Efficient Simulation of Generalized SABR and Stochastic Local Volatility Models

Duy Nguyen[1]

Joint with Zhenyu Cui (Stevens) & J. Lars Kirkby (Georgia Tech)

[1] Marist College

Overview

- SABR and generalized SABR
  - Motivation and Overview
  - Probability distribution
- Simulation framework
  - Continuous time Markov chain (CTMC) approximation
  - Characteristic functions (ChF)
  - PROJ: simulation from ChF
- Numerical examples

SABR and generalized SABR

The standard SABR model introduced in Hagan et al. (2002)[2]

$$\begin{cases} dS_t = V_t S_t^{\beta} dW_t^{(1)}, \\ dV_t = \alpha V_t dW_t^{(2)}, \end{cases} \tag{1}$$

where $\alpha > 0$ and $\beta \in [0, 1]$ are constants, and $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$ with $-1 < \rho < 1$.

- $\boxed{\beta = 1}$: $S_T = S_0 \exp\left\{-\frac{1}{2}\int_0^T V_s^2 ds + \frac{\rho}{\alpha}(V_T - V_0) + \sqrt{1-\rho^2}\int_0^T V_s dW_s^{(1)}\right\}$
- $\log(S_T)|(V_T, \int_0^T V_s ds) \sim \mathcal{N}(\log(S_0) - \frac{1}{2}\int_0^T V_s^2 ds + \frac{\rho}{\alpha}(V_T - V_0); (1-\rho^2)\int_0^T V_s^2 ds)$
- $\boxed{0 \le \beta < 1}$: It is difficult to derive the *exact* joint distribution of S_T and V_T. To our best knowledge, there is no complete answer to this question.

[2] Hagan, P. S., D. Kumar, A. S. Lesniewski, and D. E. Woodward (2002). Managing smile risk. Wilmott Magazine pp. 84-108.

Generalized SABR

Consider the following stochastic volatility model:

$$\begin{cases} dS_t = m(V_t)S_t^{\beta} dW_t^{(1)}, \\ dV_t = \alpha(V_t)dt + \gamma(V_t)dW_t^{(2)}, \end{cases} \tag{2}$$

- $\alpha > 0$ and $\beta \in [0, 1]$ are constants
- $\mathbb{E}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$ with $-1 < \rho < 1$.
- Here it is assumed that the functions $m(.), \alpha(.), \gamma(.)$ satisfy some technical conditions so the system has a unique (in law) solution.

GBM(SABR) (Hagan et al. (2002))	$m(v)=v, \quad dv_t=\alpha v_t dW_t^{(2)}$ $f(v)=v/\alpha, \quad h(v)\equiv 0$
CIR (Heston) (Heston (1993))	$m(v)=\sqrt{v}, \quad dv_t=\kappa(\theta-v_t)dt+\sigma_v\sqrt{v_t}dW_t^{(2)}$ $f(v)=v/\sigma_v, \quad h(v)=\kappa(\theta-v)/\sigma_v$
4/2 (Grasselli (2017))	$m(v)=a\sqrt{v}+b/\sqrt{v}, \quad dv_t=\eta(\theta-v_t)dt+\sigma_v\sqrt{v_t}dW_t^{(2)}$ $f(v)=\frac{(av+b\log(v))}{\sigma_v}, \quad h(v)=\frac{\eta(a\theta-b)}{\sigma_v}-\frac{a\eta v}{\sigma_v}+\left(\frac{\eta\theta b}{\sigma_v}-\frac{b\sigma_v}{2}\right)\frac{1}{v}$
Stein-Stein (Stein and Stein (1991))	$m(v)=v, \quad dv_t=\eta(\theta-v_t)dt+\sigma_v dW_t^{(2)}$ $f(v)=\frac{v^2}{2\sigma_v}, \quad h(v)=\frac{\sigma_v}{2}+\frac{\eta\theta v}{\sigma_v}-\frac{\eta v^2}{\sigma_v}$
3/2 (Lewis (2000))	$m(v)=1/\sqrt{v}, \quad d\widehat{v}_t=\widehat{\eta}[\widehat{\theta}-\widehat{v}_t]dt+\widehat{\sigma}_v\sqrt{\widehat{v}_t}dW_t^{(2)}$ $f(v)=\frac{\log(v)}{\widehat{\sigma}_v}, \quad h(v)=\left(\frac{\widehat{\eta}\widehat{\theta}}{\widehat{\sigma}_v}-\frac{\widehat{\sigma}_v}{2}\right)\frac{1}{v}-\frac{\widehat{\eta}}{\widehat{\sigma}_v}$
Hull-White (Hull and White (1987))	$m(v)=\sqrt{v}, \quad dv_t=a_v v_t dt+\sigma_v v_t dW_t^{(2)}$ $f(v)=\frac{2\sqrt{v}}{\sigma_v}, \quad h(v)=\left(\frac{a_v}{\sigma_v}-\frac{\sigma_v}{4}\right)\sqrt{v}$
Scott (Scott (1987))	$m(v)=\exp(v), \quad dv_t=\eta(\theta-v_t)dt+\sigma_v dW_t^{(2)}$ $f(v)=\frac{e^v}{\sigma_v}, \quad h(v)=e^v\left(\frac{\eta\theta}{\sigma_v}+\frac{\sigma_v}{2}-\frac{\eta v}{\sigma_v}\right)$
α-Hyper (Da Fonseca and Martini (2016))	$m(v)=\exp(v), \quad dv_t=(\eta-\theta\exp(a_v v_t))dt+\sigma_v dW_t^{(2)}$ $f(v)=\frac{e^v}{\sigma_v}, \quad h(v)=e^v\left(\frac{\eta}{\sigma_v}+\frac{\sigma_v}{2}\right)-\frac{\theta}{\sigma_v}e^{(a_v+1)v}$
CEV-SV (Andersen and Piterbarg (2007))	$m(v)=\sqrt{v}, \quad dv_t=\eta(\theta-v_t)dt+\sigma_v v_t^{\alpha}dW_t^{(2)}$ $f(v)=\frac{1}{\sigma_v(3/2-\alpha)}v^{3/2-\alpha}, \quad h(v)=\frac{\eta}{\sigma_v}(\theta-v)v^{1/2-\alpha}+\frac{\sigma_v}{2}\left(\frac{1}{2}-\alpha\right)v^{\alpha-1/2}$
$\tau/2$ models (Christoffersen et al. (2010))	$m(v)=\sqrt{v}, \quad dv_t=\eta v_t^{\omega}(\theta-v_t)dt+\sigma_v v_t^{\tau/2}dW_t^2, \quad \tau\neq 3$ $f(v)=\frac{2}{\sigma_v(3-\tau)}v^{(3-\tau)/2}, \quad h(v)=\frac{\eta}{\sigma_v}v^{(1-\tau)/2+\omega}(\theta-v)+\frac{\sigma_v}{4}(1-\tau)v^{(\tau-1)/2}$

Some selected previous works

1. Broadie, M. and O. Kaya (2006). Exact simulation of stochastic volatility and other affine jump diffusion processes. Operations Research 54(2), 217-231.
2. Cai, N., Y. Song, and N. Chen (2017). Exact simulation of the SABR model. Operations Research 65(4), 931-951.
3. Kang, C., W. Kang, and J. M. Lee (2017). Exact simulation of the Wishart multidimensional stochastic volatility model. Operations Research 65(5), 1190-1206.
4. Rhee, C.-h. and P. W. Glynn (2015). Unbiased estimation with square root convergence for SDE models. Operations Research 63(5), 1026-1043.

Our contributions

- We propose a unified efficient simulation framework for a wide class of stochastic volatility models including Heston, 3/2, 4/2, SABR, Heston-SABR,...
- Extensive numerical examples illustrate the accuracy and efficiency of our estimator, which compares favorably to existing biased and unbiased simulation estimators in the literature in terms of root mean square error (RMSE) and computational time.

The case $\beta = 1$

The model is reduced to

$$\begin{cases} dS_t = m(V_t)S_t dW_t^{(1)} \\ dV_t = \alpha(V_t)dt + \gamma(V_t)dW_t^{(2)}. \end{cases}$$

- Cholesky decomposition: $W_t^{(1)} := \rho W_t^{(2)} + \sqrt{1-\rho^2}W_t^{(3)}$, where $W_t^{(3)} \perp\!\!\!\perp W_t^{(2)}$.
- $f(v) := \int_0^v \frac{m(z)}{\gamma(z)} dz$
- $h(v) := \alpha(v)f'(v) + \frac{1}{2}\gamma^2(v)f''(v)$

The case $\beta = 1$

The model is reduced to

$$\begin{cases} dS_t = m(V_t)S_t dW_t^{(1)} \\ dV_t = \alpha(V_t)dt + \gamma(V_t)dW_t^{(2)} \end{cases}$$

Under the assumption: $\boxed{h(v) = \theta_1 + \theta_2 m^2(v)}$

$$\log(S_\Delta) \Big| \left(V_\Delta, \int_0^\Delta m^2(V_s)ds \right)$$
$$\sim \mathcal{N}\left(\log(S_0) + \zeta_1,\ (1-\rho^2)\int_0^\Delta m^2(V_s)ds \right),$$

where
$\zeta_1 := -\left(\frac{1}{2} + \rho\theta_2\right)\int_0^\Delta m^2(V_s)ds + \rho\left(f(V_\Delta) - f(V_0) - \theta_1\Delta\right)$.
Hence, conditional on $\left(V_\Delta, \int_0^\Delta m^2(V_s)ds\right)$, the underlying asset price can be simulated *exactly* with a single draw from a normal distribution

The case $\beta = 0$

The model is reduced to

$$\begin{cases} dS_t = m(V_t)[\rho dW_t^{(2)} + \sqrt{1-\rho^2} dW_t^{(3)}], \\ dV_t = \alpha(V_t)dt + \gamma(V_t)dW_t^{(2)}. \end{cases}$$

Under the assumption: $\boxed{h(v) = \theta_1 + \theta_2 m^2(v)}$ or $\boxed{\rho = 0}$:

$$\log(S_\Delta)\Big| \left(V_\Delta, \int_0^\Delta m^2(V_s)ds\right)$$
$$\sim \mathcal{N}\left(S_0 + \zeta_2\left(V_0, V_\Delta, \int_0^\Delta m^2(V_s)ds\right), \, (1-\rho^2)\int_0^\Delta m^2(V_s)ds\right).$$

Interior Case: $\beta \in (0,1)$

When $\beta \in (0,1)$ and $\rho = 0$

$$\mathbb{P}\left(S_\Delta = 0 \mid S_0 > 0, V_\Delta, \int_0^\Delta m^2(V_s)ds\right) = 1 - \chi^2_b(a;0),$$

$$\mathbb{P}\left(S_\Delta \leq \zeta \mid S_0 > 0, V_\Delta, \int_0^\Delta m^2(V_s)ds\right) = 1 - \chi^2_b(a;c),$$

where

$$a = \frac{1}{v(\Delta)}\left(\frac{S_0^{1-\beta}}{1-\beta}\right)^2, \quad b = \frac{1}{1-\beta},$$

$$c = \frac{\zeta^{2(1-\beta)}}{(1-\beta)^2 v(\Delta)}, \quad v(\Delta) = \int_0^\Delta m^2(V_s)ds,$$

$\chi^2_\delta(x;\lambda)$ is the non-central chi-square cumulative distribution function with a non-centrality parameter λ and the degree of freedom δ.

Interior Case: $\beta \in (0,1)$

When $\beta \in (0,1)$ and $\rho \neq 0$

$$\mathbb{P}\left(S_\Delta = 0 \mid S_0 > 0, V_\Delta, \int_0^\Delta m^2(V_s)ds, \int_0^\Delta h(V_s)ds\right) \approx 1 - \chi_b^2(a;0),$$

$$\mathbb{P}\left(S_\Delta \leq \zeta \mid S_0 > 0, V_\Delta, \int_0^\Delta m^2(V_s)ds, \int_0^\Delta h(V_s)ds\right) \approx 1 - \chi_b^2(a;c),$$

where

$$a = \frac{1}{v(\Delta)}\left(\frac{S_0^{1-\beta}}{1-\beta} + A_\Delta\right)^2, \quad b = 2 - \frac{(1-2\beta-\rho^2(1-\beta))}{(1-\beta)(1-\rho^2)},$$

$$c = \frac{\zeta^{2(1-\beta)}}{(1-\beta)^2 v(\Delta)}, \quad v(\Delta) = (1-\rho^2)\int_0^\Delta m^2(V_s)ds.$$

$$A_\Delta = \rho\left(f(V_\Delta) - f(V_0) - \int_0^\Delta h(V_s)ds\right).$$

Continuous Time Markov Chain (CTMC) approximation

Consider the variance process

$$dV_t = \alpha(V_t)dt + \gamma(V_t)dW_t^{(2)},$$

- We approximate V_t by a continuous time Markov chain $\tilde{V}_t^{m_0}$ taking m_0 values $\{v_0, \ldots, v_{m_0}\}$
- The generator $Q = [q_{ij}]$ satisfies the q-properties

$$q_{ij} := \begin{cases} \dfrac{\alpha^-(v_i)}{k_{i-1}} + \dfrac{\gamma^2(v_i) - (k_{i-1}\alpha^-(v_i) + k_i\alpha^+(v_i))}{k_{i-1}(k_{i-1}+k_i)}, & \text{if } j = i-1, \\ \dfrac{\alpha^+(v_i)}{k_i} + \dfrac{\gamma^2(v_i) - (k_{i-1}\alpha^-(v_i) + k_i\alpha^+(v_i))}{k_i(k_{i-1}+k_i)}, & \text{if } j = i+1, \\ -q_{i,i-1} - q_{i,i+1}, & \text{if } j = i, \\ 0, & \text{if } j \neq i-1, i, i \end{cases}$$

Proposition

Consider a uniform variance grid $\{v_j\}_{j=1}^{m_0}$*, and let* $\Psi(\cdot)$ *be a continuous function whose* $\frac{\partial^4 \Psi}{\partial x^4}$ *is bounded. Assume that at least one of the following conditions hold: i)* $\Psi(l_*) = \Psi(u_*) = 0$*, where* $[l_*, u_*]$ *is the compact support of* Ψ*; ii)* V_t *has a compact state space* $\mathbb{V}$ *with reflecting or absorbing boundaries. Then we have weak convergence of the CTMC scheme with a quadratic convergence order:*

$$\left| \mathbb{E}[\Psi(V_t) | \tilde{V}_0^{m_0} = v] - \mathbb{E}[\Psi(\tilde{V}_t^{m_0}) | \tilde{V}_0^{m_0} = v] \right| \leq C m_0^{-2}, \quad t \in [0, T],$$

for some $C > 0$ *that depends on* Ψ*.*

Proposition

For a time increment $\Delta > 0$*, and function* $g : \mathbb{R} \to \mathbb{R}$*, the following holds:*
The conditional ChF with $(\tilde{V}_0, \tilde{V}_\Delta) = (v_j, v_k)$ *is given by*

$$\mathbb{E}\left[\exp\left(i\xi \int_0^\Delta g(\tilde{V}_s)ds\right) \Big| v_j, v_k\right] = \Lambda_{j,k}(\xi), \quad j, k = 1, \ldots, m_0,$$

where $\Lambda(\xi)$ *is the matrix exponential defined by*

$$\Lambda(\xi) = \exp(\Delta(Q + i\xi \cdot diag(g(v_1), g(v_2), \ldots, g(v_{m_0})))).$$

Biorthogonal projection approach

Goal: to simulate $\left(\int_0^{\Delta} g(\tilde{V}_s)ds\right) \big| \tilde{V}_0, \tilde{V}_\Delta$ from its ChF

- Given a function φ named the *generator*.
- we construct a sequence on the support of f as

$$x_k = x_1 + (k-1)h, \quad k \in \mathbb{Z},$$

- Form the sequence $\{\varphi_{a,k}(x)\}_{k\in\mathbb{Z}} := a^{1/2}\{\varphi(a(x - x_k))\}_{k\in\mathbb{Z}}$, which generates an approximation space $\mathcal{M}_a := \overline{\text{span}}\{\varphi_{a,k}\}_{k\in\mathbb{Z}}$

$$A\|g\|_2^2 \leq \sum_{k\in\mathbb{Z}} |\langle g, \varphi_{a,k}\rangle|^2 \leq B\|g\|_2^2, \quad \forall g \in L^2(\mathbb{R}),$$

B-spline generators

1. $p = 0$: Haar generator $\quad \varphi(y) \equiv \varphi^{[0]}(y) = \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(y)$
2. $p = 1$: Linear generator

$$\varphi(y) \equiv \varphi^{[1]}(y) = \begin{cases} 1 + y, & y \in [-1, 0], \\ 1 - y, & y \in [0, 1], \end{cases}$$

3. $p \geq 2$ p-order B-spline generators are derived recursively by the convolution

$$\varphi(x) \equiv \varphi^{[p]}(x) = \varphi^{[0]} \star \varphi^{[p-1]}(x) = \int_{-\infty}^{\infty} \varphi^{[p-1]}(y-x) \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(y) dy.$$

- The orthogonal projection of any $f \in L^2(\mathbb{R})$ onto $\mathcal{M}_a$ satisfies

$$P_{\mathcal{M}_a} f(y) = \sum_{k \in \mathbb{Z}} \left(\int_{-\infty}^{\infty} f(x) \overline{\widetilde{\varphi}_{a,k}(x)} dx \right) \varphi_{a,k}(y) = \sum_{k \in \mathbb{Z}} \beta_{a,k} \cdot \varphi_{a,k}(y),$$

- $\beta_{a,k} := \mathbb{E}[\widetilde{\varphi}_{a,k}(X_1)]$.
- $\widetilde{\varphi}$ is the dual of φ. For example, for the case p-order B-spine generators

$$\widehat{\widetilde{\varphi}} \equiv \widehat{\widetilde{\varphi}}^{[p]}(\xi) = \widehat{\varphi}^{[p]}(\xi) / \Phi^{[p]}(\xi), \widehat{\varphi}^{[p]}(\xi) = \left(\frac{\sin(\xi/2)}{(\xi/2)} \right)^{p+1},$$

$$\Phi^{[p]}(\xi) = \int_{-\frac{p+1}{2}}^{\frac{p+1}{2}} \varphi^{[p]}(x)^2 dx + 2 \sum_{k=1}^{p+1} \cos(k\xi) \int_{-\frac{p+1}{2}}^{\frac{p+1}{2}} \varphi^{[p]}(x) \varphi^{[p]}(x-k) dx$$

- $||f - P_{\mathcal{M}_a} f||_2^2 \leq C ||f^{(p+1)}||_2^2 \cdot h^{2(p+1)}$

Projected density estimator

- The orthogonal projection of any $f \in L^2(\mathbb{R})$ onto $\mathcal{M}_a$ satisfies
$$P_{\mathcal{M}_a} f(y) = \sum_{k \in \mathbb{Z}} \beta_{a,k} \cdot \varphi_{a,k}(y),$$
- $\beta_{a,k} := \mathbb{E}[\widetilde{\varphi}_{a,k}(X_1)]$.
- For the linear B-spline basis:
$$\beta_{a,k} = \frac{a^{-1/2}}{\pi} \Re \left\{ \int_0^\infty \exp(-i x_k \xi) \cdot \phi_X(\xi) \cdot \widehat{\widetilde{\varphi}}(\xi/a) d\xi \right\}, \quad k \in \mathbb{Z},$$
where
$$\phi_X(\xi) = \mathbb{E}[e^{iX\xi}], \widehat{\widetilde{\varphi}}(\xi) = \frac{12 \sin^2(\xi/2)}{\xi^2(2 + \cos(\xi))}, \xi \in \mathbb{R},$$
which are respectively the ChF of X, and the Fourier transform of $\widetilde{\varphi}$.

Single step algorithm

- *Variance*. To sample from $\tilde{V}_\Delta | \tilde{V}_0 = v_{j_0} = v_0$[3], and let

$$P^\Delta_{j,k} = \boldsymbol{e}_j^\top \exp(Q\Delta)\boldsymbol{e}_k, \quad j,k = 1,\dots,m_0.$$

We store a set of cumulative conditional probabilities for the terminal state $\tilde{V}_\Delta = v_k \in \mathcal{S}_v$:

$$\mathbf{F}^v_{j_0} := \{F^v_{j_0,k}\}_{k=1}^{m_0}, \qquad F^v_{j_0,k} = \sum_{n=1}^{k} P^\Delta_{j_0,n}, \quad k = 1,\dots,m_0.$$

Simulating $\tilde{V}_\Delta | \tilde{V}_0 = v_{j_0}$ requires a single draw from a standard uniform random variable $U \sim \mathcal{U}(0,1)$.

[3]Note that given any initial v_0, it is simple to adjust the grid so that $v_{j_0} = v_0$ for some index $j_0 \in \{1,\dots,m_0\}$.

- *Integrated Variance* $I^\Delta = \int_0^\Delta g(\tilde{V}_s)ds|(\tilde{V}_0 = v_{j_0}, \tilde{V}_\Delta = v_k)$: We draw a sample $I^\Delta|(v_{j_0}, v_k)$ from its characteristic function using the projection density function.
- *Underlying*:

  $\boxed{\beta \in \{0,1\}}$

$$\tilde{S}_\Delta = \begin{cases} S_0 \exp\Big(-\left(\frac{1}{2}+\rho\theta_2\right)I^\Delta|(v_{j_0}, v_k) + f^\Delta(v_k, v_{j_0}) + Z\Big), \beta = 1, \\ S_0 - \rho\theta_2 I^\Delta|(v_{j_0}, v_k) + f^\Delta(v_k, v_{j_0}) + Z, \quad \beta = 0. \end{cases}$$

$Z \sim \mathcal{N}\left(0, (1-\rho^2)I^\Delta|(v_{j_0}, v_k)\right)$,
$f^\Delta(v_k, v_{j_0}) := \rho\left(f(v_k) - f(v_{j_0}) - \theta_1\Delta\right)$.
$\boxed{\beta \in (0,1)}$:

$$\tilde{S}_\Delta = \Big((1-\beta)^2 d(e+Z_1)^2 \cdot (1-\rho^2)I^\Delta|(v_{j_0}, v_k)\Big)^{\frac{1}{2(1-\beta)}}.$$

$Z_1 \sim \mathcal{N}(0,1)$, d, e are known.

Numerical example: Compare with Broadie and Kaya (2006)[4] and Kang et al. (2017)[5]

Test Case (Heston)	S_0	T	r	κ	θ	v_0	σ_V	ρ
Case A	100	1	0.0319	6.21	0.019	0.010201	0.61	-0.70
Case B	100	1	0.05	2.0	0.09	0.09	1.0	-0.3

Table: Baseline parameters for Heston experiments. $q = 0$, and $K = S_0$.

[4] Broadie, M. and O. Kaya (2006). Exact simulation of stochastic volatility and other affine jump diffusion processes. Operations Research 54(2), 217-231

[5] Kang, C., W. Kang, and J. M. Lee (2017). Exact simulation of the Wishart multidimensional stochastic volatility model. Operations Research 65(5), 1190-1206

	CTMC			Euler			KKL		BK	
N_{sim}	m_0	RMSE	Time	Steps	RMSE	Time	RMSE	Time	RMSE	Time
10000	20	3.70e-02	0.11	100	3.63e-01	0.03	7.80e-02	0.10	7.50e-02	3.8
40000	22	1.99e-02	0.16	200	2.27e-01	0.20	4.08e-02	0.42	3.73-02	15.2
160000	30	1.06e-02	0.33	400	1.46e-01	1.92	2.12e-02	1.63	1.86-02	60.0
640000	34	4.24e-03	0.79	800	8.09e-02	16.11	1.11e-02	6.58	9.30e-03	239.4

Table: Heston: European call option on equity, case A parameters from Table 2. Ref. price: 6.80611.

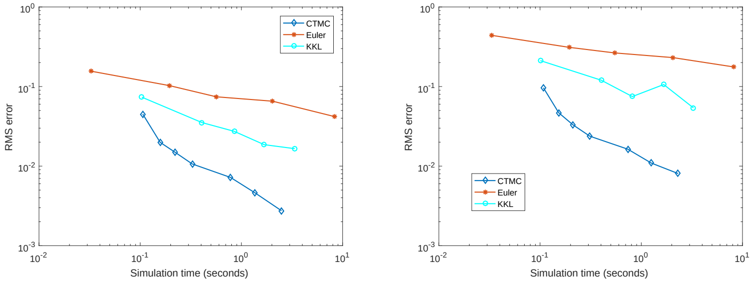

Figure: Heston: Convergence of RMS error. The Left panel corresponds to Case A, and the Right panel to Case B.

Numerical results for SABR model: Compare with Cai et al. (2017) [6]

Test Case (SABR)	S_0	T	ρ	β	α	v_0
Case A (Interest rates)	0.05	1	0	0.55	0.03	0.20
Case B (Foreign exchange)	1.10	1	0	0.70	0.10	0.20
Case C (Equity derivatives)	100	1	0	0.60	0.20	0.30

Table: Baseline parameters for SABR experiments. We also set $r = q = 0$, and $K = S_0$, unless stated otherwise.

[6] Cai, N., Y. Song, and N. Chen (2017). Exact simulation of the SABR model. Operations Research 65(4), 931-951

		CTMC, $N_{sim} = 50,000$					Alternative Methods (Price)		
Case	K	Time (sec)	Avg. Price	Avg. Err.	Std. Err.	RMSE	Exact Sim (Std.Err.)	Expansion	FDM
A	0.045	0.26	0.01732	-6.72e-05	6.86e-06	9.58e-05	0.01727 (9.19e-06)	0.01726	0.01725
	0.050	0.26	0.01513	-7.77e-05	7.09e-06	1.05e-04	0.01507 (8.69e-06)	0.01506	0.01505
	0.055	0.26	0.01319	-9.79e-05	5.59e-06	1.13e-04	0.01311 (8.20e-06)	0.01311	0.01310
B	1.0	0.24	0.14197	-5.89e-06	2.63e-05	2.62e-04	0.14188 (5.21e-05)	0.14196	0.14197
	1.1	0.24	0.08529	-4.48e-05	3.24e-05	3.25e-04	0.08518 (4.23e-05)	0.08524	0.08523
	1.2	0.24	0.04680	4.52e-05	2.91e-05	2.93e-04	0.04679 (3.20e-05)	0.04683	0.04683
C	90	0.23	10.03049	2.39e-04	1.47e-04	1.48e-03	10.02941 (1.48e-03)	10.03079	10.03078
	100	0.23	1.90258	4.13e-04	6.54e-04	6.52e-03	1.90169 (8.94e-04)	1.90301	1.90294
	110	0.23	0.04480	-1.74e-04	1.26e-04	1.69e-03	0.04444 (1.30e-04)	0.04469	0.04468

Table: SABR: European call options. The expansion column is obtained using by expansion of Hagan et al. (2002). Exact simulation results are obtained in Cai et al. (2017) using 10, 240, 000 sample paths. For CTMC, we use 50, 000 paths, and estimate the average price (CTMC) and error metrics and with respect to the finite difference method (FDM) using 100 replications.

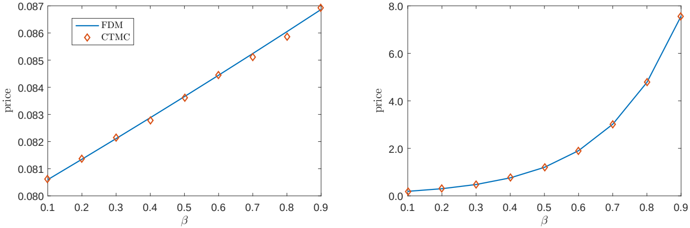

Figure: SABR: prices as function of β, for Case B (left) and Case C (right) from Table 4, with $N_{sim} = 400,000$.

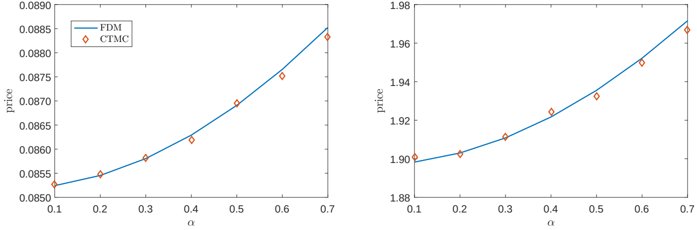

Figure: SABR: prices as function of α, for Case B (left) and Case C (right) from Table 4, with $N_{sim} = 400,000$.

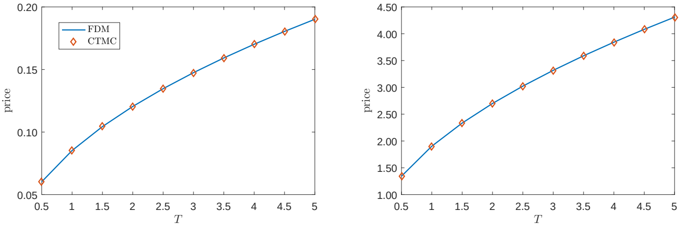

Figure: SABR: prices as function of T, for Case B (left) and Case C (right) from Table 4, with $N_{sim} = 400,000$.

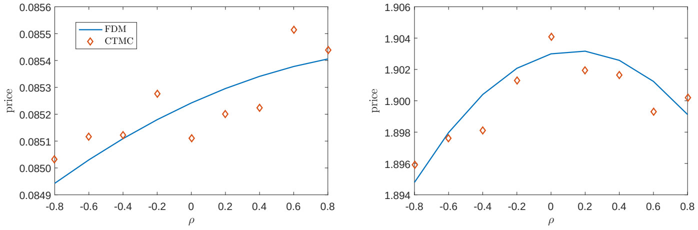

Figure: SABR: prices as function of ρ, for Case B (left) and Case C (right) from Table 4, with $N_{sim} = 400,000$.

SABR model: Compare with Rhee and Glynn (2015)[7]

	CTMC			CTMC-MS				Euler			Unbiased		
N_{sim}	m_0	RMSE	Time	m_0	Steps	RMSE	Time	Steps	RMSE	Time	IRE	RMSE	Time
10000	20	1.46e-02	0.11	20	6	1.53e-02	0.02	100	1.58e-02	0.07	0.010	2.39e-02	0.17
40000	22	7.67e-03	0.17	22	8	8.28e-03	0.10	200	8.32e-03	0.47	0.005	6.45e-03	0.66
160000	24	3.68e-03	0.35	24	10	3.21e-03	0.57	400	3.55e-03	4.33	0.002	2.47e-03	8.49
640000	26	1.76e-03	1.01	26	12	1.45e-03	2.76	800	1.82e-03	34.41	0.001	1.15e-03	43.46

Table: SABR: European call option on equity, case C parameters. Ref. price: 1.90300.

[7] Rhee, C.-h. and P. W. Glynn (2015). Unbiased estimation with square root convergence for SDE models. Operations Research 63(5), 1026-1043.

SABR model: Compare with Rhee and Glynn (2015)[8]

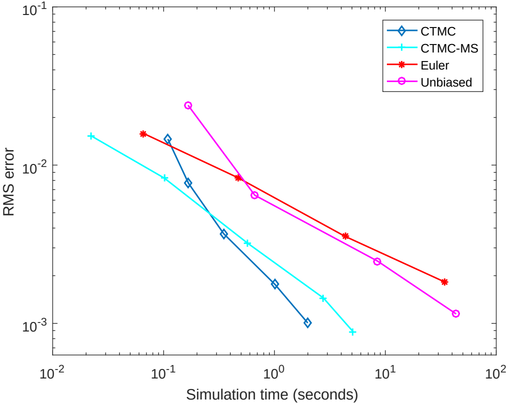

Figure: SABR: European call option on equity, case C parameters.

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[8] Rhee, C.-h. and P. W. Glynn (2015). Unbiased estimation with square root convergence for SDE models. Operations Research 63(5), 1026-1043.

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THANK YOU !!!