Factors of i.i.d. processes on graphs and groups

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Let $X$ be a set with a (discrete) group $G$ acting on it. We will be concerned in particular with the cases when

- $X = G$, so $G$ acts on itself by multiplication, and
- $X$ is an infinite graph and $G$ is a group of automorphisms of $X$. 
I.i.d. processes on groups and graphs

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- $X$ is an infinite graph and $G$ is a group of automorphisms of $X$.

We consider i.i.d. processes on $X$.

- $\xi_m = (\xi_m(x))_{x \in X}$ is the full $m$-shift on $(X, G)$ if it is i.i.d. and each $\xi_m(x)$ takes values uniformly in $\{0, \ldots, m - 1\}$.

- $\xi_{[0,1]} = (\xi_{[0,1]}(x))_{x \in X}$ is the full $[0, 1]$-shift on $(X, G)$ if it is i.i.d. and $\xi_{[0,1]}(x)$ takes values uniformly in $[0, 1]$.
Groups act on processes

Let $\xi$ be an i.i.d. process on $(X, G)$. Then $G$ acts on configurations of $\xi$:

$$g \cdot (\xi(x))_{x \in X} = (\xi(g \cdot x))_{x \in X}, \quad g \in G$$
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Examples:

- $X = G = \mathbb{Z}$. Then $\xi_2$ is the 2-sided i.i.d. Bernoulli-(1/2) process. $\mathbb{Z}$ acts on this process by shifting sequences.

  → 0 → 0 → 1 → 0 →
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- $X = G = F_2 = \text{the free group on 2 generators}$:

  ![Diagram of the free group on 2 generators](image)
Question

Under what conditions on \((X, G)\) do there exist \(m < n\) such that there is a \(G\)-factor from \(\xi_m\) to \(\xi_n\) on \((X, G)\)?

Definition of a \(G\)-factor:

- Let \(\xi, \zeta\) be processes on a set \(X\) with \(G\) acting.
- A map \(F : \xi \to \zeta\) is a \(G\)-factor if \(F\) commutes with the \(G\)-action on \(\xi\) and \(F(\xi) = \zeta\):

\[
g \cdot F(\xi) = F(g \cdot \xi).
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\[
(\xi(x))_{x \in X} \overset{g \in G}{\longrightarrow} (\xi(g \cdot x))_{x \in X}
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Motivation

Consider case where $X = G = \mathbb{Z}$ and $\mathbb{Z}$ acts on a process $\xi$ by shifting:

$$n \cdot (\ldots, \xi(-1), \xi(0), \xi(1) \ldots) \mapsto (\ldots, \xi(n - 1), \xi(n), \xi(n + 1) \ldots)$$

**Theorem.** There is a $\mathbb{Z}$-factor from $\xi_m$ to $\xi_n$ on $\mathbb{Z}$ $\iff m \geq n$.

- In particular, the full 4-shift is not a $\mathbb{Z}$-factor of the full 2-shift.
- This is a baby version of Sinai’s Factor Theorem, which can be considered part of Ornstein’s Isomorphism Theory. It is a fundamental result in ergodic theory.
Motivation

Now consider the case where $X = G = \mathbb{F}_2$, the free group on two generators, $a$ and $b$.

**Proposition.** (Ornstein-Weiss, 1987) There is an $\mathbb{F}_2$-factor from $\xi_2$ to $\xi_4$ on $\mathbb{F}_2$.

**Proof.** Define the factor map $F$ on the full 2-shift $\xi_2$ by:

$$F(\xi_2)(x) = (\xi_2(a \cdot x) \oplus \xi_2(x), \xi_2(b \cdot x) \oplus \xi_2(x))$$
Observation (Adam Timar):
If there exists a $G$-factor from $\xi_2$ to $\xi_4$ on $(X, G)$, then there exists a $G$-factor from $\xi_2$ to $\xi_{[0,1]}$ on $(X, G)$.

**Proof.** We will code a number in $[0, 1]$ in binary.
Let $F$ be a $G$-factor taking $\xi_2$ to $\xi_4$ on $(X, G)$. We use it to build a $G$ factor $F'$ from $\xi_2$ to $\xi_{[0,1]}$.

- Apply $F$ to $\xi_2$ to get $\xi_4$.
- Code the coordinates of $\xi_4 = F(\xi_2)$ as bits: 00, 01, 10, 11.
- Let the first bit of $F'(\xi_2)(x)$ be the first bit of $F(\xi_2)(x)$.
- Apply $F$ again to the second bits of $F(\xi_2)$.
- Repeat the previous 2 steps (infinitely many times).
Amenability

**Definition.** A discrete group $G$ is *amenable* if for every finite set $C \subset G$ and $\epsilon > 0$, there exists a finite $K \subset G$ such that

$$|CK \triangle K| < \epsilon |K|.$$  

**Definition.** A graph $G$ is *amenable* if for every $\epsilon > 0$, there exists a finite set of vertices $K \neq \emptyset$ such that

$$\frac{\text{# edges in } G \text{ w/ exactly 1 endpt in } K}{\text{# vertices in } K} < \epsilon.$$
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Example. $\mathbb{Z}^2$ is amenable: take $K$ a large square.
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**Example.** $\mathbb{Z}^2$ is amenable: take $K$ a large square.

**Example.** $\mathbb{F}_2$ is not amenable: easy to see balls don’t work.
Existence of factors

Take \( X = G \).

**Theorem.** (B.) A finitely-generated group \( G \) is nonamenable if and only if there exists \( m > 0 \) such that a \( G \)-factor taking \( \xi_{2m} \) on \( G \) to \( \xi_{2m+1} \) on \( G \).

**Notes:**

- (\( \Leftarrow \)) follows easily from entropy considerations for amenable group actions.

- If \( G \) is nonamenable and has a subgroup isomorphic to \( F_2 \), then (\( \Rightarrow \)) can be proved using the Ornstein-Weiss factor.

- There are nonamenable groups which do not have an \( F_2 \)-subgroup, as proved by Ol’shanskii. These are mysterious objects.
Recall the Ornstein-Weiss factor:

\[ F(\xi_2)(x) = (\xi_2(a \cdot x) \oplus \xi_2(x), \xi_2(b \cdot x) \oplus \xi_2(x)) \]

The independence of the random variables depends on the fact that the Cayley graph of \( \mathbb{F}_2 \) above is a many-ended tree.
Strategy

For a general finitely generated nonamenable graph \( G \), our goal will be to split the information in \( \xi_{2m} = (\xi_{2m-1}, \xi_2) \) and

Step 1. Let \( X \) be a Cayley graph of \( G \) and choose at least one tree \( T \subseteq X \) as a \( G \)-factor of \( \xi_{2m-1} \).

We want to choose trees which

- have no leaves (vertices of degree one) and
- have at least three ends (paths to infinity).
Step 2.

- Modify the O-W factor above to give an $\text{Aut}(T)$-factor $F : \xi_2 \rightarrow \xi_4$ on a tree $T$ with bounded degree, no leaves, and $\geq 3$ ends.

- Use Adam’s observation to get infinitely many indep. bits at each $x \in T$. 
Step 3. Distribute the bits to the rest of the $x \in X$. 

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Step 1. Trees as factors

Goal: To choose a percolation of $G$ containing a tree with at least three ends as a $G$-factor of $\xi_{2^{m-1}}$.

We appeal to some results from percolation theory. Let $X$ be a graph. Then define

- $p_c(X) = \inf\{p : \text{Bernoulli}(p) \text{ has an infinite comp.}\}$
- $p_u(X) = \inf\{p : \text{Bernoulli}(p) \text{ has a unique } \infty\text{-comp.}\}$

Theorem. (Pak, Smirnova-Nagnibeda 2000) If $G$ is a nonamenable, f.g. group, then there exists a Cayley graph $X$ of $G$ such that $p_c(X) < p_u(X)$.

Theorem. (Benjamini, Schramm 1996) Consider $\text{Bernoulli}(p)$ percolation with $p_c(X) < p < p_u(X)$. For every $n$ there is an infinite component with more than $n$ ends.
Step 1. Trees as factors

Let $X$ be a Cayley graph of $G$ with $p_c(X) < p_u(X)$.

Choose $m$ large enough that there is a value of $k$ such that

$$p_c(X) < k/2^{m-1} < p_u(X).$$

Use $\xi_{2^{m-1}}$ to generate a Bernoulli($k/2^{m-1}$) percolation on $X$ which has a component $C$ with at least three ends!

$C$ has $\geq 3$ ends:

$\exists$ a finite set $K \subset C$ s.t. $C \setminus K$ has at least 3 components.
Step 1. Trees as factors

Have: A component with at least 3 ends.
Want: A tree with at least 3 ends.

Use minimal spanning forests:
- Let $X$ be a graph with distinct labels $L(e)$ at each edge $e$.
- Remove an edge $e$ from $X$ if $L(e)$ is the largest label along some cycle of $X$. 
Step 1. Trees as factors

Have: A component with at least 3 ends.
Want: A tree with at least 3 ends.

Use minimal spanning forests:

Let $X$ be a graph with distinct labels $L(e)$ at each edge $e$.

Remove an edge $e$ from $X$ $\iff$ $L(e)$ is the largest label along some cycle of $X$.
Step 1. Trees as factors

Label the edges of $X$. Generate a continuous r.v. $U(e)$ at each edge $e$ in $X$ as a $G$-factor of $\xi_{2^{m-1}}$:

- order the elements of $G$, $g_1 < g_2 < \ldots$,
- form $U(x, y)$ ($x \sim y$) by concatenating $\xi(x \cdot g_1) \oplus \xi(y \cdot g_1)$, $\xi(x \cdot g_2) \oplus \xi(y \cdot g_2)$, $\ldots$,
- the $U(e)$ are dependent, but a.s. distinct.

[Diagram of a tree structure with nodes labeled $x$ and $y$.]
Step 1. Trees as factors

- Use the $U(e)$ and minimal spanning forest to trim to a tree:
  - form the minimal spanning tree inside the red circles
  - identify the red circles points and generate the minimal spanning forest on the resulting graph

This ensures that one of the resulting trees has at least three ends.
Consider $X$ a nonamenable transitive graph, $G = \text{Aut}(X)$. 
($X$ is transitive if for $x, y \in X$, $\exists \gamma \in G$ with $\gamma(x) = y$.)

**Question.** Does there always exist $m \leq n$ such that there is a $G$ factor from $\xi_m$ to $\xi_n$ on $X$?

**Answer.** No. (Though often there is.)

Potential problems with proof outlined above for groups:

- Is $p_c(X) < p_u(X)$?
  (To get a component with at least 3 ends.)
  This is conjectured to be true by Benjamini and Schramm.

- Can we generate distinct edge labels as a $G$-factor of $\xi_m$ for some $m$? (To use MSF to trim component trees.)
A counterexample

Let $S_3$ be the graph formed by taking the regular tree $T_3$ of degree 3 and:

- replacing each vertex $x \in T_3$ be two vertices $x_1, x_2$
- replacing each edge $(x, y) \in T_3$ by four edges $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)$.

$S_3$ is transitive, has $\infty$ many ends, and is unimodular.

Note: there is an automorphism $\gamma_x$ of $S_3$ which interchanges $x_1$ and $x_2$ but fixes the rest of the vertices.
A counterexample

Let $G_3$ be the full automorphism group of $S_3$. Fix $m, n$ and let $F : \xi_m \to \xi_n$ be a $G_3$-factor. Then:

- If $\xi_m(x_1) = \xi_m(x_2)$, then $(F(\xi_m))(x_1) = (F(\xi_m))(x_2)$

\[ 1 = m \quad \text{and} \quad 1 = n \]

Therefore, $1 = m \quad \text{and} \quad 1 = n$. 

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A counterexample

Let $G_3$ be the full automorphism group of $S_3$. Fix $m, n$ and let $F : \xi_m \rightarrow \xi_n$ be a $G_3$-factor. Then:

- If $\xi_m(x_1) = \xi_m(x_2)$, then $(F(\xi_m))(x_1) = (F(\xi_m))(x_2)$
- $P(\xi_m(x_1) = \xi_m(x_2)) = 1/m$ and $P(\xi_n(x_1) = \xi_n(x_2)) = 1/n$
A counterexample

Let $G_3$ be the full automorphism group of $S_3$. Fix $m, n$ and let $F : \xi_m \to \xi_n$ be a $G_3$-factor. Then:

- If $\xi_m(x_1) = \xi_m(x_2)$, then $(F(\xi_m))(x_1) = (F(\xi_m))(x_2)$
- $P(\xi_m(x_1) = \xi_m(x_2)) = 1/m$ and $P(\xi_n(x_1) = \xi_n(x_2)) = 1/n$
- Therefore, $1/m \leq 1/n \Rightarrow m \geq n.$
How to generate distinct edge labels?

"Theorem". Let $X$ be a nonamenable graph with transitive automorphism group $G$. If

- $X$ has at least 3 ends or $p_c(X) < p_u(X)$, and
- for $j$ large enough, there is a $G$-factor of $\xi_j$ which puts distinct labels on each edge of $X$,

then for $m$ sufficiently large, there is a $G$-factor: $\xi_{2m} \rightarrow \xi_{2m+1}$.

Sufficient condition for the second bullet:
There do not exist $x_1, x_2 \in X$ and $\gamma : x_1 \rightarrow x_2$ such that for all but finitely many $y$,

$$\text{Stab}_G(x_1) \cdot y = \text{Stab}_G(x_2) \cdot \gamma y.$$
Open questions

Question 1. Is there a pair \((X, G)\) and \(m \neq n\) such that there is an invertible \(G\) factor from \(\xi_m\) to \(\xi_n\) on \((X, G)\)?

Question 2. (R. Lyons) \(T_4\) is the regular tree of degree 4.

- We have seen that \(\mathbb{F}_2\) acts on \(T_4\).
  (By giving \(T_4\) the structure of the Cayley graph of \(\mathbb{F}_2\).)

- \(\text{Aut}(T_4)\) is larger than \(\mathbb{F}_2\) since it allows \(a\) and \(b\) edges to be interchanged.

Is there an \(\text{Aut}(T_4)\)-factor of \(\hat{\xi}_{[0,1]}\) on \(T_4\) which gives \(T_4\) the structure of the Cayley graph of \(\mathbb{F}_2\)?