

On the Statistics of Dimension: Multi-fractals and Fractal Modulation

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We consider a new approach to modelling a non-stationary or multi-fractal stochastic field $u(x, t)$ using a fractional partial differential equation of (time dependent) order $q(t)$ given by

$$\left[\frac{\partial^2}{\partial x^2} - \tau^{q(t)} \frac{\partial^{q(t)}}{\partial t^{q(t)}} \right] u(x, t) = -F(x, t), h - \infty < q(t) < \infty, h \forall t$$

where τ is a constant and F and q are stochastic functions. The theoretical background and ideas upon which this equation is based are presented. This includes a brief overview of random fractal walks, Lévy flights, fractional calculus and fractional dynamics. A general solution to this equation is then considered using a Green's function method from which an asymptotic solution is derived for $x \rightarrow 0$. Numerical algorithms for computing this solution are introduced and results presented to illustrate its characteristics which depend on the random behaviour of $q(t)$. One important and interesting aspect of this approach is concerned with the effect of changing the statistics [i.e. the Probability Density Function (PDF) of $q(t)$] used to 'drive' the solution. Since q is a dimension (the 'Fourier dimension' which is related to the fractal dimension), the introduction of a PDF associated with $q(t)$ leads directly to the notion of the 'statistics of dimension'. In the context of $q(t)$ being a random variable, the asymptotic solution for $u(x, t)$ when $x \rightarrow 0$ yields special behaviour when $q = 2$. This is characterised by events that are analogous to Lévy-type flights which we call 'Brownian transients'. We also address the inverse problem in which a discrete solution is found for $q(t)$ from a known stochastic field $u(x, t)$ when $x \rightarrow 0$. A single application is considered - Fractal Modulation - where $q(t)$ is assigned just two states for all t . This approach is analogous to Frequency Modulation but where a bit stream is forced to 'look like' background fractal noise for applications in (covert) digital (multimedia or otherwise) communication systems.