Learning Algebraic Geometry: Lessons from the String Landscape

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IMA Workshop: SageMath and Macaulay2
- An Open Source Initiative, July, 2019
The Geometrization Programme

- Algebraic/differential geometry/topology: the right language for physics
  - Gravity $\sim$ Ricci 2-form of the Tangent bundles;
  - Elementary Particles $\sim$ irreducible representations of the Lorentz group and sections of bundles with Lie structure group;
  - Interactions $\sim$ Tensor products of sections . . .
- String theory: brain-child of gauge-gravity geometrization tradition

- A new exciting era for synergy with (pure & computational) geometry, group theory, combinatorics, number theory: Sage and M2 indispensible tools
- Interdisciplinary enterprise: cross-fertilisation of particle/string theory, phenomenology, pure mathematics, computer algorithms, data-bases, . . .
10 = 4 + 3 \times 2

Superstring Theory 9+1 d

I. 6 Large Dim
- AdS/CFT
- Brane World

II. 6 small dim
- Compactification

1. Reduce Dim: 10 = 6+4
2. Break SUSY

Our world 3+1d
- SU(3) x SU(2) x U(1) SM + GR

Unified theory of quantum gravity
Compact Calabi-Yau Threefolds

String Compactification
Triadophilia: A 30-year search

- Heterotic string [Gross-Harvey-Martinec-Rohm]: $E_8 \times E_8$ or $SO(32)$, 1984 - 6

- String Phenomenology [Candelas-Horowitz-Strominger-Witten]: 1986
  - Het $E_8$ string theory on $X \times \mathbb{R}^{1,3}$ for smooth, compact Calabi-Yau 3-fold $X \sim \mathcal{N} = 1$ SUSY $E_6$-GUT theory on $\mathbb{R}^{1,3}$
  - $E_6$ commutant of $SU(3)$ in $E_8$ and $SU(3)$ is holonomy of $T_X$
  - $\mathcal{N} = 1$ follows from CY condition (existence of covariantly constant spinor)
  - $\#$ net generations $= |h^{1,1}(X) - h^{2,1}(X)| = \frac{1}{2}|\chi(X)|$

- Believed to be ToE: het $E_8$ on CY3 gives GUT + gravity in 4-D
  (cf. Greene-Ross, Distler, et al)

- Need to construct CY3 with $\chi = \pm 6$ explicitly and to work out details
Complete Intersection Calabi-Yau (CICY) 3-folds

- immediately: (generic homog.) Quintic $Q$ in $\mathbb{P}^4$ is CY3
  (by Euler sequence $5 = 4 + 1 \sim c_1(T_X) = 0$), but $Q^{h^{1,1}, h^{2,1}} = Q^{1,101}_{-200}$ so too may generations (even with quotient $-200 \not\in 3\mathbb{Z}$)
  - dim(Ambient space) - #(defining Eq.) = 3 (complete intersection)
- $M = \begin{bmatrix} n_1 & q_1^1 & q_1^2 & \cdots & q_1^K \\ n_2 & q_2^1 & q_2^2 & \cdots & q_2^K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n_m & q_m^1 & q_m^2 & \cdots & q_m^K \end{bmatrix} \quad m \times K$
  - $K$ eqns of multi-degree $q_j^i \in \mathbb{Z}_{\geq 0}$
  - embedded in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$
  - $c_1(X) = 0 \sim \sum_{j=1}^{K} q_j^i = n_r + 1$
  - $M^T$ also CICY

Famous Examples
Problem: *classify all configuration matrices*; employed the best computers at the time (*CERN supercomputer*); q.v. magnetic tape and dot-matrix printout in Philip’s office

- 7890 matrices from $1 \times 1$ to $\max(\text{row}) = 12$, $\max(\text{col}) = 15$; with $q_{ij} \in [0, 5]$
- 265 distinct Hodge pairs $(h^{1,1}, h^{2,1}) = (1, 65), \ldots, (19, 19)$
- 70 distinct Euler $\chi \in [-200, 0]$ (all negative)
- [V. Braun, 1003.3235] : 195 have freely-acting symmetries (quotients), 37 different finite groups (from $\mathbb{Z}_2$ to $\mathbb{Z}_8 \rtimes H_8$)

[Candelas-Lynker-Schimmrigk, 1990] *Hypersurfaces in Weighted P4*

- generic homog deg $= \sum_{i=0}^{4} w_i$ polynomial in $W_{\mathbb{P}^4}^{[w_0:w_1:w_2:w_3:w_4]} \cong \mathbb{C}^5 - \{0\}/(x_0, x_1, x_2, x_3, x_4) \sim (\lambda^{w_0} x_0, \lambda^{w_1} x_1, \lambda^{w_2} x_2, \lambda^{w_3} x_3, \lambda^{w_4} x_4)$
- 7555 inequivalent 5-vectors $w_i$, 2780 Hodge pairs, $\chi \in [-960, 960]$
The age of data science in mathematical physics/string theory/geometry not as recent as you might think.
The Kreuzer-Skarke Dataset

Generalize WP4, take reflexive polytope $\Delta_n \in \mathbb{R}^n$ (lattice convex polytope with a single interior point, take as origin); Need to classify up to $SL(n; \mathbb{Z})$;

- **THM [Batyrev-Borisov, 1994]** hypersurface in toric variety $X(\Delta_n)$ is CY(n-1)
- Classically known: $n = 2$, there are 16 $\Delta_2 \sim$ elliptic curves
- **Kreuzer†-Skarke 1997-2002**: 4319 $\Delta_3$ and 473,800,776 $\Delta_4$
  - 30,108 distinct Hodge pairs, $\chi \in [-496, 496]$;
  - Dual polytope $\Delta \leftrightarrow \Delta^\circ =$ mirror symmetry
  - Cornell (McCallister, Stillman)
- **OPEN**: 1, 16, 4319, 473800776, > 185269499015, ???
The Compact CY3 Landscape

- 20 years of research by mathematicians and physicists
- 500 million data-points (and growing)
- Horizontal $\chi = 2(h^{1,1} - h^{2,1})$ vs. Vertical $h^{1,1} + h^{2,1}$: a Georgia O’Keefe Plot

![Diagram showing Calabi-Yau Threefolds, Elliptic Fibration, KS Toric Hypersurface, and CICY](image)
**CY3 Compactification: Recent Development**

- $E_6$ GUTs less favourable, $SU(5)$ and $SO(10)$ GUTs: general embedding
  - Instead of $TX$, use (poly-)stable holomorphic vector bundle $V$
  - Gauge group $(V) = G = SU(n)$, $n = 3, 4, 5$, gives $H = \text{Commutant}(G, E_8)$:

<table>
<thead>
<tr>
<th>$E_8 \rightarrow G \times H$</th>
<th>Breaking Pattern</th>
</tr>
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<tbody>
<tr>
<td>$SU(3) \times E_6$</td>
<td>$248 \rightarrow (1, 78) \oplus (3, 27) \oplus (\overline{3}, 27) \oplus (8, 1)$</td>
</tr>
<tr>
<td>$SU(4) \times SO(10)$</td>
<td>$248 \rightarrow (1, 45) \oplus (4, 16) \oplus (\overline{4}, 16) \oplus (6, 10) \oplus (15, 1)$</td>
</tr>
<tr>
<td>$SU(5) \times SU(5)$</td>
<td>$248 \rightarrow (1, 24) \oplus (5, 10) \oplus (\overline{5}, 10) \oplus (10, 5) \oplus (\overline{10}, 5) \oplus (24, 1)$</td>
</tr>
</tbody>
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- MSSM: $H \xrightarrow{\text{Wilson Line}} SU(3) \times SU(2) \times U(1)$

- Issues in low-energy physics $\sim$ Precise questions in Alg Geo of $(X, V)$
  - Particle Content $\sim$ (tensor powers) $V$ Bundle Cohomology on $X$
  - LE SUSY $\sim$ Hermitian Yang-Mills connection $\sim$ Bundle Stability
  - Yukawa $\sim$ Trilinear (Yoneda) composition
  - Doublet-Triplet splitting $\sim$ representation of fundamental group of $X$
Algorithmic Compactification

- Schön Quotients: **Penn Model** [Braun-YHH-Ovrut-Pantev 2005] exact MSSM
- Searching the MSSM, *Sui Generis*?
  - **culminating in** .. Stable Sum of Line Bundles over CICYs (Oxford-Penn-Virginia 2012-)
    - Anderson-Gray-Lukas-Ovrut-Palti \( \sim 200 \) in \( 10^{10} \) MSSM
Local (affine) CY3

AdS/CFT, Holography, Brane worlds, Gauge Theory
A Geometer's AdS/CFT

- Rep. Variety (Quiver) $\sim$ VMS (SUSY QFT) $\sim$ affine/singular variety
  - e.g. $\mathcal{N} = 1$ Quiver variety $=$ vacuum of F- & D-flatness $=$ non-compact CY3

- $\mathcal{N} = 4$ $U(N)$ Yang-Mills
  - 3 adjoint fields $X, Y, Z$ with superpotential $W = \text{Tr}(XYZ - XZY)$

  \[ X \]
  \[ Y \]
  \[ Z \]

  - Original AdS/CFT [Maldacena, '98]
  - $N$ D3-branes (w.v. is $\mathcal{N} = 4$ in $\mathbb{R}^{3,1}$) $\perp \mathbb{R}^6$
    $\sim \mathbb{C}^3 = $ Vacuum Moduli Space

- QUIVER $=$ Finite graph (label $=$ rank(gauge factor)) $+$ relations from $W$
  - Matter Content: Nodes $+$ arrows
  - Relations (F-Terms): $D_i W = 0 \Rightarrow [X, Y] = [Y, Z] = [X, Z] = 0$

  - Here $\mathbb{C}^3$ is real cone over $S^5$ (simplest Sasaki-Einstein 5-manifold), others?
Next Simplest Example, Orbifolds: $\mathcal{M} = \mathbb{C}^3/(\Gamma \subset SU(3))$

- Gimon-Polchinski, Douglas-Moore (1996); Greene-Morrison-Plesser, Johnson-Myers (1997);
  $$\Gamma \subset SU(2), \text{ ADE (McKay Correspondence)} \ \Gamma \subset SU(3) \text{ (Hanany-YHH 9811183):}$$
  Projection of parent $\mathbb{C}^3$ theory

- *Geometry of $\mathcal{M}$ and w.v. physics* $\sim \text{Rep}(\Gamma) = \{r_i\}$
  $$\mathcal{R} \otimes r_i = \bigoplus_j a^\mathcal{R}_{ij} r_j, \ a^\mathcal{R}_{ij} \sim \text{adjacency matrix of quiver}$$

- Sasaki-Einstein base $=$ Lens space of $S^5$

- Discrete finite subgroups of $SU(n)$ classified up to $n = 8$, usual pattern:
  1. $\mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_{n-1}}$;
  2. a few non-Abelian infinite families generalizing Dihedral group;
  3. a finite # Exceptional cases;
Non-Compact Toric CY3

- Combinatorially: $\text{CY}_n = \text{convex lattice polytope in } \mathbb{R}^{n-1}$

- No known *compact* CY3 metrics, all known (non-compact) CY3 metrics are toric so far (SE Cone $U(1)^3$ isometry): infinite families $Y^{p,q}$, $L^{abc}$ (conifold, special case); [Candelas-de la Ossa, Cvetic, Hanany, Pope, Sparks, Waldram 1990’s–…]

- By far the largest class known and studied (Use Witten’s GLSM Aspinwall, Beasley, Cachazo, Diaconescu, Douglas, Greene, Katz, Morrison, Plesser, Vafa et al., 1997-2000)

  Feng-Hanany-YHH 0003085: Inverse Algorithm toric diag $\leadsto$ gauge theory

- [Franco-Hanany-Kennaway-Sparks-Vegh 2005] gauge theory $\equiv$ bipartite brane tiling $T^2$
  
  1. **Mirror symmetry**: Feng-Kennaway-YHH-Vafa 2005
  2. Recent generalizations to SUSY gauge theories in other dimensions

Franco-Seong-Lee-Vafa 2016-7
Standard Model and Geometrical Selection

- With Jejjala, Matti, Nelson, Stillman: 2009 -

- MUCH more complicated than simple quivers ($n = 49$ component fields)

<table>
<thead>
<tr>
<th>INDICES</th>
<th>FIELDS</th>
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<tbody>
<tr>
<td>$i, j, k, l = 1, 2, 3$</td>
<td>$u^i_a$, $d^i_a$, $Q^i_{a,\alpha}$, $L^i_{\alpha}$, $e^i$, $H^\alpha$, $H^\alpha$</td>
</tr>
<tr>
<td>$a, b, c, d = 1, 2, 3$</td>
<td>$SU(3)_C$ color indices</td>
</tr>
<tr>
<td>$\alpha, \beta, \gamma, \delta = 1, 2$</td>
<td>$SU(2)_L$ indices</td>
</tr>
</tbody>
</table>

Flavour indices

- Superpotential: all renormalisable terms compatible with R-parity:

$$W_{\text{renorm}} = C^0 \sum_{\alpha, \beta} H^\alpha \overline{H}^\beta \epsilon_{\alpha\beta} + \sum_{i,j} C^1_{ij} \sum_{\alpha, \beta, \alpha} Q^i_{a,\alpha} u^j_a H_{\beta} \epsilon_{\alpha\beta} + \sum_{i,j} C^2_{ij} \sum_{\alpha, \beta, a} Q^i_{a,\alpha} d^j_a \overline{H}_{\beta} \epsilon_{\alpha\beta} + \sum_{i,j} C^3_{ij} \sum_{\alpha, \beta, \alpha} L^i_{\alpha} \overline{H}_{\beta} e^j \epsilon_{\alpha\beta},$$

- List of GIO $k = |D| = 991$; Too much for a head-on GB calculation

- Subsectors (set quarks = 0) $\leadsto$ EW-sector: get cone over Veronese Surface
The Landscape of non-Compact (affine, local, sing) CY3

a 15-year prog. joint with A. Hanany et al.

World-Volume = Quiver Gauge Theory

Local CY3

1 [Davey-Hanany-Pasukonis, Franco, Mekareeya, Torri, Sparks, Benishti, Musiker, Broomhead, King, YHH] toric CY3,4: 2009-

2 Quiver Mutation = Seiberg Duality [Fomin-Zelevinsky, Feng-Hanany-YHH-Uranga, 2000]

3 Use reflexive polytope: Hanany-Seong, 2014 Done in SAGE YHH-Seong-Yau, 2017
Databases and algorithms in physics and mathematics: SAGE, M2, etc., indispensable tools (cf. “Periodic table of shapes Project” classify Fanos, grdb)

Archetypical Problems

- Classify configurations (typically integer matrices: polytope, adjacency, ...)
- Compute geometrical quantity algorithmically
  - toric $\sim$ combinatorics;
  - quotient singularities $\sim$ rep. finite groups;
  - generically $\sim$ ideals in polynomial rings;
  - Numerical geometry (homotopy continuation);
  - Cohomolgy (spectral sequences, Adjunction, Euler sequences)
Where we stand . . .

**The Good**  Last 10-15 years: several international groups have been biting the bullet Oxford, London, Vienna, Blacksburg, Boston, Johannesburg, Munich, . . . typically using SAGE and M2 and compiled into various databases

Landscape Data \((10^9 \sim 10^{10} \text{ entries typically})\)

**The Bad**  Generic computation HARD: dual cone algorithm (exponential), triangulation (exponential), Gröbner basis (double-exponential) . . . e.g., how to construct stable bundles over the \(\gg 473\) million KS CY3? Sifting through for SM not possible . . .

**The ???**  Borrow new techniques from “Big Data/Al” revolution
Deep-Learning the Landscape
Typical Problem in String Theory/Algebraic Geometry:

\[
\text{INPUT} \quad \text{integer tensor} \quad \rightarrow \quad \text{OUTPUT} \quad \text{integer}
\]

Q: Can (classes of problems in computational) Algebraic Geometry be "learned" by AI?

1706.02714 (PLB 774, 2017) YHH: Experimentally, it seems to be the case for many situations

cf. YHH, CY manifolds: from geom to phys to ML, book, Springer, 2020(?).
CICYs: a Colourful Example

- An image = a matrix (pixels) with entries denoting shade/colour; NN really good at images (e.g. hand-writing) [RMK: not using CNN here]
- CICY is a (padded) $12 \times 15$ matrix with 6 colours $\sim$ CICY is an image

\[ \text{(a) typical CICY; (b) average CICY} \]

- Input more sophisticated, so greater accuracy expected: e.g. in learning large number of Kahler parametres $h^{1,1} > 5$:

  learns 4000 samples ($< 50\%$) in $\sim 5$ min; validate against 7890-4000: 97% accuracy, $d_C = 0.98$, $\phi = 0.87$. 
CICYs: Detailed Analysis


- **TensorFlow** Python’s implementation of NNs and DL
- Compare NNs with Decision Trees, Support Vector Machines, etc

Can one learn the FULL information on Hodge numbers? \(h^{1,1} \in [0, 19]\) so can set up 20-channel NN classifier, regressor, as well as SVM
$h^{1,1}$ for NN, Regressor, SVM at 20 and 80% training
Success Stories

- **Distinguishing Elliptic Fibrations** [YHH-SJ. Lee 1904.08530]: test in CICY which are elliptically fibred (bypass Oguiso-Kollar-Wilson Theorem/Conjecture)

- **Identifying Simple Finite Group** [YHH-MH. Kim 1905.02263]: “Look” at the Cayley multiplication table (bypass Sylow and Noether Theorems)

- **Sanity Check**: predicting primes BAD
Summary and Outlook

**PHYSICS**
- The string landscape now solidly resides in the **age of Big Data**
- Use Machine-Learning s as
  1. **Classifier** deep-learn and categorize landscape data
  2. **Predictor** estimate results beyond computational power

**MATHS**
- somewhat bypassing the expensive steps of long sequence-chasing, Gröbner bases, dual cones/combinatorics and getting the right answer. how is AI guessing maths more efficiently without knowing any maths?
- problems in geometry, combinatorics, etc, good; number theory, not so good. Only probabilistically doing NP-Hard
THANK YOU

- **Boris Zilber** [Merton Professor of Logic, Oxford]: “you’ve managed syntax without semantics…”

- Try your favourite problem and see

- **2017:**
  - First non-human citizen (2017, Saudi Arabia)
  - First non-human with UN title (2017)
  - First String Data Conference (2017)

Sophia (Hanson Robotics, HK)
Famous CICYs

- **The Quintic** \( Q = [4|5]^{1,101}_{1,010} \) (or simply \([5]\));

- **Tian-Yau Manifold**: \( TY = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}^{14,23}_{-18} \)
  - no CICY has \( \chi = \pm 6 \)
  - TY has freely-acting \( \mathbb{Z}_3 \sim (TY/\mathbb{Z}_3)^{6,9} \);
  - central to early string pheno

- **Schön Manifold**: \( S = \begin{pmatrix} 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{pmatrix}^{19,19}_0 \) has \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) freely acting symmetry
  - explored more recently;
  - The quotient is \( M^0_{3,3} \).
Archetypal Quiver Gauge Theory and Moduli Space

- Archetypal example: $\mathbb{C}^3/\mathbb{Z}_3$ with action $(1,1,1) \sim U(1)^3$ quiver theory

```
R=ZZ/101[X_{1},..,X_{9}];
S=ZZ/101[y_{1},..,y_{27}];
fterms = {-X_{6}*X_{8} + X_{5}*X_{9}, X_{6}*X_{7} - X_{4}*X_{9}, -(X_{5}*X_{7} + X_{4}*X_{8} + X_{3}*X_{9}) , X_{2}*X_{7} - X_{1}*X_{8}, -(X_{3}*X_{5}) + X_{2}*X_{6}, X_{3}*X_{4} - X_{1}*X_{6}, -(X_{2}*X_{4}) + X_{1}*X_{5}};
dterms = {X_{1}*X_{4}*X_{7}, X_{1}*X_{4}*X_{8}, X_{1}*X_{4}*X_{9}, X_{1}*X_{5}*X_{7}, X_{1}*X_{5}*X_{8}, X_{1}*X_{5}*X_{9}, X_{1}*X_{6}*X_{7}, X_{1}*X_{6}*X_{8}, X_{1}*X_{6}*X_{9}, X_{1}*X_{7}, X_{1}*X_{8}, X_{1}*X_{9}, X_{2}*X_{4}*X_{7}, X_{2}*X_{4}*X_{8}, X_{2}*X_{4}*X_{9}, X_{2}*X_{5}*X_{7}, X_{2}*X_{5}*X_{8}, X_{2}*X_{5}*X_{9}, X_{2}*X_{6}*X_{7}, X_{2}*X_{6}*X_{8}, X_{2}*X_{6}*X_{9}, X_{2}*X_{7}, X_{2}*X_{8}, X_{2}*X_{9}, X_{3}*X_{4}*X_{7}, X_{3}*X_{4}*X_{8}, X_{3}*X_{4}*X_{9}, X_{3}*X_{5}*X_{7}, X_{3}*X_{5}*X_{8}, X_{3}*X_{5}*X_{9}, X_{3}*X_{6}*X_{7}, X_{3}*X_{6}*X_{8}, X_{3}*X_{6}*X_{9}};
fterms = intersect decompose trim ideal fterms;
V = ker map(R/fterms,S,dterms);
```

$$W = \epsilon_{\alpha\beta\gamma}X_{12}^{(\alpha)}X_{23}^{(\beta)}X_{31}^{(\gamma)},$$

- Moduli space: 27 quadrics in $\mathbb{C}^{10}$ as local Calabi-Yau 3-fold

$\mathbb{C}^3/\mathbb{Z}_3 \leftarrow \text{Tot}(\mathcal{O}_{\mathbb{P}^2}(-3))$
Refined Structure in KS Data

- DATABASES:
  - http://hep.itp.tuwien.ac.at/~kreuzer/CY/
  - http://www.rossealtman.com/

- Altman-Gray-YHH-Jejjala-Nelson 2014-17 triangulate $\Delta_4$ (orders more than 1/2-billion): up to $h^{1,1} = 7$

- Candelas-Constantin-Davies-Mishra 2011-17 special small Hodge numbers

- Taylor, Johnson, Wang et al. 2012-17 elliptic fibrations

- YHH-Jejjala-Pontiggia 2016 distribution of Hodge, $\chi$, Pseudo-Voigt
\[ r = h^{1,1} + h^{1,2} \]

**Pseudo-Voigt distribution**

\[
(1 - \alpha) \frac{A}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \alpha \frac{A}{\pi} \frac{\sigma^2}{(x-\mu)^2 + \sigma^2}
\]

**Planck distribution**

\[
\frac{A}{x^n} \frac{1}{e^{h/(x-c)} - 1}
\]

He, VJ, Pontiggia (2015)
Can build monad with **generic map** on Quintic in $\mathbb{P}^4$:

\[
R = \mathbb{Z}/101[x_0..x_4]; \text{cy} = \text{Proj}(R/(\text{random}(5,R))); \quad -- \text{Quintic}
\]

\[
\text{ocy} = \text{OO_cy}; \quad -- \text{Structure Sheaf}
\]

\[
A = \text{module ocy}^5(0); B = \text{module ocy}(5); \quad -- \text{A rank 4 bundle}
\]

\[
f = \text{map}(B, A, \text{random}(B, A)) \quad -- \text{random map } f : A \rightarrow B
\]

-- Define $SU(4)$ Bundle $0 \rightarrow V \rightarrow O_X(1)^{\oplus 5} \xrightarrow{f} O_X(5) \rightarrow 0$

\[
V = \text{sheaf ker}(f);
\]

-- Faster to do Cokernel

\[
V_{\text{dual}} = \text{coker}(\text{map}(\text{dual}(A), \text{dual}(B), \text{random}(\text{dual}(A), \text{dual}(B))));
\]

**Cohomologies can be computed, albeit slow:**

\[
\text{HH}^1(V) \quad \text{HH}^1(V \ast V_{\text{dual}}) \quad \text{HH}^1(\text{exteriorPower}(2, V))
\]

**QUESTION:** **GENERICITY** of map $f$ does NOT require calculation of detailed maps; can we simplify the corresponding sheafCoh algorithms