a solution on the time interval $[0, T]$ in the sense of Definition 5.1.

Remark. i) The geometric compatibility condition (G_{eom}) formally follows from the fact that if the evolution is smooth the velocity $\dot{m}(t)$ of the triple junction $m(t)$ has to satisfy

$$\dot{m}(0) \cdot N_0^i = \sigma^i l^i \kappa_{0,ss}^i, \quad i = 1, 2, 3.$$

This, however, is only possible if (G_{eom}) is fulfilled.

ii) A similar such theorem may be stated for triple junction motion within the context of motion by mean curvature (see Bronsard and Reitich [8]). The analogous geometric

compatibility condition would be then

$$\sigma^1 \kappa_0^1 + \sigma^2 \kappa_0^2 + \sigma^3 \kappa_0^3 = 0,$$

and a proof of the sufficiency of such a condition can be given following closely the arguments of the proof of Theorem 5.2 given below.

Proof of Theorem 5.2. To apply the local existence theorem of the preceding section, the parametrization of the initial curves need to fulfill the compatibility conditions of order 0 of the initial boundary value problem $((PDE_L), (I_L)-(IX_L))$. Since the initial curves fulfill (B) and (C) the remaining compatibility conditions are (47), (CI_1) , (CI_2) and (CVI_i) , $i = 1, 2, 3$. In the terminology of (44)–(46), the compatibility conditions (CI_1) and (CI_2) at the triple junction can be stated as

$$\sigma^1 l^1 (\kappa_{0,ss}^1 N_0^1 + \beta_0^1 T_0^1) = \sigma^2 l^2 (\kappa_{0,ss}^2 N_0^2 + \beta_0^2 T_0^2) = \sigma^3 l^3 (\kappa_{0,ss}^3 N_0^3 + \beta_0^3 T_0^3), \quad (49)$$

where the subscript 0 has been used to indicate that the quantities have been evaluated at time $t = 0$. Using the geometric compatibility condition (G_{geom}) , it is possible to check that in order to fulfill (CI_1) and (CI_2) the β_0^i , $i = 1, 2, 3$, must be given by

$$\beta_0^i = -\frac{1}{(\sigma^i)^2 l^i} \left[(\sigma^j)^2 l^j \kappa_{0,ss}^j \frac{T_0^i \cdot T_0^k}{N_0^i \cdot T_0^k} + (\sigma^k)^2 l^k \kappa_{0,ss}^k \frac{T_0^i \cdot T_0^j}{N_0^i \cdot T_0^j} \right] \quad (50)$$

for permutations (i, j, k) of $(1, 2, 3)$. We remark that the right hand side of (50) is invariant under reparametrization. Similarly at points of intersection of an interface with the external boundary the compatibility conditions (CVI_i) , $i = 1, 2, 3$ may be written as

$$\sigma^i l^i (\kappa_{0,ss}^i N_0^i + \beta_0^i T_0^i) \cdot T_0^i = 0, \quad (51)$$

i.e., we must require that

$$\beta_0^i = 0, \quad i = 1, 2, 3, \quad (52)$$

at points of intersection of an initial interface with the external boundary.

Let us now choose a sufficiently smooth reparametrization

$$\tilde{X}_0^i(p) = X_0^i(\xi^i(p))$$

of the initial curves in such a way that the \tilde{X}_0^i , $i = 1, 2, 3$, fulfill (47), (50) and (51) at the end points. Noting the definition (44) of β^i , it is easy to see that a prescription of β^i at $p = 0$ or $p = 1$ via (50) or (51) and the fulfillment of condition (47) can be achieved by solving a fourth order ODE for ξ^i locally near the ends of $[0, 1]$. In this manner, the initial parametrization may be redefined to yield a parametrization fulfilling the compatibility conditions of order 0. The statement of the theorem now follows from Theorem 4.2. \square

In terms of uniqueness for the geometric problem, our results may be stated as follows

Theorem 5.3. *Assume $C^{4+\alpha}$ -initial curves which fulfill the boundary conditions (B), (C), and the geometric compatibility condition (G_{eom}) are given. Let $X_0^i \in C_p^{4+\alpha}([0, 1], \mathbb{R}^2)$, $i = 1, 2, 3$, be parametrizations of the initial curves which fulfill the compatibility conditions of order 0 and let $(X^1, X^2, X^3) \in \Xi(T, M)$ be the solution of ((PDE), (I)–(IX)) for the initial data $(X_0^i)_{i=1,2,3}$ which was obtained in Theorem 4.2. Furthermore, assume that (Y^1, Y^2, Y^3) is a solution of the geometric evolution problem (for the same initial curves) in the sense of Definition 5.1. Then there exists a time $T_1 > 0$ and functions*

$$\xi^i \in C_{t,p}^{1+\frac{\alpha}{4}, 4+\alpha}([0, T_1] \times [0, 1], [0, 1])$$

which satisfy $\xi^i(t, 0) = 0$, $\xi^i(t, 1) = 1$ and $\frac{\partial}{\partial p}(\xi^i)(t, p) > 0$ for all $p \in [0, 1], t \in [0, T_1]$ such that

$$X^i(t, p) = Y^i(t, \xi^i(t, p)).$$

Remark. i) In the proof of Theorem 5.2 we demonstrated that $C^{4+\alpha}$ -initial curves which satisfy (B), (C) and (G_{eom}) can be parametrized by functions $X_0^i \in C_p^{4+\alpha}([0, 1], \mathbb{R}^2)$, $i = 1, 2, 3$, such that the compatibility conditions of order zero are satisfied.

ii) Theorem 5.3 shows that the geometric evolution problem defined by (A)–(C) in Section 3 has only one geometrically distinct smooth solution in the sense that two smooth parametrizations of evolving curves which fulfill (A)–(C) and which parametrize the same curves at the initial time, parametrize the same curves at later times throughout their common interval of existence. The maximal interval of existence may depend on the parametrization, though it is possible to formulate a maximal “geometric” interval of existence if one allows for reparametrizations.

Proof of Theorem 5.3. Our approach is to derive an initial boundary value problem for reparametrizations $(\xi^i)_{i=1,2,3}$ such that

$$Z^i(t, p) := Y^i(t, \xi^i(t, p)) \tag{53}$$

is a solution of ((PDE), (I)–(IX)) whenever ξ^i is a sufficiently smooth solution of this initial boundary value problem. The uniqueness statement in Theorem 4.2 then implies that $X^i = Z^i$, $i = 1, 2, 3$, and proves the theorem.

Let us formally derive an initial boundary value problem which should be satisfied by the functions ξ^i . Our goal is to find functions ξ^i such that Z^i , $i = 1, 2, 3$, satisfies

$$Z_t^i = -\sigma^i l^i (\kappa_{,ss}^i N^i + \beta^i T^i), \tag{54}$$

where β^i is defined as in (44) in terms of the parametrization Z^i . Noting (44) through (46), we observe that equation (54) is equivalent to the equation (PDE). For Z^i , the

following identities hold:

$$\begin{aligned}
Z_t^i(t, p) &= Y_t^i(t, \xi^i(t, p)) + Y_{,p}^i(t, \xi^i(t, p))\xi_t^i(t, p), \\
Z_{,p}^i &= Y_{,p}^i \xi_{,p}^i, \\
Z_{,pp}^i &= Y_{,pp}^i (\xi_{,p}^i)^2 + Y_{,p}^i \xi_{,pp}^i, \\
Z_{,ppp}^i &= Y_{,ppp}^i (\xi_{,p}^i)^3 + 3Y_{,pp}^i \xi_{,p}^i \xi_{,pp}^i + Y_{,p}^i \xi_{,ppp}^i, \\
Z_{,pppp}^i &= Y_{,pppp}^i (\xi_{,p}^i)^4 + 6Y_{,ppp}^i (\xi_{,p}^i)^2 \xi_{,pp}^i + 3Y_{,pp}^i (\xi_{,pp}^i)^2 + 4Y_{,pp}^i \xi_{,p}^i \xi_{,ppp}^i + Y_{,p}^i \xi_{,pppp}^i,
\end{aligned}$$

where the arguments in the last four lines have been omitted for simplicity. Since

$$\begin{aligned}
Z_t^i &= Y_t^i + Y_{,p}^i \xi_t^i \\
&= (Y_t^i \cdot N^i) N^i + (Y_t^i \cdot T^i) T^i + Y_{,p}^i \xi_t^i \\
&= -\sigma^i l^i \kappa_{,ss}^i N^i + (Y_t^i \cdot T^i) T^i + Y_{,p}^i \xi_t^i,
\end{aligned}$$

equation (54) is fulfilled if

$$0 = |Y_{,p}^i| \xi_t^i + \sigma^i l^i (\beta^i - \bar{\beta}^i),$$

where we defined $\bar{\beta}^i := -\frac{1}{\sigma^i l^i} Y_t^i \cdot T^i$ which is given in terms of Y^i . The term β^i can be expressed in terms of Y^i and ξ^i . In fact it is easy to check that

$$\begin{aligned}
\beta^i &= T^i \cdot \frac{Z_{,pppp}^i}{|Z_{,p}^i|^4} + \tilde{\beta}(Z_{,p}^i, Z_{,pp}^i, Z_{,ppp}^i) \\
&= \frac{Y_{,p}^i \xi_{,p}^i}{|Y_{,p}^i \xi_{,p}^i|} \cdot \frac{Y_{,pppp}^i \xi_{,p}^i}{|Y_{,p}^i \xi_{,p}^i|^4} + \hat{\beta}(Y_{,p}^i, Y_{,pp}^i, Y_{,ppp}^i, Y_{,pppp}^i, \xi_{,p}^i, \xi_{,pp}^i, \xi_{,ppp}^i)
\end{aligned}$$

for suitable functions $\tilde{\beta}$ and $\hat{\beta}$.

Hence, it is sufficient to require that ξ^i , $i = 1, 2, 3$, satisfy the initial boundary value problem

$$\xi_t^i + \frac{\sigma^i l^i}{|Y_{,p}^i|^4} \frac{\xi_{,pppp}^i}{(\xi_{,p}^i)^4} + \frac{\sigma^i l^i}{|Y_{,p}^i|} \left(\hat{\beta}(Y_{,p}^i, Y_{,pp}^i, Y_{,ppp}^i, Y_{,pppp}^i, \xi_{,p}^i, \xi_{,pp}^i, \xi_{,ppp}^i) - \bar{\beta}^i \right) = 0, \quad (55)$$

$$\xi^i(t, 0) = 0, \quad \xi^i(t, 1) = 1, \quad \text{and} \quad \xi_{,p}^i(t, p) = \frac{1}{|Y_{,p}^i(t, p)|}, \quad p = 0, 1 \text{ and } t \in [0, T], \quad (56)$$

$$\xi^i(0, p) = \xi_0^i(p) \quad \text{for } p \in [0, 1]. \quad (57)$$

Either by noting the geometric nature of the conditions (B) and (C) which implies invariance under reparametrization or by direct verification, it is easy to see that the boundary conditions (B) and (C) are satisfied by Z^i , $i = 1, 2, 3$. This and the third equation in (56) imply that the boundary conditions (I)–(IX) are satisfied by Z^i , $i = 1, 2, 3$. Hence, $X^i = Z^i$ on $[0, T_1] \times [0, 1]$ provided we can show the existence of a solution

$$\xi^i \in C_{t,p}^{1+\frac{\alpha}{4}, 4+\alpha}([0, T_1] \times [0, 1], [0, 1])$$

of the initial boundary value problem (55)–(57). However, this initial boundary value problem can be solved locally in time by using the same methods as in Section 4.

We do not carry through the details of the proof, but we demonstrate that the compatibility conditions of order 0 are satisfied and that the data are regular enough to apply the contraction argument. Since formally $\xi_t^i(0, p) = 0$ has to hold for $p = 0, 1$, one must guarantee that

$$\beta^i(0, p) = \bar{\beta}^i(0, p) \quad \text{for } p = 0, 1.$$

But this follows for $p = 1$ since the regularity of $Y^i(0, p)$ and the complementary condition for X_0^i imply that $\beta^i(0, 1)$ and $\bar{\beta}^i(0, 1)$ are given by the right hand side of (50). Similarly we get that

$$\beta^i(0, 0) = 0 = \bar{\beta}^i(0, 0).$$

The other compatibility conditions are easy to verify. We remark that i) and ii) in Definition 5.1 guarantee that $\bar{\beta}^i \in C^{\frac{\alpha}{4}, \alpha}([0, T] \times [0, 1], \mathbb{R})$. This ensures that we can prove the existence of a unique solution to (55)–(57) on a time interval $[0, T_1]$ provided T_1 is sufficiently small. \square

6. CONCLUSIONS

Employing formal asymptotic expansions we determined the asymptotic limit ($\gamma = \varepsilon^2$, $t \rightarrow \varepsilon^2 t$, $\theta = \mathcal{O}(\varepsilon)$, $\varepsilon \searrow 0$) of a Cahn–Hilliard system with a degenerate mobility matrix. The limiting motion is a sharp interface model where the evolution of interfaces is governed by motion by surface diffusion. At triple junctions Young’s law, a condition required by the continuity of the chemical potentials and mass balance conditions hold. At points of an intersection with the external boundary, a Neumann type right angle condition and a no flux condition have been derived.

For this limiting system we prove local in time existence and uniqueness in the Hölder space $C_{t,p}^{1+\frac{\alpha}{4}, 4+\alpha}$. The existence result is obtained by solving a nonlinear parabolic boundary value problem for parametrizations of the interfaces via linearization and a contraction argument. To demonstrate uniqueness of a solution of the geometric problem the contraction argument alone did not suffice; we also needed to show uniqueness under admissible reparametrizations. We remark that in general one cannot expect to establish results on long time existence with our approach since topological changes might occur; e.g., the length of one of the interfaces may converge to zero.

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