

Nonlinear Hydrodynamic Description of non-Newtonian Fluids

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I. STRAIN TENSOR DESCRIPTION

In this section we review the hydrodynamic description [1] of nonlinear elasticity and its generalization to viscoelasticity in terms of the Eulerian strain tensor U_{ij} .

Definitions: \mathbf{a} is the (initial) coordinate of a point in a stress-free elastic body, while \mathbf{r} is that coordinate if the body is displaced, rotated, compressed and sheared.

Eulerian description: $\mathbf{a}(\mathbf{r})$ leading to the

Eulerian strain tensor $U_{ij} = \frac{1}{2}[\delta_{ij} - (\nabla_j a_k)(\nabla_i a_k)]$, which can be written in the more familiar form $U_{ij} = \frac{1}{2}[\nabla_j u_i + \nabla_i u_j - (\nabla_i u_k)(\nabla_j u_k)]$ using the displacement field $u_i(\mathbf{r}) = r_i - a_i(\mathbf{r})$.

Since the dynamics for \mathbf{a} is simply $d\mathbf{a}/dt = 0$ in the absence of phenomenological currents, the dynamics for the Eulerian strain tensor reads [2, 3]

$$\frac{d}{dt}U_{ij} - A_{ij} + U_{ki}\nabla_j v_k + U_{kj}\nabla_i v_k = -\alpha_{ijkl}\Psi_{kl} \quad (1)$$

with $d/dt \equiv \partial/\partial t + \mathbf{v} \cdot \nabla$, where v_i is the velocity and $A_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$ its symmetrized gradient describing deformational flow.

This equation contains, apart from the transport derivative $\sim \mathbf{v} \cdot \nabla$ due to Galilean translational invariance, a linear and a nonlinear coupling to flow. The former ($\sim A_{ij}$) reflects the spontaneously broken translational symmetry in elastic media (in an equation for the displacement vector the equivalent term subtracts the solid body translations, which must not change the elastic state of the fluid). The nonlinear ones are of the lower convected type and result from the freedom to choose the orientation of a material-fixed frame independently from that of the laboratory frame. For a Lagrangian strain tensor the upper convected derivative would be obtained. On the right hand side there is the phenomenological relaxation $-\alpha_{ijkl}\Psi_{kl}$, describing viscoelasticity. Here Ψ_{ij} is the *elastic* stress tensor, the thermodynamic conjugate to the strain tensor. The strain tensor U_{ij} is symmetric, in order to exclude solid body rotations, since the latter must not change the elastic state.

The second relevant dynamic equation is the momentum balance or generalized Navier-Stokes equation

$$\rho \frac{d}{dt}v_i + \nabla_i p + \nabla_j (-\Psi_{ij} + \Psi_{ki}U_{jk} + \Psi_{kj}U_{ik} - \nu_{ijkl}A_{kl}) = 0 \quad (2)$$

with the isotropic pressure p . The linear and nonlinear contributions of the *elastic* stress tensor Ψ_{ij} are the counterterms to the linear and nonlinear flow contributions in Eq. (1) that are necessary to cancel any contribution to the entropy production, since all these terms are reversible. Thus, these parts of the stress tensor are completely fixed by general physical principles. The phenomenological part of the stress tensor is $-\nu_{ijkl}A_{kl}$ and describes in the simplest form Newtonian viscosity. The stress tensor is symmetric (or can be made so) in order to guarantee angular momentum conservation.

We assume a linear relation between the "forces" and the "fluxes" (linear irreversible thermodynamics). By allowing nonlinear dependences of the "fluxes" on the "forces", one would leave the solid grounds of well-established statistical physics, since not very much is known on the validity range of such theories. Nevertheless, these contributions are nonlinear in the sense that the material tensors depend on the variables of the systems, in particular on temperature T , pressure and the strain tensor. The dependence on the scalar quantities is rather trivial and will not be shown explicitly, while the strain tensor dependence is more complicated. Starting from an equilibrium state $U_{ij} = 0$ we can expand the material tensors into powers of U_{ij} . Up to quadratic order in Eqs.(1,2), which is what we need for the comparison with traditional constitutive equations, below, we find for the general form of the rank 4 material tensors

$$\begin{aligned} \alpha_{ijkl} = & \frac{\alpha_1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\alpha_2}{2}(U_{ik}\delta_{jl} + U_{jk}\delta_{il} + U_{il}\delta_{jk} + U_{jl}\delta_{ik}) + \alpha_3\delta_{ij}\delta_{kl} + \\ & \alpha_4(\delta_{ij}U_{kl} + \delta_{kl}U_{ij}) + \alpha_5\delta_{ij}\delta_{kl}U_{pp} + \alpha_6(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})U_{pp} + O(3) \end{aligned} \quad (3)$$

For the viscosity tensor ν_{ijkl} the form (3) can be simplified, since for incompressibility $A_{kk} = 0$ and only the coefficients $\nu_{1,2,6}$ appear in Eq.(2). Note that neither U_{ij} , nor Ψ_{ij} , nor the total stress tensor σ_{ij} have to be traceless, despite A_{ij} being so. If desired, the expansion (3) can easily be continued to arbitrary order.

The static phenomenological properties are expressed by a relation between the strain and the elastic stress and derived from an energy functional

$$\Psi_{ij} \equiv \delta \int \epsilon dV / \delta U_{ij} = K_1 U_{ij} + 2K_2 U_{ik} U_{jk} + K_3 \delta_{ij} U_{kk} + K_4 (\delta_{ij} U_{kl} U_{kl} + 2U_{ij} U_{kk}) + K_5 \delta_{ij} U_{kk} U_{ll} + O(3) \quad (4)$$

where again a power series expansion is used. There are two linear elastic moduli $K_{1,3}$ and three quadratic ones $K_{2,4,5}$ involved.

II. ORIENTATIONAL ORDER PARAMETER TENSOR DESCRIPTION

In this section we review the hydrodynamic description of transient orientational order in terms of the order parameter tensor Q_{ij} .

Definitions: Q_{ij} is the orientational order parameter tensor, symmetric, traceless and of second rank ($Q_{ij} = Q_{ji}$ and $Q_{ii} = 0$). In contrast to the case of a nematic phase with spontaneous and permanent orientational order, there is no nematic order in equilibrium and a director does not exist. The dynamics of a transient Q_{ij} can be written as [4]

$$\dot{Q}_{ij} + v_k \nabla_k Q_{ij} + Q_{jk} \Omega_{ki} + Q_{ik} \Omega_{kj} - \lambda_{ijkl} A_{kl} = -\alpha_{ijkl} \psi_{kl} \quad (5)$$

with $2A_{ij} = \nabla_j v_i + \nabla_i v_j$ and $2\Omega_{ij} = \nabla_j v_i - \nabla_i v_j$ the symmetric and antisymmetric velocity gradients characterizing deformational and rotational flow, respectively. The nonlinear reversible coupling terms to flow are a priori of the corotational or Jaumann derivative type (containing only Ω_{ij} the rotational flow – suitably for the orientational order involved), but there is in addition a phenomenological reversible coupling to symmetric velocity gradients that makes the effective convective derivative material dependent. The phenomenological material tensor λ_{ijkl} (a kind of generalized flow alignment tensor) is given as a power series expansion in Q_{ij}

$$\begin{aligned} \lambda_{ijkl} = & \lambda_1 (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il} - \frac{2}{3} \delta_{ij} \delta_{kl}) + \lambda_3 \delta_{kl} Q_{ij} \\ & + \lambda_2 (\delta_{ik} Q_{jl} + \delta_{jk} Q_{il} + \delta_{jl} Q_{ik} + \delta_{il} Q_{jk} - \frac{4}{3} \delta_{ij} Q_{kl}) + O(2) \end{aligned} \quad (6)$$

where higher order terms $O(2)$ are not needed here. It contains one phenomenological, material dependent, reversible reactive coefficient in linear, and two additional ones in quadratic order. If the Jaumann terms are combined with the quadratic contribution (6) for the special value $\lambda_2 = \frac{1}{2}$ ($= -\frac{1}{2}$) one gets something that looks like an upper (lower) convected derivative – with some additional correction terms that ensure $\dot{Q}_{ii} = 0$. However, there is no general reason why such a relation should hold for all different materials nor can it hold for all temperatures and pressures, since $\lambda_{1,2,3}$ generally depend on all scalar state variables, like ρ , σ (or p , T) and on the invariants $Q_{ij} Q_{ij}$ and $Q_{ij} Q_{jk} Q_{ki}$. Within the quadratic approximation used here, the latter dependencies do not show up.

In Eq.(5) the relaxation of Q_{ij} is described by a dissipative material tensor α_{ijkl} that can be written as a power series expansion in Q_{ij} (with $\alpha_{iikl} = 0 = \alpha_{ijkk}$)

$$\alpha_{ijkl} = \alpha_1 (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il} - \frac{2}{3} \delta_{ij} \delta_{kl}) + \alpha_2 (\delta_{ik} Q_{jl} + \delta_{jk} Q_{il} + \delta_{jl} Q_{ik} + \delta_{il} Q_{jk} - \frac{4}{3} [\delta_{ij} Q_{kl} + \delta_{kl} Q_{ij}]) + O(2) \quad (7)$$

with the relaxation parameters $\alpha_{1,2}$ being functions of the scalar state variables. It should be noted that we stay very well inside the framework of "linear irreversible thermodynamics" that has a solid foundation in statistical mechanics, although the expressions (6,7) are genuinely nonlinear due to the dependence on state variables.

The orientational elastic stress tensor ψ_{kl} is defined as the thermodynamic conjugate to Q_{ij} . It has to be taken as symmetric and traceless, since only that part enters the Gibbs relation and has a physical meaning. It is derived from an energy functional and is given in quadratic order by

$$\psi_{ij} \equiv \delta \int \epsilon dV / \delta Q_{ij} = c_1 Q_{ij} + c_2 (Q_{ik} Q_{jk} - \frac{1}{3} \delta_{ij} Q_{kl} Q_{kl}) + O(2) \quad (8)$$

neglecting gradient terms. Near the nematic phase transition the rotational elastic moduli c_1, c_2 can be interpreted as Landau parameters and the expansion usually is driven to higher powers.

For the momentum balance or generalized Navier-Stokes equation this leads to

$$\rho \frac{d}{dt} v_i + \nabla_i p - \nabla_j (\lambda_{kl ij} \psi_{kl} + \nu_{ijkl} A_{kl}) = 0 \quad (9)$$

The counter term to the linear deformational flow term in (5), $\sim \lambda_{kl ij}$, leads to a symmetric part of the stress tensor, while there are no counter terms to the nonlinear Jaumann terms, since the latter do not at all contribute to the entropy production due to $\psi_{ik} Q_{jk} = \psi_{jk} Q_{ik}$.

In the incompressible limit the viscosity tensor, expanded in Q_{ij} , reads

$$\nu_{ijkl} = \frac{\nu_1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \frac{\nu_2}{2} (Q_{ik} \delta_{jl} + Q_{jk} \delta_{il} + Q_{il} \delta_{jk} + Q_{jl} \delta_{ik}) \quad (10)$$

with the viscosities $\nu_{1,2}$ generally being functions of the scalar state variables.

III. GENERAL FORM OF AN EFFECTIVE STRESS TENSOR DESCRIPTION [CONSTITUTIVE EQUATION]

In the preceding pages viscoelastic hydrodynamics was expressed either by the strain tensor or by the orientational order parameter tensor. We will now rewrite these equations with the stress tensor as variable by replacing the strain tensor or the orientational order parameter tensor (and their time derivative) by the stress tensor (and its time derivative). This can only be done in an approximate way, in the form of a power series up to quadratic order, since the equations are nonlinear. Of course, the resulting equations are less general than the starting ones and only applicable, if quadratic nonlinearities are sufficient for the problem at hand. The general form for the constitutive equations is [5, 6]

$$\tau_1 \frac{D_a}{Dt} \sigma_{ij} + \sigma_{ij} = -\nu_\infty A_{ij} - \nu_1 \tau_1 \frac{D_b}{Dt} A_{ij} + \frac{1}{3} \delta_{ij} \Sigma + \frac{r}{K} \sigma_{ik} \sigma_{jk} + \frac{\tau_1 \nu_2}{K} \left([\sigma_{jk} + \nu_1 A_{jk}] \frac{\partial}{\partial t} A_{ik} + [\sigma_{ik} + \nu_1 A_{ik}] \frac{\partial}{\partial t} A_{jk} \right) + O(3) \quad (11)$$

with the definition

$$\frac{D_a}{Dt} T_{ij} \equiv \frac{d}{dt} T_{ij} - a(T_{ik} A_{jk} + T_{jk} A_{ik}) - (T_{ik} \Omega_{jk} + T_{jk} \Omega_{ik}) \quad (12)$$

for any tensor T_{ij} and number a . For $a = -1$ ($a = +1$) D_a/Dt is the lower (upper) convected derivative, for $a = 0$ the Jaumann or corotational derivative, while for a general a a linear combination of those is invoked.

For the two cases of relaxing elasticity (U) and transient orientational order (Q) the parameters involved in the effective constitutive Eq.(11) are related to the original hydrodynamic parameters (viscosity, elasticity, relaxation, flow alignment) by

$$\frac{1}{\tau_1} \frac{U}{=} \alpha_1 K_1 \quad \frac{Q}{=} 2c_1 \alpha_1 \quad \nu_\infty \frac{U}{=} \nu_1 + \tau_1 K_1 \quad \frac{Q}{=} \nu_1 + 4c_1 \tau_1 \lambda_1^2 \quad (13)$$

$$\frac{1}{\tau_2} \frac{U}{=} 2\alpha_1 K_2 + 2\alpha_2 K_1 \quad \frac{Q}{=} 2c_2 \alpha_1 + 4c_1 \alpha_2 \quad r \frac{U}{=} \frac{\tau_1}{\tau_2} - \frac{K'_2}{K_1} \quad \frac{Q}{=} \frac{\tau_1}{\tau_2} + \frac{c_2}{c_1} + 2 \frac{\lambda_2}{\lambda_1} \quad (14)$$

$$a \frac{U}{=} -1 + \frac{\nu_1}{K_1 \tau_2} - \frac{K'_2}{K_1^2} \frac{\nu_\infty}{\tau_1} \frac{Q}{=} -2\lambda_1 \left(\frac{c_2}{c_1} + 3 \frac{\lambda_2}{\lambda_1} \right) - \frac{\nu_1 r}{2c_1 \lambda_1 \tau_1} \quad b \frac{U}{=} -1 + \frac{\nu_1}{2K_1 \tau_2} - \frac{K'_2}{2K_1^2} \frac{\nu_\infty}{\tau_1} \frac{Q}{=} -2\lambda_1 \left(\frac{c_2}{c_1} + 3 \frac{\lambda_2}{\lambda_1} \right) - \frac{\nu_1 r}{4c_1 \lambda_1 \tau_1} \quad (15)$$

with $K'_2 = 2K_1 - 2K_2$ and $K_1 \stackrel{U}{=} K \stackrel{Q}{=} 2c_1 \lambda_1$. The isotropic part Σ , which can be interpreted as an addition to the thermodynamic pressure, is zero in the U case and $\Sigma \stackrel{Q}{=} \sigma_{kk} + x A_{kl} A_{kl} + y \sigma_{kl} \sigma_{kl} + z \sigma_{kl} A_{kl}$ in the Q case (for the definition of x, y, z cf. [6]). In the U case the trace-related coefficients $\alpha_{3,4,5,6}$, ν_6 , and $K_{3,4,5}$ have been neglected.

Splitting Eq.(11) into the trace and the traceless part, it is easy to see that σ_{kk} and $(d/dt)\sigma_{kk}$ are of $O(2)$, since $A_{kk} = 0$, and do not influence the dynamics of the traceless part σ_{ij}^0 up to second order. For the trace σ_{kk} the relaxation equation reads

$$\tau_1 \frac{\partial}{\partial t} \sigma_{kk} + B \sigma_{kk} = \left(\frac{r}{K} + y \right) \sigma_{kl}^0 \sigma_{kl}^0 + (2\nu_1 \tau_1 b + x) A_{kl} A_{kl} + (2\tau_1 a + z) \sigma_{kl}^0 A_{kl} + \frac{2\nu_2 \tau_1}{K} (\nu_1 A_{kl} + \sigma_{kl}^0) \frac{\partial}{\partial t} A_{kl} + O(3) \quad (16)$$

and is completely determined by σ_{ij}^0 and A_{ij} in lowest order. Here, $1 \stackrel{U}{=} B \stackrel{Q}{=} 0$ and x, y, z are zero in the U case, anyhow.

For a comparison with traditional constitutive equations cf. [5, 6].

IV. TWO-FLUID DESCRIPTION OF [VISCO]ELASTIC MEDIA

Two-fluid models have been employed in many different physical contexts, e.g. flow of bubbly liquids, fluid suspensions of particulates, and binary mixtures of simple fluids. Other examples in condensed matter physics include two-fluid models for superfluid helium, dynamics of plasmas, transport in superconductors, viscoelasticity of concentrated fluid emulsions, flow-induced ordering of wormlike micelle solutions, flow of colloidal suspensions. Two-fluid models have been used extensively to model a wide range of dynamical phenomena in polymer solutions and binary blends, including the hydrodynamic modes of quiescent polymer solutions, kinetics of polymer dissolution, hydrodynamics and rheology of polymer solutions and blends, and polymer migration and phase separation under flow.

As a general feature, two distinct species or coexisting phases with mass densities ρ_1 and ρ_2 ($\rho \equiv \rho_1 = \rho_2$ and $\phi \equiv \rho_1/\rho$), which are conserved individually in the absence of chemical reactions, move with distinct velocities \mathbf{v}_1 and \mathbf{v}_2 , respectively. Due to (usually strong) internal friction, the momenta of the constituent species, $\rho_1 \mathbf{v}_1 \equiv \mathbf{g}_1$ and $\rho_2 \mathbf{v}_2 \equiv \mathbf{g}_2$, are not conserved individually. Of course, total momentum is conserved. In most cases of fluid mixtures the friction is so strong that the velocity difference $\mathbf{v}_1 - \mathbf{v}_2 \equiv \mathbf{w}$ is nonzero for very short times only, i.e. it is a very rapidly relaxing quantity that is not included in the hydrodynamic description for binary mixtures. However, there are systems and situations, where the relaxation of the relative momenta is slow enough to have a significant influence even on the hydrodynamic time scale. Then a two-fluid description is appropriate and useful.

In this section we focus on a nonlinear two-fluid description of complex fluids, where one species is a viscous Newtonian fluid and the other a polymer or elastomer. The Gibbs relation

$$d\epsilon - Td\sigma - \Phi_{ij} dU_{ij} = \mu_1 d\rho_1 + \bar{\mu}_2 d\rho_2 + \mathbf{v}_1 \cdot d\mathbf{g}_1 + \mathbf{v}_2 \cdot d\mathbf{g}_2 = \Pi' d\phi + \mu d\rho + \mathbf{v} \cdot d\mathbf{g} + \mathbf{m} \cdot d\mathbf{w} \quad (17)$$

contains the variables and defines the conjugates (e.g. $\mathbf{v} = \phi \mathbf{v}_1 + (1 - \phi) \mathbf{v}_2 = \rho^{-1}(\mathbf{g}_1 + \mathbf{g}_2)$ and $\mathbf{m} = \rho(1 - \phi)\phi \mathbf{w} = (\rho_2 \mathbf{g}_1 - \rho_1 \mathbf{g}_2)\rho^{-1}$). Here we want to display explicitly 2-fluid hydrodynamics under the following assumptions:

- a) convection with natural velocities (for U_{ij} , \mathbf{g}_2 , ρ_2 and \mathbf{g}_1 , ρ_1 this is \mathbf{v}_2 and \mathbf{v}_1 , respectively);
- b) the linearized elastic force acts on the elastomeric fluid (index 2) only;
- c) global incompressibility, $\delta\rho = 0$ (i.e. $\delta\rho_1 = -\delta\rho_2$); and
- d) linearizing the phenomenological dissipative currents, but keeping quadratic nonlinearities otherwise. Then the following set of equations is obtained:

The incompressibility condition (in 3 equivalent versions),

$$0 = \text{div } \mathbf{v} = \mathbf{w} \cdot \nabla \rho_1 + \rho_1 \text{div } \mathbf{v}_1 + \rho_2 \text{div } \mathbf{v}_2 = \mathbf{w} \cdot \nabla \phi + \phi \text{div}(1 - \phi)\mathbf{w} - (1 - \phi) \text{div} \phi \mathbf{w} \quad (18)$$

the concentration dynamics (in 3 equivalent versions),

$$\dot{\phi} + \nabla_i (\phi v_i + \phi(1 - \phi)w_i) - d_{ij} \nabla_i \nabla_j (\mu_1 - \bar{\mu}_2) - \phi(1 - \phi) d_{ij}^{(T)} \nabla_j \nabla_i T = 0 \quad (19)$$

$$\dot{\rho}_1 + \mathbf{v}_1 \cdot \nabla \rho_1 + \rho_1 \text{div} \mathbf{v}_1 - \rho d_{ij} \nabla_i \nabla_j (\mu_1 - \bar{\mu}_2) - \frac{\rho_1 \rho_2}{\rho} d_{ij}^{(T)} \nabla_i \nabla_j T = 0 \quad (20)$$

$$\dot{\rho}_2 + \mathbf{v}_2 \cdot \nabla \rho_2 + \rho_2 \text{div} \mathbf{v}_2 + \rho d_{ij} \nabla_i \nabla_j (\mu_1 - \bar{\mu}_2) + \frac{\rho_1 \rho_2}{\rho} d_{ij}^{(T)} \nabla_i \nabla_j T = 0 \quad (21)$$

the entropy dynamics (heat conduction equation),

$$\dot{\sigma} + v_i \nabla_i \sigma + \frac{\beta}{\rho} \nabla_i (\rho_1 \rho_2 w_i) - \kappa_{ij} \nabla_i \nabla_j T - \frac{\rho_1 \rho_2}{\rho} d_{ij}^{(T)} \nabla_i \nabla_j (\mu_1 - \bar{\mu}_2) = 0 \quad (22)$$

and the elasticity dynamics

$$\dot{U}_{ij} + v_k^{(2)} \nabla_k U_{ij} - \frac{1}{2} (\nabla_j v_i^{(2)} + \nabla_i v_j^{(2)}) - \frac{\rho_1}{2} (w_i \nabla_j + w_j \nabla_i) \ln \frac{\rho_2}{\rho} + U_{ki} \nabla_j v_k^{(2)} + U_{kj} \nabla_i v_k^{(2)} + \zeta_l \delta_{ij} \Phi_{kk} \quad (23)$$

$$+ \zeta_{tr} (\Phi_{ij} - \frac{1}{3} \delta_{ij} \Phi_{kk}) - \xi_1 \delta_{ij} \Delta \Phi_{kk} - \xi_2 \Delta \Phi_{ij} - \xi_3 (\nabla_i \nabla_j \Phi_{kk} + \delta_{ij} \nabla_k \nabla_l \Phi_{kl}) - \xi_4 (\nabla_i \nabla_k \Phi_{jk} + \nabla_j \nabla_k \Phi_{ik}) = 0$$

For the momentum balance of the two different species we get

$$\rho_1 \dot{v}_i^{(1)} + \rho_1 v_j^{(1)} \nabla_j v_i^{(1)} + \frac{\rho_1}{\rho} \nabla_i (p + \frac{1}{2} \rho_2 w_j (v_j^1 + v_j^2)) + \frac{\rho_1 \rho_2}{\rho} \nabla_i (\mu_1 - \bar{\mu}_2) + \frac{\rho_1}{\rho} \Phi_{kj} \nabla_i U_{kj} - \rho_1 \Phi_{ij} \nabla_j \ln \frac{\rho_2}{\rho} + \frac{\rho_1 \rho_2}{\rho} \beta \nabla_i T + \xi \rho_1 \rho_2 w_i - \nu_{ijkl}^{(1)} \nabla_j \nabla_l v_k^{(1)} - \nu_{ijkl}^{(2)} \nabla_j \nabla_l v_k^{(2)} = 0 \quad (24)$$

$$\rho_2 \dot{v}_i^{(2)} + \rho_2 v_j^{(2)} \nabla_j v_i^{(2)} + \frac{\rho_2}{\rho} \nabla_i (p - \frac{1}{2} \rho_1 w_j (v_j^1 + v_j^2)) - \frac{\rho_1 \rho_2}{\rho} \nabla_i (\mu_1 - \bar{\mu}_2) - \frac{\rho_1}{\rho} \Phi_{kj} \nabla_i U_{kj} + \rho_1 \Phi_{ij} \nabla_j \ln \frac{\rho_2}{\rho} - \frac{\rho_1 \rho_2}{\rho} \beta \nabla_i T - \nabla_j \Phi_{ij} + \nabla_j (\Phi_{jk} U_{ik} + \Phi_{ik} U_{jk}) - \xi \rho_1 \rho_2 w_i - \nu_{ijkl}^{(2)} \nabla_j \nabla_l v_k^{(2)} - \nu_{ijkl}^{(1)} \nabla_j \nabla_l v_k^{(1)} = 0 \quad (25)$$

The ξ -terms describe the coupling of the the two momenta due to the difference of the two velocities (friction). Note that although we made the approximation that the linear elastic stress does only act on fluid 2, there are inevitably nonlinear contributions to fluid 1, too. There is also a (nonlinear) coupling of fluid 1 to the concentration, if elastic distortions are present.

We also give eqs.(24,25) as dynamic equations for the total momentum and for the relative velocity

$$\rho \dot{v}_i + \nabla_i p + \nabla_j \left(\rho v_i v_j + \frac{\rho_1 \rho_2}{\rho} w_i w_j \right) + 2 \nabla_j (\Phi_{jk} U_{ik}) - \nabla_j \Phi_{ij} - \nu_{ijkl} \nabla_j \nabla_l v_k - \frac{\rho_1 \rho_2}{\rho} \nu_{ijkl}^{(c)} \nabla_j \nabla_l w_k = 0 \quad (26)$$

$$\begin{aligned} \dot{w}_i + \left(v_j + \frac{\rho_2 - \rho_1}{\rho} w_j \right) \nabla_j w_i + \rho \xi w_i + \nabla_i \left(\mu_1 - \bar{\mu}_2 + \mathbf{v} \cdot \mathbf{w} + \frac{\rho_2 - \rho_1}{2\rho} \mathbf{w}^2 \right) + \nabla_j \frac{1}{\rho_2} \Phi_{ij} \\ + \frac{1}{\rho_2} \Phi_{kj} \nabla_i U_{kj} - \frac{2}{\rho_2} \nabla_j (\Phi_{kj} U_{ik}) - \frac{\rho_1 \rho_2}{\rho} \nu_{ijkl}^{(m)} \nabla_l \nabla_j w_k - \nu_{ijkl}^{(c)} \nabla_j \nabla_l v_k = 0 \end{aligned} \quad (27)$$

Convective velocities: A more general theory avoiding the assumptions listed above. has been set up recently [7]. The comparison with the general theory reveals implicit and explicit assumptions, approximations and possible generalizations of a given model. A “simple” or “natural” choice in a given model may not be mandatory, but rather imply a presumption.

Quite generally we find that neither the velocity, with which a certain variable is convected, nor the stress division between the different fluids can be determined by general principles, but is rather system or material dependent. On the other hand, there are certain restrictions and interrelations among the convective velocities and other physical effects that limit the possible choices. For the two densities ρ_1, ρ_2 e.g., the “natural” choice for the convection velocities (taken in the equations above) seems to be their native velocities \mathbf{v}_1 and \mathbf{v}_2 , respectively. This implies that the total density is convected with the mean velocity \mathbf{v} (as required by mass transport, but manifest only, if the incompressibility assumption is lifted), while the concentration ϕ is convected with $(1/\rho)(\rho_2 \mathbf{v}_1 + \rho_1 \mathbf{v}_2)$. Another obvious (“simple”) choice would be the mean velocity as convection velocity for both, the total density as well as the concentration implying that also ρ_1 and ρ_2 are convected with \mathbf{v} . However, the actual convection velocity depends on the value of a material dependent (reactive) flow parameter (call it γ) and is not restricted to the two choices mentioned above ($\gamma = 0$ and $= -1$ for the “natural” and the “simple” one).

In the case of visco-elastic and elastic media, which are described by a dynamic equation for the (Eulerian) strain tensor U_{ij} , there are two velocities involved. One is the usual convection velocity ($v_k \nabla_k U_{ij}$) and the other one occurs in the “lower convected” part ($U_{kj} \nabla_i v_k + U_{ki} \nabla_j v_k$). There is no fundamental reason for the two to be equal, nor to be one of the obvious choices (\mathbf{v} or \mathbf{v}_2).

In the equations above it was assumed that (linear) elastic stress is carried only by fluid 2. Generally, however, the distribution of the elastic stress among the two fluids is governed by a phenomenological coefficient (call it $\lambda^{(U)}$). For, respectively, $2\lambda^{(U)} = 1/\rho_2, = 0$, or $= -1/\rho_1$, the elastic stress is carried only by fluid 2, is equally distributed between 1 and 2, or carried only by fluid 1; but $\lambda^{(U)}$ can have any value in between.

For the evolution equations of the momenta special care has to be taken to get a description, which is compatible with general laws. The currents and quasi-currents that enter the description in terms of either the total momentum and the velocity difference or in terms of the two individual momenta are not the same comparing eqs.(24, 25) with (26, 27). In particular, the momenta $\mathbf{g}_{1,2}$ are convected with $\mathbf{v}_{1,2}$ implying that the total momentum \mathbf{g} and the relative velocity \mathbf{w} are convected with \mathbf{v} and $(1/\rho)(\rho_2 \mathbf{v}_1 + \rho_1 \mathbf{v}_2)$, respectively. Instead of this “natural” choice there are other possibilities governed by some phenomenological parameters (independent from γ), e.g. the “simple” choice that all 4 quantities are convected with \mathbf{v} .

A prominent feature of the 2-fluid description is the coupling of the concentration dynamics to the velocity difference. Linearizing and Fourier transforming the dynamic equations, thus eliminating \mathbf{w} from e.g. the concentration dynamics and neglecting fourth order gradient terms we get

$$i\omega\phi - d^{eff} \Delta \Pi - \frac{\rho_1 \rho_2}{\rho^2} d^{(T)eff} \Delta T - 2\lambda^{(\phi)} \nabla_i \nabla_j \Phi_{ij} = 0 \quad (28)$$

with frequency dependent effective diffusion and thermo-diffusion coefficients

$$d^{eff} = d + \frac{\rho_1 \rho_2}{\rho^2} \frac{(\gamma + 1)^2}{\rho \xi + i\omega} \quad \text{and} \quad d^{(T)eff} = d^{(T)} + \frac{\beta(\gamma + 1)}{\rho \xi + i\omega} \quad (29)$$

and the dynamic coupling to the elastic degree of freedom by

$$\lambda^{(\phi)} = \frac{\rho_1 \rho_2}{\rho} \lambda^{(U)} \frac{1 + \gamma}{\rho \xi + i\omega} \quad (30)$$

where the dispersion step around $\omega \approx \rho \xi$ is due to the friction ($\sim \xi$) between the two fluid momenta. Again these possible additions to the concentration dynamics, however, depend on the choices for the convection velocities.