Introduction to the Cauchy problem for the Einstein equations

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Abstract

The Cauchy problem for the Einstein equations has a number of special features when compared with that for other partial differential equations. These issues are briefly discussed, starting with model equations which illustrate some of these features in a less complicated context. Next the Cauchy problem for the Einstein equations is formulated. The standard approach to proving local existence is presented. Finally, some remarks are made on the 3+1 decomposition.

1 The Cauchy problem for the Einstein equations

1.1 The local and global Cauchy problems

These notes are mainly concerned with the local Cauchy problem for the Einstein equations, where 'local' means 'local in time'. The precise meaning of the terms 'local' and 'global' can only be explained once the necessary mathematical tools have been introduced but it is important to have some intuitive idea of the distinction right at the beginning. To illustrate this, consider an ordinary differential equation $du/dt = F(u)$. It is well known that, given an initial value u_0, there exists a unique solution of the equation with $u(0) = u_0$. This solution exists locally, that is on some time interval containing $t = 0$. However there is in general no solution of the equations defined for all values of t with the given initial value; the local solution becomes singular in finite time. A simple example is the case $F(u) = u^2$ where the only initial value for which no singularity occurs in the solution is $u_0 = 0$.

If we have a partial differential equation of evolution type for a function u of t and x then the Cauchy problem is roughly speaking the following: given a function $u_0(x)$, does there exist a solution $u(t, x)$ of the given equation with $u(0, x) = u_0(x)$ and if so is this solution unique? It may be, depending on the exact type of equation, that time derivatives of u should also be prescribed at $t = 0$. In the local Cauchy problem a solution is sought which exists on some neighbourhood of the initial surface $t = 0$. The example above shows that this is the best which can be hoped for in general. In the global Cauchy problem one looks for a solution defined for all values of t and x. Note that the distinction between local and global Cauchy problems is made necessary by the consideration of nonlinear equations. For a linear equation the Cauchy problem can usually be solved globally if it can be solved locally.

The notion of a global solution becomes more subtle when the Einstein equations are considered, as will be explained in detail later. The question whether the Cauchy problem for the Einstein equations can be solved globally is intimately related to the important physical question of the existence and nature of spacetime singularities in general relativity. In particular, the strong cosmic censorship hypothesis, which is one of the most interesting open questions in general relativity, is most naturally expressed in terms of the global Cauchy problem. The local Cauchy problem for the Einstein equations is quite well understood. The global problem is characterised by open questions and will not be treated here.

1.2 Why study the Cauchy problem?

General relativity is the appropriate tool for describing the dynamics of self-gravitating matter in a fully relativistic way. It also describes the dynamics of spacetime geometry. The basic equations of general relativity are the Einstein equations. For this reason it is of interest to understand what kind of solutions these equations possess. It is not enough only to have a few explicit solutions. To understand the equations properly it is necessary to know something about the most general solutions. A convenient way of parametrising these solutions is also very useful for physics since it allows one to characterise those solutions which are appropriate for the description of a given physical system. The Cauchy problem provides a way of achieving both of these goals. It deals with the most general solutions of the equations being considered and parametrises them by their initial data on some appropriate initial hypersurface. Furthermore, it is the only known method for achieving these goals. Once the local problem has been solved one can then proceed to ask about the global qualitative behaviour of solutions. The Cauchy problem provides a useful framework in which questions of this type can be studied.

There is also a close connection to practical physics. When modelling an astrophysical system it is seldom that an exact solution of the Einstein equations can be used. It is necessary to use analytic approximations (expansions in a small parameter) or numerical calculations. The latter are of course also a type of approximation. Observational data, such as those from the binary pulsar, are compared with calculations which use these approximation methods. Thus strictly speaking we can only say that these data allow the predictions of general relativity to be compared with observation if there exists a solid theoretical understanding of the relation between solutions of the Einstein equations and solutions of equations which approximate them in some sense. Once again, it is the Cauchy problem which provides a mathematical basis for this kind of understanding.

1.3 Some model equations

When compared with other partial differential equations the Einstein equations present some special difficulties. So as not to have to confront all these difficulties at once when discussing the Cauchy problem I will illustrate certain facts first using some model equations. Each of these shares some features with the Einstein equations but does not involve all the same difficulties. These model equations will now be introduced. The Einstein equations are essentially hyperbolic equations, that is they describe the propagation of wavelike phenomena. Thus it is natural to start with the prototype of all hyperbolic equations, the ordinary wave equation. This is given by

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta u = 0, \tag{1}$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and u is a function of $t = x^0$ and x^a. (Here and in the following Latin and Greek indices take the values 1,2,3 and 0,1,2,3 respectively.) The next example is the source-free Maxwell equations in Minkowski space:

$$\partial_\alpha F^{\alpha\beta} = 0, \;\; \partial_{[\alpha}F_{\beta\gamma]} = 0. \tag{2}$$

These equations can also be written using the 4-potential A_α, in terms of which $F_{\alpha\beta} = A_{[\alpha,\beta]}$. That version of the equations can be generalised to the Yang-Mills equations. The definition of the Yang-Mills equations requires a Lie algebra to be chosen. Suppose for definiteness that we take this to be the Lie algebra $su(2)$, which can be realised as the vector space of two by two skew-Hermitian matrices. The basic unknown is the gauge potential A_α which is a one-form with values in $su(2)$. This can be described concretely in terms of coordinate components as follows. A one-form (covector) on a four-dimensional manifold is represented in local coordinates by a set of four real-valued functions. If we instead take four functions whose values are two by two skew-Hermitian matrices we get a coordinate representation of a one-form with values in $su(2)$. A one-form of this type has an associated curvature which is given by the following formula. This can be interpreted, via an appropriate kind of abstract indices, as defining a two-form with values in $su(2)$ or can be thought of more concretely as defining a set of functions with values in $su(2)$ indexed by α and β. The formula is

$$F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]} + [A_\alpha, A_\beta]. \tag{3}$$

Here the large square brackets denote the commutator of matrices, which is also the Lie bracket in this case. The Yang-Mills equation is then

$$\partial_\alpha F^{\alpha\beta} + [A_\alpha, F^{\alpha\beta}] = 0. \tag{4}$$

The Einstein equations are equations for a Lorentz metric $g_{\alpha\beta}$ of the general form $G_{\alpha\beta} = T_{\alpha\beta}$, where $G_{\alpha\beta}$ is the Einstein tensor of $g_{\alpha\beta}$ and $T_{\alpha\beta}$ is the energy-momentum tensor. In order to have a determined system of equations it remains to specify the matter content of spacetime. Mathematically, this means that we have to specify the variables which describe the matter, how the energy-momentum tensor is built from these variables and which equations they are supposed to satisfy. There are many different kinds of matter model which are of interest in general relativity. The simplest choice of all is vacuum. In that case there are no matter variables at all and the energy-momentum tensor is zero. Other matter models may be divided roughly into two types which might be called field theoretic and phenomenological. In the field theoretic type of model the matter variables are thought of as fundamental physical fields or at least as simplified models for such fields. A simple example is to take a massless scalar field.

There the matter is described by a single scalar function ϕ and the energy-momentum tensor is

$$T_{\alpha\beta} = \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2}(g^{\gamma\delta}\nabla_\gamma \phi \nabla_\delta \phi) g_{\alpha\beta} \tag{5}$$

The field equation is the curved space version of (1), namely $\nabla_\alpha \nabla^\alpha \phi = 0$. Similarly, it is possible to couple the Maxwell or Yang-Mills equations to the Einstein equations. The explicit form of the energy-momentum tensor for those cases will not be needed in the following. Phenomenological matter models are intended to model the macroscopic behaviour of matter without worrying about its fundamental constituents. These are the matter models which are required for astrophysical applications. Two important examples are the perfect fluid and collisionless matter. In the case of an isentropic perfect fluid the matter can be described in terms of the four-velocity of the fluid, U^α, the energy density ρ and the pressure p. The vector U^α is assumed normalised, $U^\alpha U_\alpha = -1$ and an equation of state $p = f(\rho)$ is postulated. The energy-momentum tensor is

$$T_{\alpha\beta} = (\rho + p) U_\alpha U_\beta + p g_{\alpha\beta} \tag{6}$$

and the equations for the matter are given by the conservation law for energy-momentum, namely $\nabla_\alpha T^{\alpha\beta} = 0$. In the case of collisionless matter a kinetic description is used. If we consider a collisionless gas of particles with unit mass then the 4-momentum of an individual particle is given by a vector p^α satisfying the condition $g_{\alpha\beta} p^\alpha p^\beta = -1$. Let P be the mass shell i.e. the set of all vectors satisfying this condition. The basic matter variable in the case of a collisionless gas is the distribution function f, a function on P which decribes the density of particles at a given spacetime point with given momentum. The mass shell can be coordinatised by x^α and p^a. The equation which f must satisfy is the Vlasov equation

$$p^\alpha \partial f / \partial x^\alpha - \Gamma^a_{\beta\gamma} p^\beta p^\gamma \partial f / dp^a = 0, \tag{7}$$

and the energy-momentum tensor is

$$T_{\alpha\beta} = -\int f p_\alpha p_\beta \sqrt{-g} / p_0 dp^1 dp^2 dp^3. \tag{8}$$

1.4 The Cauchy problem

Consider the wave equation (1). The Cauchy problem for this equation is as follows. Given initial data consisting of two functions ϕ_0 and ϕ_1 on $\mathbf{R}^3$ does there exist a solution ϕ of (1) with $\phi(0, x^a) = \phi_0(x^a)$ and $\partial_t \phi(0, x^a) = \phi_1(x^a)$? If so, is this solution uniquely determined by the initial data? It is well known that, under appropriate differentiability assumptions on the data, the answer to both these questions is yes. It is not necssary to make a distinction between local and global solutions in this case since the equation is linear. If we now move from the wave equation to the Maxwell equations a new element appears in the problem, namely the existence of constraints. To see this it is useful to do a 3+1 decomposition of the equations, i.e. to split spacetime into space and time using the hypersurfaces $t =$ const. in Minkowski space and to decompose the 4-dimensional field tensor into parts according to this splitting. Let t^α denote the vector in Minkowski space with components $(1,0,0,0)$. Define

$$E_\alpha = F_{\alpha\beta}t^\beta, \quad B_\alpha = \epsilon_{\alpha\beta\gamma\delta}F^{\beta\gamma}t^\delta \tag{9}$$

Because of the antisymmetry of $F_{\alpha\beta}$ the zeroth components of E^α and B^α vanish and we can think of them as purely spatial vectors. They are just the electric and magnetic fields which occur in the standard description of electrodynamics and writing (2) in terms of these variables gives the traditional form of the Maxwell equations in terms of E and B fields. In that form the Maxwell equations fall into two groups of two equations each. The equations of the first group contain time derivatives and are referred to as the evolution equations. The equations of the second group contain no time derivatives and are referred to as constraints because they constrain the initial data for the evolution equations. If E_0 and B_0 are vector valued functions on $\mathbf{R}^3$ satisfying appropriate differentiability assumptions then there exists a unique solution of the evolution equations with $E(0,x^a) = E_0(x^a)$ and $B(0,x^a) = B_0(x^a)$. However this will not in general be a solution of all Maxwell equations. A necessary condition for all Maxwell equations to be satisfied is that the constraints be satisfied on the initial hypersurface $t = 0$. In fact it is also a sufficient condition because of the following fact: if the constraints are satisfied on the initial surface and the evolution equations are satisfied everywhere then the constraints are satisfied everywhere. This is usually described by saying that the constraints propagate. We will see that a similar phenomenon occurs in the case of the Einstein equations.

Next the initial value problem for the Yang-Mills equations will be looked at. This will illustrate another idea which is important for the Einstein equations, namely geometric uniqueness. This concept becomes necessary because of the gauge invariance of the equations. If A_α satisfies the Yang-Mills equations then so does $B_\alpha = g^{-1}A_\alpha g + g^{-1}\partial_\alpha g$ for any function g with values in $SU(2)$, the group of unitary two by two matrices with determinant one. This immediately shows that uniqueness in the Cauchy problem in the ordinary sense must fail for the Yang-Mills equations. For if A_α is a solution of the Yang-Mills equations with given initial data on $t = 0$ we need only choose some non-trivial function g which vanishes to sufficiently high order on the initial hypersurface. Then the gauge-tranformed solution B_α has the same initial data as A_α. The resolution of this problem is simple. In Yang-Mills theory gauge equivalent potentials, i.e. potentials related as A_α and B_α above, are just different mathematical descriptions of the same physical situation. Hence the natural formulation of existence and uniqueness in the Cauchy problem for the Yang-Mills equations is as follows. Let two $su(2)$-valued 1-forms $A_{0,\alpha}$ and $A_{1,\alpha}$ be given on $\mathbf{R}^3$ which satisfy the constraints. Then there exists a solution A_α of the Yang-Mills equations defined on a neighbourhood of the initial hypersurface with $A_\alpha(0,x^a) = A_{0,\alpha}(x^a)$ and $\partial_t A_\alpha(0,x^a) = A_{1,\alpha}(x^a)$. If B_α is any other solution with the same initial data then there exists a function g which is the identity on the initial hypersurface and which is such that B_α and A_α are related by the gauge transformation defined by g as above. There are two points here which require comment. Firstly, as implied by the formulation of this statement, the Yang-Mills equations have constraints similar to those of the Maxwell equations; they also propagate. Secondly, since the Yang-Mills equations are nonlinear the statement is only about local solutions.

In the above only the case has been discussed where data are given on the hypersurface $t = 0$ but actually data can be given on much more general hypersurfaces. The relevant condition is that the hypersurface should be spacelike i.e. $\eta_{\alpha\beta}x^\alpha x^\beta > 0$ for any non-zero vector x^α tangent to it. This requirement on the admissible initial hypersurfaces is expressed in terms of the non-dynamical Minkowski metric $\eta_{\alpha\beta}$. Since in the case of the Einstein equations the spacetime metric itself is an unknown in the equation it is clear that some care will be needed when formulating the corresponding condition. Furthermore, when solving the Einstein equations it is not appropriate to think of the spacetime manifold as something given in advance. In some sense this manifold is part of the solution. This fact together with the fact that geometric uniqueness is the appropriate uniqueness concept means that the Cauchy problem for the Einstein equations must be formulated in a somewhat different way from conventional Cauchy problems.

1.5 Local existence and uniqueness

It was already mentioned above that the Einstein equations are equations for a Lorentzian metric and it is appropriate at this point to recall some facts from Lorentzian geometry. A non-zero vector x^α on a manifold where a Lorentz metric $g_{\alpha\beta}$ is defined is called spacelike, null or timelike if $g_{\alpha\beta}x^\alpha x^\beta$ is positive, zero or negative respectively. A hypersurface is called spacelike if its normal vector is timelike. Equivalently, a hypersurface is spacelike if all vectors tangent to it are spacelike. A vector is called causal if it is timelike or null. A curve is called causal if its tangent vector is everywhere causal. This terminology is due to the fact that the causal curves are precisely those along which causal influences can propagate. A spacelike hypersurface is called a Cauchy surface if each inextendible causal curve hits it precisely once. It turns out that Cauchy surfaces are the correct places to give data for the Cauchy problem. Not every spacetime possesses a Cauchy surface. Those which do are called globally hyperbolic. A limitation of the applicability of the Cauchy problem in constructing spacetimes is that only globally hyperbolic spacetimes can be obtained in this way. This is, however, perhaps not so serious as it at first seems. If the strong cosmic censorship hypothesis is true then only globally hyperbolic spacetimes are physically significant; other spacetimes are then just mathematical artefacts.

If S is a spacelike hypersurface in a Lorentz manifold M with metric $g_{\alpha\beta}$ then the metric induces two geometrical objects on S. These are the induced metric and the second fundamental form. If u^α is a unit vector normal to S then define $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ and $k_{\alpha\beta} = \nabla_\gamma u_\delta h^\gamma_\alpha h^\delta_\beta$. It is a standard fact that $k_{\alpha\beta}$ is symmetric, $k_{\alpha\beta} = k_{\beta\alpha}$. Both of these objects have vanishing contractions with u^α and so they can be identified with tensors h_{ab} and k_{ab} intrinsic to S. The latter are the induced metric and second fundamental form of S; they constitute the initial data for the Einstein equations. Very roughly they correspond to the spatial components of the metric together with their time derivatives. The fact that no data have to be given for the other metric components has to do with the diffeomorphism invariance of the Einstein equations, as will be shown in more detail later. As has already been mentioned, data for the Einstein equations cannot be given freely; they must satisfy constraints. These constraints are

$$R - k_{ab}k^{ab} + (h^{ab}k_{ab})^2 = 2\rho \tag{10}$$

$$\nabla^b k_{ab} - \nabla_a(h^{bc}k_{bc}) = j_a \tag{11}$$

Here R is the scalar curvature of the metric h_{ab} and ρ and j_a are projections of the energy-momentum tensor. They are the energy density and matter current measured on the Cauchy surface. Since when solving the Einstein equations the manifold is part of the solution it is appropriate to use the concept of an abstract initial data set. An abstract initial data set consists of a 3-dimensional manifold S, a Riemannian (i.e. positive definite) metric h_{ab} on S, a symmetric tensor k_{ab} on S and some initial data for matter fields. This collection of objects is assumed to satisfy the constraint equations (10). Notice that S is not assumed to be a submanifold of a 4-dimensional manifold.

When solving the Cauchy problem for the Einstein equations it is necessary to establish a connection between the 3-dimensional manifold where the data live and the 4-dimensional manifold underlying the solution. This is accomplished by the following definition.

Definition Let an abstract initial data set for the Einstein equations coupled to a definite matter model be given. A *Cauchy development* of these data is an embedding ϕ of S as a Cauchy surface into a 4-dimensional manifold M on which are defined a globally hyperbolic Lorentz metric and matter fields satisfying the coupled Einstein and matter equations which induce on the image of S the correct induced metric, second fundamental form and matter data.

A Cauchy development ϕ is called maximal if it has the following property: given any other Cauchy development ϕ' of the same initial data there exists an isometry ψ of the second spacetime onto a subset of the first which preserves the matter fields and satisfies $\phi = \psi \circ \phi'$. Intuitively this says that any other Cauchy development is a subset of the original one, up to diffeomorphisms. The fundamental facts about the Cauchy problem for the Einstein equations coupled to appropriate matter and with appropriate assumptions on the differentiability of the initial data and solutions are:

1. (Existence) For every abstract initial data set there exists a maximal Cauchy development.

2. (Geometric uniqueness) If ϕ_1 and ϕ_2 are two maximal Cauchy developments of the same initial data then there exists a diffeomorphism ψ of the one spacetime onto the other which satisfies $\phi_2 = \psi \circ \phi_1$.

This statement contains nothing which would correspond intuitively to information about 'how big' the maximal Cauchy development is for given initial data; it is essentially a local statement. Note also that no assumptions were made about the manifold S or the asymptotic behaviour of h_{ab} and k_{ab}. Such assumptions, which incorporate the physically relevant boundary conditions, are important when it comes to looking at global questions. They are also important for the question of the existence of solutions of the constraint equations; the above statement is useless unless we know that abstract initial data sets exist in situations of interest.

There are two types of boundary conditions which are usually imposed on initial data sets. They correspond to two different physical situations. The first type may be called ‘cosmological boundary conditions‘. The physical picture in that case is that the solutions of the Einstein equations should describe the universe as a whole. One possible mathematical definition of cosmological boundary conditions is that the manifold S should be closed i.e. compact without boundary. The other kind of boundary conditions which are very important are the asymptotically flat ones. The physical picture in that case is that the solution of the Einstein equations should describe an isolated system, like the solar system. The idea is that outside a compact subset the topology of S should be that of $\mathbf{R}^3$ with a ball removed and that the metric h_{ab} looks like the flat Euclidean metric near infinity. It is also assumed that k_{ab} goes to zero at infinity and that the same is true of the matter variables.

One further concept which should be mentioned is that of the domain of dependence. Consider a spacetime with Cauchy surface S and let p be a point to the future of S (i.e. there exists a future-directed timelike curve from S to p). Define $I^-(p)$ to be the union of all past-directed timelike curves starting at p and let V be the part of $I^-(p)$ consisting of points on or in the future of S. Then it can be shown that $S \cap V$ is a Cauchy surface for V so that if the spacetime is a solution of the Einstein equations the restriction of the metric to V is determined uniquely (up to diffeomorphism) by the restriction of the initial data to $S \cap V$. By continuity the metric at p is also determined uniquely by this restriction of the data. Thus we see that the solution of the Einstein equations at p is determined by the initial data on a subset of S which is sometimes called the domain of dependence of p. This is typical of hyperbolic equations.

2 Solving the evolution equations

2.1 Local existence

The Einstein equations are equivalent to the equations

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}(g^{\gamma\delta} T_{\gamma\delta}) g_{\alpha\beta}, \tag{12}$$

where $R_{\alpha\beta}$ is the Ricci tensor of the Lorentz metric $g_{\alpha\beta}$. The properties of a system of differential equations are often determined by its principal part i.e. by the part containing the highest order derivatives. For the Einstein equation the principal part can be determined by looking at the part of the Ricci tensor containing second derivatives:

$$R_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}(\partial_\gamma \partial_\delta g_{\alpha\beta} + \partial_\alpha \partial_\beta g_{\gamma\delta} - \partial_\alpha \partial_\gamma g_{\beta\delta} - \partial_\beta \partial_\delta g_{\alpha\gamma}) + \ldots \tag{13}$$

This differential operator does not belong to any standard type such as elliptic, hyperbolic etc. We also know that the best that can be hoped for in the way of uniqueness is geometric uniqueness. These two problematic aspects of the situation can be taken care by one device. The idea is to choose preferred coordinates in any solution and so to freeze the diffeomorphism freedom. The standard choice, due to Choquet-Bruhat [1, 2] is harmonic coordinates. Note that, by definition coordinates on some region of spacetime are four scalar functions x^α whose derivatives are everywhere linearly independent. The harmonic condition is that in the given coordinates $\Gamma^\mu = 0$, where

$$\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu_{\beta\alpha} = g^{\mu\nu}g^{\alpha\beta}(\partial_\beta g_{\alpha\nu} - \frac{1}{2}\partial_\nu g_{\alpha\beta}) \tag{14}$$

Equivalently, the coordinates x^α should satisfy the wave equation with respect to the given metric. When such coordinates have been introduced the last three explicitly written terms in (13) drop out and the principal part becomes identical to that of the wave equation. The equations so obtained are hyperbolic and are called the reduced Einstein equations. They are equivalent to the full Einstein equations provided the coordinates are harmonic.

What about the existence of harmonic coordinates in a given spacetime? Since these coordinates are, in particular, functions which solve the wave equation they can be constructed by solving a Cauchy problem for that equation. Let S be a Cauchy surface (so we are assuming at this point that the spacetime is globally hyperbolic) and let the unit normal to S be denoted by u^α. The initial data can now be chosen so that $x^0 = 0$, $u^\alpha\partial_\alpha x^0 = 1$ and $u^\alpha\partial_\alpha x^a = 0$ there. Standard linear hyperbolic theory then gives the existence of functions x^α with these initial data satisfying the wave equation. The choice of initial data ensure that the derivatives of these functions are independent on the Cauchy surface and hence, by continuity, on a neighbourhood of it. In these coordinates $g_{00} = -1$ and $g_{0a} = 0$ on the initial hypersurface.

Now the logic of the proof of existence and (geometric) uniqueness for the Einstein equations can be presented. Let us start with an abstract initial data set. From this it is necessary to produce an initial data set for the reduced equations and there is some freedom in doing this due to the diffeomorphism invariance. One way of proceeding is as follows. First choose coordinates x^a on S. There are no global coordinates on S but it is sufficient to solve the Cauchy problem locally in space and then to piece together the local solutions using the domain of dependence mentioned at the end of section 1. Now identify the piece of S covered by a given coordinate patch with an open subset U of the hyperplane $t = 0$ in $\mathbf{R}^4$. The aim is to solve the reduced equations on some open neighbourhood of U in $\mathbf{R}^4$. At the moment however we do not have a complete initial data set for the reduced equations, but only g_{ab} and k_{ab}. The solution will be constructed in such a way that the Cartesian coordinates on $\mathbf{R}^4$ become harmonic coordinates of the kind discussed above for the solution. In particular $g_{00} = -1$ and $g_{0a} = 0$ on the initial hypersurface and this gives part of the remaining data needed for the reduced Einstein equations. It can be calculated that when the components $g_{0\alpha}$ of the metric have this form $\partial_t g_{ab} = -2k_{ab}$. This allows another part of the initial data for the reduced equations to be calculated. It remains to determine the time derivatives $\partial_t g_{0\alpha}$ on the initial surface. These are fixed by the equation $\Gamma^\alpha = 0$. In this way a full initial data set $g_{\alpha\beta}, \partial_t g_{\alpha\beta}$ for the reduced equations is obtained.

The reduced Einstein equations in vacuum form a hyperbolic system to which standard local existence and uniqueness results apply. A reasonable matter model will also be such that the local in time Cauchy problem can be solved in any given globally hyperbolic spacetime. When the Einstein equations are coupled to matter it remains to be checked that the local existence and uniqueness for the coupled system can also be proved by standard methods. Assuming that this is so for a given matter model, there exists a unique solution of the reduced equations coupled to the matter equations which has the given initial data. It is however not yet clear that the spacetime obtained in this way satisfies the Einstein equations. For the reduced Einstein equations and the full Einstein equations are only equivalent if the coordinates are harmonic i.e. if $\Gamma^\alpha = 0$ everywhere. At the moment we only know that this equation holds on the initial hypersurface and so another step is necessary in order to complete the existence proof. This step makes use of the fact that whenever the reduced equations are satisfied the quantities Γ^α satisfy a second order homogeneous linear hyperbolic equation. Uniqueness in the Cauchy problem for that equation shows that if Γ^α and $\partial_t\Gamma^\alpha$ vanish on the initial hypersurface then $\Gamma^\alpha = 0$ everywhere. The construction of the initial data for the reduced equations ensures that $\Gamma^\alpha = 0$ on the initial surface. The reduced equations and the constraints together imply that $\partial_t\Gamma^\alpha = 0$ there. Thus the fact that the initial data satisfy the constraints ensures that the solution of the reduced equations which has been constructed is actually a solution of the Einstein equations. This completes the sketch of the existence part of the proof.

I do not want to discuss the uniqueness proof in any detail but just to mention the basic idea. Suppose we have two Cauchy developments of the same initial data. Choose some coordinates on the manifold S. Based on these we can uniquely construct harmonic coordinates as indicated above in each of the two spacetimes. Call them x^α and $\bar{x}^\alpha$. Define a mapping ψ from one spacetime to the other by the condition that $\bar{x}^\alpha = \psi \circ x^\alpha$. Then the first metric $g_{\alpha\beta}$ and the metric obtained by pulling back the second metric $g'_{\alpha\beta}$ with ψ both solve the reduced equations and induce the same data on the initial hypersurface. Thus, by uniqueness in the Cauchy problem for the reduced equations, they must be equal. This gives geometric uniqueness.

2.2 Remarks on the 3+1 decomposition

The 3+1 decomposition was already mentioned above in the context of the Maxwell equations. There a standard time coordinate in Minkowski space was used for this purpose. In the case of the Einstein equations there are no preferred coordinates and so the 3+1 decomposition (also known as the ADM decomposition in that context) involves considerations about the choice of coordinates. The last section showed that the Cauchy problem for the Einstein equations can be treated without ever introducing a 3+1 decomposition. However in most work on solving the Einstein equations numerically it is unavoidable.

Choosing a time coordinate in general relativity can be broken down into two steps. The first is to choose a foliation of part of spacetime by spacelike hypersurfaces and the second is to choose a labeling of these hypersurfaces. When this has been done a function t is obtained whose value on a given hypersurface is the real number chosen to label that hypersurface. When a foliation has been chosen there is an induced metric and second fundamental form for each leaf but there is no way of comparing these objects on different leaves without introducing extra structure. If l is the length of dt then the lapse function α is defined to be l^{-1}. In coordinates $\alpha = (-g^{00})^{-1/2}$. If the labeling is changed by replacing t with a smooth monotone function of t then α is multiplied by the derivative of this function. In particular the restriction of α to one hypersurface is multiplied by a constant. This shows to what extent α depends on t and to what extent it only depends on the foliation. The component n_0 is equal to $-\alpha$.

Hence $\rho = \alpha^2 T^{00}$ and $j_a = \alpha T^0_a$.

Once a time coordinate (or just the corresponding foliation) has been fixed choosing a space coordinate can be broken down into two steps. The first is to specify which points on different hypersurfaces are to have the same spatial coordinates. This is equivalent to specifying a congruence of curves transverse to the foliation. The second is to label these curves. This can be done by specifying a coordinate system on one leaf of the foliation (which in general will only be possible locally on the leaf). Given a fixed time coordinate t, a preferred parametrisation of the congruence of curves is fixed. Let t^α be the field of tangent vectors to the curves when they are parametrised in this way. The shift vector β^a is the image of t^α under the projection h^α_β. It depends genuinely on the time coordinate and not only on the foliation which the time coordinate defines. A quantity which depends only on the foliation is $\alpha^{-1}\beta^a$.

References

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