Pure Euler-Poincaré reduction – general theory

(Marsden, Holm, Ratiu, Bloch and co-workers)

**Left-invariant** Lagrangian: \( L : TG \rightarrow \mathbb{R} \) (Lie group \( G \), Lie algebra \( g \))

Reduced Lagrangian: \( l : g \rightarrow \mathbb{R}, \quad l(\xi) := L(e, \xi) = L(g^{-1}g, g^{-1}g) = L(g, \dot{g}) \)

Then the following four statements are equivalent:

1. The variational principle \( \delta \int_a^b L(g(t), \dot{g}(t)) dt = 0 \) \( ((g(t), \dot{g}(t)) \in TG) \) holds, as usual, for variations \( \delta g(t) \) of \( g(t) \) vanishing at the end points.

2. The curve \( g(t) \) satisfies the Euler-Lagrange equations for \( L \) on \( G \).

3. The variational principle \( \delta \int_a^b l(\xi(t)) dt = 0 \) \( (\xi(t) = g(t)^{-1} \dot{g}(t) \in g) \)
holds, using (induced) variations of the form \( \delta \xi = \dot{\eta} + [\xi, \eta] \), where \( \eta(t) \) is an arbitrary path in \( g \) that vanishes at the end points, i.e., \( \eta(a) = \eta(b) = 0 \).

4. The (pure) Euler-Poincaré equations hold:

\[
\frac{d}{dt} \frac{\delta l}{\delta \xi} = \text{ad}^*_\xi \frac{\delta l}{\delta \xi} \quad (\text{ad}^*_\xi(\eta) = [\xi, \eta])
\]

Reconstruction: \( \dot{g}(t) = g(t)\xi(t), \quad g(0) = g_0 \quad (\xi(0) = \xi_0 := g_0^{-1}v_0, \quad v_0 = \dot{g}(0)) \)
Extensions for quasi-rods

Rigid body case ($G = SO(3)$):

$$L(R, \dot{R}) = \frac{1}{2} \text{tr} \left( \dot{R} J \dot{R}^T \right) = \frac{1}{2} \text{tr} \left( \hat{\omega} J \hat{\omega}^T \right) = \frac{1}{2} \omega^T I \omega = l(\hat{\omega})$$

$$\frac{\delta l}{\delta \omega} = I \omega, \quad [\hat{\omega}, \hat{\eta}] = \omega \times \eta \quad \Rightarrow \quad I \dot{\omega} = I \omega \times \omega \quad \text{(Euler equation)}$$

Need two extensions: (i) forced (symmetry-broken) case, (ii) second-order case

(i) Forced case:

Various views:

1. Euler-Poincaré with advected parameters ($L = L_F(R, \dot{R})$): extended configuration space to $SO(3) \times (\mathbb{R}^3)^*$ (semidirect extension)

2. Lagrange-d’Alembert principle, generalised forces (also covers dissipative systems; e.g., ferromagnetic rigid body (Bloch et al., 1996))

3. Generalised Euler-Poincaré: $L = L(g(t), \dot{g}(t)) = \bar{L}(g(t), \xi(t))$ (special case of Hamel equations)
Either way, for SO(3) we get:

\[
\frac{d}{dt} \left( \frac{\delta l}{\delta \omega} \right) = \frac{\delta l}{\delta \omega} \times \omega + \frac{\delta l}{\delta k} \times k
\]

For the heavy top (\(d_3 = (0, 0, 1)\)):

\[
L(R, \dot{R}) = K - V = \frac{1}{2} \text{tr} \left( \dot{R} J \dot{R}^T \right) - mgh \langle k, Rd_3 \rangle
\]

\[
= \frac{1}{2} \omega^T I \omega - mgh \langle k, d_3 \rangle = l(\hat{\omega}, k)
\]

Thus

\[
\frac{\delta l}{\delta \omega} = I \omega, \quad \frac{\delta l}{\delta k} = -mgh d_3 \quad \implies \quad I \dot{\omega} = I \omega \times \omega - mgh d_3 \times k
\]

Also

\[
Rk = k \quad \implies \quad \dot{R}k + R \dot{k} = 0
\]

and hence

\[
\dot{k} = -R^{-1}Rk = -\hat{\omega}k = -\omega \times k
\]

or

\[
\dot{k} = k \times \omega
\]
(ii) **Second-order case:**

Consider a left-invariant Lagrangian: \( L : T^{(2)} G \to \mathbb{R}, \ L = L(g, \dot{g}, \ddot{g}) = l(\xi, \dot{\xi}) \)

Hamilton's principle: \( \delta \int_a^b L(g, \dot{g}, \ddot{g}) \, dt = 0, \quad \delta g = \delta \dot{g} = 0 \) at end points

Let

\[
M := \frac{\delta l}{\delta \xi} - \frac{d}{dt} \frac{\delta l}{\delta \dot{\xi}}
\]

Then, taking (induced) variations (now \( \eta \) and \( \dot{\eta} \) and hence \( \delta \xi \) zero at end points),

\[
\delta \int_a^b l(\xi, \dot{\xi}) \, dt = \int_a^b \left( \frac{\delta l}{\delta \xi} \delta \xi + \frac{\delta l}{\delta \dot{\xi}} \delta \dot{\xi} \right) \, dt = \int_a^b \left( \frac{\delta l}{\delta \xi} - \frac{d}{dt} \frac{\delta l}{\delta \dot{\xi}} \right) \delta \xi \, dt
\]

\[
= \int_a^b M \delta \xi \, dt = \int_a^b M(\dot{\eta} + \text{ad}_\xi \eta) \, dt = \int_a^b (-\dot{M} + \text{ad}^*_\xi M) \eta \, dt
\]

Stationarity gives

\( \dot{M} = \text{ad}^*_\xi M \)

Thus, for SO(3) (rigid body/quasi-rod; \( M \) generalises angular momentum \( l\omega \)):

\[
\dot{M} + \omega \times M = 0, \quad M = \frac{\delta l}{\delta \omega} - \frac{d}{dt} \frac{\delta l}{\delta \dot{\omega}}
\]
Quasi-rod equilibrium equations (in Euler-Poincaré form)

Lagrangian:

\[ \mathcal{L}(\omega, \omega') = \kappa^2 (1 + \eta^2)^{\frac{1}{2}} \log \left( \frac{1 + w\eta'}{1 - w\eta'} \right) + M_2\omega_2 - F \cdot t \]

Euler-Lagrange equations:

\[ \begin{align*}
F' + \omega \times F &= 0 \quad (F' = 0) \\
M' + \omega \times M + t \times F &= 0 \quad (M' + r' \times F = 0) \\
M_j &= \frac{\partial \mathcal{L}}{\partial \omega_j} - \frac{d}{ds} \frac{\partial \mathcal{L}}{\partial \omega'_j}, \quad j = 1, 2, 3 \quad \text{('constitutive relations', DAE)}
\end{align*} \]

Contact transformation:

\[ \begin{align*}
(\omega_1, \omega_3) &= (\tau, \kappa) \quad \longrightarrow \quad (\eta, \kappa) = (\tau/\kappa, \kappa)
\end{align*} \]

Reconstruction of the centreline:

Frenet-Serret equations:

\[ \begin{align*}
t' &= \kappa \ n, \quad n' = -\kappa \ t + \tau \ b, \quad b' = -\tau \ n
\end{align*} \]

Centreline equation:

\[ r' = t \]

15D ODE
Hamiltonian

\[ \mathcal{H}(\pi_1, \omega, \pi_2, F) = \pi_1 \cdot \omega + \pi_2 \cdot \omega' - l(\omega, \omega', F) \]

where

\[ \pi_1 = \frac{\partial l}{\partial \omega} - \frac{d}{ds} \left( \frac{\partial l}{\partial \omega'} \right) = M, \quad \pi_2 = \frac{\partial l}{\partial \omega'} \]

(reduced Ostrogradsky momenta)

In our variables

\[ \mathcal{H} = \kappa (M_1 \eta + M_3) + \kappa^2 (1 + \eta^2)^2 \left[ \frac{1}{1 - (w\eta')^2} - \frac{1}{w\eta'} \log \left( \frac{1 + w\eta'}{1 - w\eta'} \right) \right] + F_1 \]
Euler-Poincaré theory applied to the pendulum \((S^1)\)

\[ S^1 = \{ q \in \mathbb{R}^2 | q^T q = 1 \} \]

\[ \simeq SO(2) = \{ q \in \mathbb{R}^{2\times2} | q^T q = I, \ \det q = 1 \} \]

\[
\begin{pmatrix}
q_1 \\
q_2
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
q_1 & -q_2 \\
q_2 & q_1
\end{pmatrix}
\quad (q_1^2 + q_2^2 = 1)
\]

**Lagrangian:**

\[ L : SO(2) \times \mathfrak{so}(2) \simeq S^1 \times \mathbb{R} \rightarrow \mathbb{R} : \quad L(q, \omega) = T - V = \frac{1}{2} mL^2 \omega^2 - mgL e_2^T q \]

**Kinematics:**

\[ \dot{q} = \omega S q, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\omega = (q \times \dot{q}) \cdot e_3) \]

**Variations:**

\[ \delta q = \eta S q, \quad \eta \text{ a scalar} \quad (q \cdot q = 1 \implies \delta q \cdot q = 0) \]

\[ \delta \omega = \dot{\eta} \quad (\delta(\dot{q}) = (\delta q) \cdot ; \text{ note that for } \mathfrak{so}(2), [\omega, \eta] = 0) \]
Taking variations:

\[ 0 = \delta \int L(q, \omega) \, dt = \int \left( \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \omega} \delta \omega \right) \, dt \]

\[ = \int \left( \frac{\partial L}{\partial q} \eta Sq + \frac{\partial L}{\partial \omega} \eta \right) \, dt = \int \left( \frac{\partial L}{\partial q} Sq - \frac{d}{dt} \left( \frac{\partial L}{\partial \omega} \right) \right) \eta \, dt \]

Euler-Lagrange equation on \( S^1 \):

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \omega} \right) = \frac{\partial L}{\partial q} Sq \]

Applied to the pendulum:

\[ \frac{\partial L}{\partial q} = -mgL e_2, \quad \frac{\partial L}{\partial \omega} = mL^2 \omega \quad \implies \quad mL^2 \dot{\omega} + mgL e_2^T Sq = 0 \]

or, with \( e_2^T Sq = e_1^T q \), \[ mL^2 \dot{\omega} + mgL e_1^T q = 0 \]

In coordinates (such that \( \theta = 0 \) corresponds to pendulum hanging down):

\[ q = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}, \quad \omega = \dot{\theta}, \quad e_1^T q = \sin \theta \]

Thus \[ mL^2 \ddot{\theta} + mgL \sin \theta = 0 \]
Equilibrium equations (for a quasi-rod)

Force and moment balance:

\[
\begin{align*}
F' + \omega \times F &= 0 \\
M' + \omega \times M + t \times F &= 0 \quad \text{with} \quad \omega = (\kappa \eta, 0, \kappa)
\end{align*}
\]

‘Constitutive’ equations

\[
g(\kappa, \eta, \eta') = \kappa^2 (1 + \eta^2)^2 \frac{1}{2w\eta'} \log \left( \frac{1 + w\eta'}{1 - w\eta'} \right)
\]

\[
\begin{align*}
\frac{\partial g}{\partial \kappa} - \eta M_1 - M_3 &= 0 \\
\frac{\partial g}{\partial \eta} - \frac{d}{ds} \left( \frac{\partial g}{\partial \eta'} \right) - \kappa M_1 &= 0
\end{align*}
\]

First equation is algebraic in \((\kappa, \eta, \eta') \rightarrow \text{DAE}\)

Reconstruction of the centreline:

Frenet-Serret equations:

\[
\begin{align*}
t' &= \kappa n, \quad n' = -\kappa t + \tau b, \quad b' = -\tau n
\end{align*}
\]

Centreline equation:

\[
r' = t
\]

15D ODE
Symmetries

The equilibrium equations are invariant under the reversing involutions

\[ R_1: \]
\[ F_1 \rightarrow F_1, \quad F_2 \rightarrow -F_2, \quad F_3 \rightarrow F_3, \quad M_1 \rightarrow M_1, \quad M_2 \rightarrow -M_2, \quad M_3 \rightarrow M_3, \]
\[ \kappa \rightarrow \kappa, \quad \eta \rightarrow \eta, \quad s \rightarrow -s \]

\[ R_2: \]
\[ F_1 \rightarrow F_1, \quad F_2 \rightarrow F_2, \quad F_3 \rightarrow -F_3, \quad M_1 \rightarrow M_1, \quad M_2 \rightarrow M_2, \quad M_3 \rightarrow -M_3, \]
\[ \kappa \rightarrow -\kappa, \quad \eta \rightarrow -\eta, \quad s \rightarrow -s \]

and the non-reversing involution

\[ S: \]
\[ F_1 \rightarrow F_1, \quad F_2 \rightarrow F_2, \quad F_3 \rightarrow -F_3, \quad M_1 \rightarrow -M_1, \quad M_2 \rightarrow -M_2, \quad M_3 \rightarrow M_3, \]
\[ \kappa \rightarrow \kappa, \quad \eta \rightarrow -\eta, \quad s \rightarrow s \]

Note that \( R_2 \) requires \( \kappa \) to be interpreted as the signed curvature.
Randrup & Røgen (1996) conditions for a symmetric developable Möbius strip:

odd number of switching points where the curvature changes sign and the Frenet frame flips $180^\circ$; at these points: $\kappa = 0$, $\tau = 0$ and $\eta = \tau/\kappa = 0$

Solve for half a strip

Boundary conditions: $\eta'(0) = 0$ (cylindrical point), $\kappa(L/2) = 0$ (switching point), $F_n(0) = 0 = M_n(0)$ ($\Rightarrow \eta(L/2) = 0$)
Centrelines of Möbius strips of various widths

Gert van der Heijden (UCL)
Equilibrium shapes of rectangular strips
Randrup & Røgen (1996) conditions satisfied (except in limit \( w = 0 \))

Sadowsky limit \( w \to 0 \) singular: discontinuous \( \kappa \) and \( \eta \to 1 \) (i.e., \( \beta = 45^\circ \)) at switching point, \( s = \pi \) (both anticipated by Sadowsky, by experimentation)

Corrected Sadowsky functional (Freddi, Hornung, Mora & Paroni, 2016):

\[
g(\kappa, \eta) = \begin{cases} 
\kappa^2 (1 + \eta^2)^2 & \text{if} \quad \eta < 1 \\
4\eta^2 \kappa^2 & \text{if} \quad \eta \geq 1
\end{cases}
\]

avoids singularity (allows \( \eta \), and hence \( \tau \), to be 0 at inflection point, where \( \kappa = 0 \)) (see also Paroni & Tomassetti (2019), using convexification)
3D shapes and bending energy density

Colour indicates the bending energy density (violet for low bending, red for high bending) — flat triangular region radiating out from singular point

Limiting aspect ratio \( \frac{w}{L} = \sqrt{3}/6 \approx 0.2887 \) (flat limit of triple-covered equilateral triangle; \( \delta \)-functions for \( \kappa \) and \( \tau \))

Halpern & Weaver (1977) conjecture: isometric embedding in \( \mathbb{R}^3 \) of smooth developable Möbius strip only possible if \( \frac{w}{L} < \sqrt{3}/6 \)
Planar development with edge of regression
Recall bending energy density

\[ U = \frac{D}{2} \int_0^L \int_{-w}^w \kappa^2(1 + \eta^2)^2 \frac{1}{1 + t\eta'} \, dt \, ds = Dw \int_0^L g(\kappa, \eta, \eta') \, ds \]

\[ g(\kappa, \eta, \eta') = \kappa^2 (1 + \eta^2)^2 \frac{1}{2w\eta'} \log \left( \frac{1 + w\eta'}{1 - w\eta'} \right) \]

When \( \eta' = 0 \), the nonzero principal curvature \( \kappa_1 \) is constant along generators ('cylindrical' points)

Away from cylindrical points (\( \eta' = 0 \)), \( \kappa_1 \) becomes singular when \( t = -1/\eta' \).

The curve of such singular points defines the **edge of regression**:

\[ x_e(s) = r(s) - \frac{1}{\eta'(s)} \left[ b(s) + \eta(s) \, t(s) \right] \]

It's where nearby generators intersect, so the strip cannot be wider than the critical value of \( t \), i.e., we require that \( w|\eta'| \leq 1 \)

Note: the developable surface consists of the tangents to the edge of regression (it's its tangent developable)
Singularities

**Removable singularity** at $s = 0$, where $\eta' = 0$

**Logarithmic singularity** at $s = L/2$ (switching point), where we always find $|\eta'| \to 1/w$ (i.e., principal curvature $\kappa_1 \to \infty$)

The edge of regression touches the edge of the strip
Computed solution superimposed on paper strip ($L/2w = 2\pi$)
Chen & Fried (and Fosdick) confusion


Claim that shapes computed from parametrisation for \( s \in [0, L], \nu \in [-w, w] \) are not rectangular (in fact, trapezoidal) and that hence the deformation is not isometric and therefore the Wunderlich reduction (integrating along generators) invalid ("covering problem").

However, the parametrisation is in non-orthogonal coordinates \((s, \nu)\) and the reference configuration is not rectangular.

Covering problem not an issue for closed strips, or for open strips if the end generators are controlled (i.e., fixed-\( \eta \) boundary conditions). **Problem not just kinematics!**
Continuum mechanics: deformation and strain

Deformation (material coordinates $\mathbf{X}$, current coordinates $\mathbf{x}$):

$$\mathbf{x} = \chi(\mathbf{X})$$

(Material) deformation gradient:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \chi(\mathbf{X})}{\partial \mathbf{X}}, \quad F_{ij} = \frac{\partial x_i}{\partial X_j}$$

Explicitly, in terms of the material and spatial basis vectors $\{\mathbf{E}_J\}$ and $\{\mathbf{e}_i\}$:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial X_J} \otimes \mathbf{E}_J = \frac{\partial x_i}{\partial X_J} \mathbf{e}_i \otimes \mathbf{E}_J$$

Thus

$$\mathbf{F}d\mathbf{X} = \left(\frac{\partial x_i}{\partial X_J}\right)\mathbf{e}_i \otimes \mathbf{E}_J(d\mathbf{X}_M\mathbf{E}_M) = \left(\frac{\partial x_i}{\partial X_J}\right)dX_J\mathbf{e}_i = d\mathbf{x}$$

So $\mathbf{F}$ maps infinitesimal line elements $d\mathbf{X}$ in the reference configuration to infinitesimal line elements $d\mathbf{x}$ in the current configuration.
Now consider two line elements $dX^{(1)}$ and $dX^{(2)}$ in the reference configuration, which are mapped to $dx^{(1)}$ and $dx^{(2)}$; then

$$dx^{(1)} \cdot dx^{(2)} = (F dX^{(1)}) \cdot (F dX^{(2)}) = dX^{(1)} (F^T F) dX^{(2)} = dX^{(1)} C dX^{(2)}$$

where $C$ is the **right Cauchy-Green strain tensor** defined by

$$C = F^T F, \quad C_{ij} = F_{ki} F_{kj} = \frac{\partial x_k}{\partial X_i} \frac{\partial x_k}{\partial X_j}$$

So $C$ gives the square of the local change in distances due to deformation

Explicitly, in terms of basis vectors,

$$C = \left( \frac{\partial x_k}{\partial X_l} e_k \otimes e_l \right) \left( \frac{\partial x_m}{\partial X_j} e_m \otimes e_j \right) = \frac{\partial x_k}{\partial X_l} \frac{\partial x_k}{\partial X_j} e_l \otimes e_j$$

$C$ is symmetric and positive definite (so has real positive eigenvalues)
Example (axes coincident):

Deformation:

\[ x = \chi(X) = -6X_2 e_1 + \frac{1}{2}X_1 e_2 + \frac{1}{3}X_3 e_3 \]

Then

\[ F = \frac{\partial x_i}{\partial X_j} = \begin{pmatrix}
0 & -6 & 0 \\
1/2 & 0 & 0 \\
0 & 0 & 1/3
\end{pmatrix} \]

and hence

\[ C = F^T F = \begin{pmatrix}
0 & 1/2 & 0 \\
-6 & 0 & 0 \\
0 & 0 & 1/3
\end{pmatrix} \begin{pmatrix}
0 & -6 & 0 \\
1/2 & 0 & 0 \\
0 & 0 & 1/3
\end{pmatrix} = \begin{pmatrix}
1/4 & 0 & 0 \\
0 & 36 & 0 \\
0 & 0 & 1/9
\end{pmatrix} \]
Calculation of the right Cauchy-Green strain tensor

Parametrisation of the stress-free reference configuration:

\[ \mathbf{X} = u \mathbf{e}_3 + v \mathbf{e}_1 = s \mathbf{e}_3 + v(\mathbf{e}_1 + \eta \mathbf{e}_3) \]

\[ u = s + v \cot \theta, \quad \eta = \tau / \kappa = \cot \theta \]

Deformation:

\[ \mathbf{x} = \chi(\mathbf{X}) = \mathbf{x}(s, v) = \mathbf{r}(s) + v[\mathbf{b}(s) + \eta(s)\mathbf{t}(s)] \]

Let:

\[ \mathbf{G}_i = \frac{\partial \mathbf{X}}{\partial \xi^i}, \quad \mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}, \quad (\xi^1, \xi^2) = (s, v) \]

Deformation gradient:

\[ \mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial \mathbf{x}}{\partial \xi^i} \otimes \mathbf{G}_i = \mathbf{g}_i \otimes \mathbf{G}_i \]

Right Cauchy-Green strain tensor (\( g_{ij} \) metric tensor):

\[ \mathbf{C} = \mathbf{F}^T \mathbf{F} = (\mathbf{G}_i \otimes \mathbf{g}_i) \cdot (\mathbf{g}_j \otimes \mathbf{G}_j) = (\mathbf{g}_i \cdot \mathbf{g}_j)(\mathbf{G}_i \otimes \mathbf{G}_j) = g_{ij} \mathbf{G}_i \otimes \mathbf{G}_j \]
We have (using \(\tau = \eta \kappa\) [developability] and Frenet-Serret equations):

\[
\frac{\partial \mathbf{x}}{\partial \xi^1} = \mathbf{r}' + \nu (\mathbf{b}' + \eta' \mathbf{t} + \eta \mathbf{t}') = (1 + \nu \eta') \mathbf{t}, \quad \frac{\partial \mathbf{x}}{\partial \xi^2} = \mathbf{b} + \eta \mathbf{t}
\]

Thus \(g_{11} = (1 + \nu \eta')^2, \quad g_{12} = g_{21} = \eta (1 + \nu \eta'), \quad g_{22} = 1 + \eta^2\)

and hence

\[
\mathbf{C} = \mathbf{g}_{ij} \mathbf{G}^i \otimes \mathbf{G}^j = (1 + \nu \eta')^2 \mathbf{G}^1 \otimes \mathbf{G}^1 + \eta (1 + \nu \eta')(\mathbf{G}^1 \otimes \mathbf{G}^2 + \mathbf{G}^2 \otimes \mathbf{G}^1) + (1 + \eta^2) \mathbf{G}^2 \otimes \mathbf{G}^2
\]

Exactly Chen & Fried’s strain tensor but on the \(\{\mathbf{G}^1, \mathbf{G}^2\}\) basis rather than the (orthogonal) \(\{\mathbf{e}_3, \mathbf{e}_1\}\) basis. They simply missed the transformation \(u = s + \nu \cot \theta\). **The reference configuration is not rectangular!**

Identity (i.e., isometry) only if \(\eta \equiv 0\), i.e., \(\tau \equiv 0\), i.e., planar centreline (cylindrical deformed surface)

But \(\mathbf{G}_i = \mathbf{g}_{ij} \mathbf{G}^j\) and \(\det \mathbf{g} = (1 + \nu \eta')^2\) and thus

\[
\mathbf{G}^1 = \frac{1}{(1 + \nu \eta')^2} \left[ (1 + \eta^2) \mathbf{G}_1 - \eta (1 + \nu \eta') \mathbf{G}_2 \right]
\]

\[
\mathbf{G}^2 = \frac{1}{(1 + \nu \eta')^2} \left[ -\eta (1 + \nu \eta') \mathbf{G}_1 + (1 + \nu \eta')^2 \mathbf{G}_2 \right]
\]
where

\[ G_1 = \frac{\partial X}{\partial \xi^1} = e_3 + v \eta' e_3 = (1 + v \eta') e_3, \quad G_2 = \frac{\partial X}{\partial \xi^2} = e_1 + \eta e_3 \]

Hence, finally,

\[ C = g_{ij} G^i \otimes G^j = G^i \otimes G_i = e_1 \otimes e_1 + e_3 \otimes e_3 \]
M.C. Escher, Möbius Band II, woodcut, 1963 (left), Möbius Band I, woodcut, 1961 (right)
(Collection Gemeentemuseum, The Hague, Netherlands)
We can rotate at either a cylindrical or a switching point (constant colour)

There are in fact two cylindrical points ($S_0$ and $S_1$)

Two symmetric closure conditions on interval $[0, L/(2n)]$:

1) coplanarity of $n(0)$, $b(L/(2n))$ and $r(L/(2n)) - r(0)$:

$$\left(n(0) \times b(L/(2n))\right) \cdot \left(r(L/(2n)) - r(0)\right) = 0$$

2) closure:

$$n(0) \cdot b(L/(2n)) = \cos \left( m \frac{\pi}{n} \right)$$

$D_n$-symmetric solutions consisting of a sequence of $2n$ congruent pieces, either of short ($S_1 I$) or long ($S_0 I$) type
$D_3$-symmetric solution for $(m, n) = (2, 3)$
Two-sided strip for \((m, n) = (1, 2)\), with \(Lk = 1\)

\(D_2\)-symmetric solution (figure-of-eight):

Self-contacting solutions constructed from short \((S_1 I)\) piece of Möbius strip:

Last shape is at first self-intersection \((L/(2w) = 4.218)\)
Contact force:

Limiting aspect ratio (flat triple-covered rhombus): $L/(2w) = 2\sqrt{3} \approx 3.4641$
$D_2$-symmetric solutions for $(m, n) = (1, 2)$, with $Lk = 1$

Solution constructed from long $(S_0 I)$ piece of Möbius strip (four views):

No self-contact
One-sided strip for \((m, n) = (2, 3)\), with \(Lk = 3/2\)

First self-contact at \(w/L = 0.106\)

Contact condition: \(y_c = w\) (\(y_c\) intersection of \(D_n\)-symmetry axis with \(y\) axis)
$D_3$-symmetric solutions for $(m, n) = (2, 3)$, with $Lk = 3/2$

Solution constructed from long $(S_0I)$ piece of Möbius strip (four views):
$D_n$-symmetric solutions for $m = 3, 4, 5, \ n = m + 1$

Solutions constructed from short ($S_1I$) piece of Möbius strip (two views):

Limiting flat shapes at aspect ratio $L/(2w) = n \cot \frac{\pi}{n}$
$D_n$-symmetric solutions for $m = 1, n = 3, 4, 5$

Solutions constructed from long ($S_0I$) piece of Möbius strip (two views):
$D_n$-symmetric solutions for $(m, n) = (2, 5)$ and $(3, 5)$

Top and side views:
All generators through single point (apex):  \( \eta = \sqrt{2} - s/w \quad \Rightarrow \quad \eta' = -1/w \)
Recall:

\[ U = \frac{D}{2} \int_0^L \int_{-w}^w \frac{\kappa^2 (1 + \eta^2)^2}{1 + t\eta'} \, dt \, ds = Dw \int_0^L g(\kappa, \eta, \eta') \, ds \]

\[ g(\kappa, \eta, \eta') = \kappa^2 (1 + \eta^2)^2 \, V(w\eta'), \quad V(w\eta') = \frac{1}{2w\eta'} \log \left( \frac{1 + w\eta'}{1 - w\eta'} \right) \]

Diverges!

Regularisation (cut off tip): integrate \( t \in [-w, (1 - \epsilon)w] \), \( 0 < \epsilon \ll 1 \)

Then \( V \) is a multiplicative constant, \( V = -\frac{1}{2} \log \frac{\epsilon}{2} \), which does not enter the equations

New regularised (partial) Lagrangian (\( \lambda(s) \) Lagrange multiplier):

\[ \tilde{g}(\kappa, \eta) = \kappa^2 (1 + \eta^2)^2 + \lambda(s) \left( \eta - \sqrt{2} + s/w \right) \]

Can derive equations as before and formulate a BVP, but cannot satisfy \( \eta'(0) = 0! \) (generator continuous but not smooth)
Reformulation for different reference curve

The generators induce a one-to-one mapping between the strip’s centreline and the circular arc, \( s \rightarrow s_c \) for \( s \in [0, w\sqrt{2}] \), \( s_c \in [0, s_{ce}] \), \( s_{ce} = w \arctan \sqrt{2} \), given by

\[
\sqrt{2} - \frac{s}{w} = \tan \frac{s_{ce} - s_c}{w} = \tan \left( \arctan(\sqrt{2}) - \frac{s_c}{w} \right), \quad \text{ds} = (1 + \eta^2)\text{ds}_c
\]

Also, letting \( \kappa_N \) be the normal curvature of the new reference curve (taking \( t = t_c \)):

\[
t_c = w(1 - \sin \beta) = w(1 - 1/\sqrt{1 + \eta^2}), \quad \kappa_N = -\kappa(1 + \eta^2)^{3/2}
\]

Then bending energy becomes

\[
\int_0^{w\sqrt{2}} \kappa(s)^2 \left( 1 + \eta(s)^2 \right)^2 \text{ds} = \int_0^{s_{ce}} \kappa_N^2(s_c) \text{ds}_c
\]

‘Elastica’!
Darboux vector at circular arc has \( \hat{\omega}_c = R_c \frac{dR_c}{ds_c}, R_c(s_c) := (T, N, U) \):

\[
\omega_{c1} = \tau_g \equiv 0, \quad \omega_{c2} = \kappa_g \equiv \frac{1}{W}, \quad \omega_{c3} = \kappa_N
\]

Lagrangian \( (M_{c1}(s_c) \text{ and } M_{c2}(s_c) \text{ Lagrange multipliers}) \):

\[
\mathcal{L}_c = \omega_{c3}^2 + M_{c1} \omega_{c1} + M_{c2} \left( \omega_{c2} - \frac{1}{W} \right) - F_{c1}
\]

Constitutive relation:

\[
M_{c3} = 2 \kappa_N
\]

Hamiltonian:

\[
\mathcal{H}_c = \kappa_N^2 + \frac{M_{c2}}{W} + F_{c1}
\]

Balance equations give:

\[
\frac{d^2 \kappa_N}{ds_c^2} + \frac{\kappa_N}{2} \left( \kappa_N^2 - H_c + \frac{2}{W^2} \right) = 0
\]

Indeed curvature equation for the Euler elastica
Fabric draping

(from: Cerda et al., *PNAS* 101, 2004)
Assembling the tricone

Kinematics equations:

\[
\frac{d\psi_c}{ds_c} = -\frac{1}{w} \sin \varphi_c \csc \vartheta_c, \quad \frac{d\vartheta_c}{ds_c} = -\frac{1}{w} \cos \varphi_c, \quad \frac{d\varphi_c}{ds_c} = \kappa_N + \frac{1}{w} \sin \varphi_c \cot \vartheta_c
\]

BVP: 6 boundary conditions for a 5th-order system with one free parameter \( H_c \)

Geometry decouples from mechanics

Since the only parameter \( w \) can be scaled out, we have a universal triconical shape \( (H_c w^2 = -1.830652) \)

Surface normal continuous, curvature discontinuous at generators connecting apices
Paper crumpling and garment wrinkling

Bending energy localisation along ridges bounding flat triangular regions

Generic feature of twisted elastic sheets
Tearing a sheet of paper

Singularities may indicate points of tearing:
Tearing a sheet of paper

Singularities may indicate points of tearing:

Tearing a deck of cards (or a telephone book):
Möbius strip crystals

Topological crystals (Tanda et al., Nature 417, 2002):

Figure 1: Scanning electron microscopic images of NbSe₂ crystal topology. a–c, The three types of topology, classified by the nature of their twisting (shown schematically beneath images; white ribbons represent NbSe₂ crystals, red spheres represent selenium droplets). a, Ring structure (0π twist). b, Möbius strip (1π twist). c, Figure-of-eight strip (2π twist). d, NbSe₂ fibres (white streaks) bend into rings around a selenium drop of diameter about 50 μm. The NbSe₂ ribbon is spooled onto a selenium droplet by surface tension until its ends bind together. e, High-magnification image showing a twist in a ribbon of NbSe₂ crystals; spooling of the ribbon can produce a twist, as in the formation of a Möbius crystal, owing to its anisotropic elastic properties. Scale bars, 10 μm.
Möbius graphene nanoribbons

(Caetano et al., J. Chem. Phys. 128, 2008)

First stable Möbius aromatic hydrocarbon molecule synthesised in 2003 (Ajami et al., Nature 426) – predicted by Heilbronner in 1964
Open strips with stress localisation

A new buckling pattern of twisted inextensible strips

Buckling under twist and high tension into a regular pattern consisting of helically stacked nearly-flat triangular facets

Alternating sequence of flat triangles with stress concentrations on the edge of the strip

Nature’s way of achieving global twisting by means of local bending and minimal stretch (unlike helicoid)
Other strip buckling patterns

Contrast with buckling of twisted *extensible* strips

Wrinkling of pulled sheet:

Sinusoidal buckling pattern of twisted strip:

Crispino & Benson (1986)  
Mockensturm & Mote (2001)
Use Möbius cut-out and symmetry:

Rotate about normal $\mathbf{n}$ at $s = 0 \ (\eta' = 0, \ F_n = M_n = 0)$ and about binormal $\mathbf{b}$ at $s = L \ (\eta = 0, \ F_b = M_b = 0)$ – force and moment balance preserved!
Strip solutions for $n = 8$

Length of strip $2nL$

Define displacements:
- end-to-end distance $|\mathbf{e}|$, where $\mathbf{e} = \mathbf{r}(2nL) - \mathbf{r}(0)$
- end-to-end angle $\alpha$ (angle between end binormals projected $\perp \mathbf{e}$)

Corresponding loads: $\vec{F} = \mathbf{F}(L) \cdot \hat{\mathbf{e}}$, $\vec{M} = \mathbf{M}(L) \cdot \hat{\mathbf{e}}$ ($\hat{\mathbf{e}} = \mathbf{e}/|\mathbf{e}|$)
Load-displacement diagrams

\( n = 2 : \)

\( n = 4 : \)
Instability - mode jumping

\( n = 8 \):

Jump into higher modes under increase of tension:
Twisted strip – phase diagram

(Chopin & Kudrolli, *PRL* 111, 2013)

L=45 cm, W=25.4 mm, h=127 µm