Valuing Equity-Linked Insurance Products

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Motivation

- To value guarantees and options in Variable Annuities
- Variable Annuities
  $=$ Investment Funds (Mutual Funds)
  $+$
  Rider(s) : Guaranteed Minimum Benefits

Equity-linked death benefit

- x age at issue of policy (time 0)
- T_x time of death
- payment at time T_x
- depends on $S(T_x)$,
- or more generally on $S(t)$, $0 \leq t \leq T_x$

Goal:

- Calculate $\mathrm{E}[e^{-\delta T_x} \times \text{payment}]$
- the expectation of the discounted value of the payment

  δ valuation force of interest

- Equity-linked products are very popular in the market nowadays.
- Example: Guaranteed Minimum Death Benefits
- Payoff:

$$\max(S(T_x), K) = S(T_x) + [K - S(T_x)]_+ = K + [S(T_x) - K]_+,$$

where T_x is the time-until-death random variable for a life age x, $S(t)$ is the price of equity-index at time t, and K is the guaranteed amount.

Mathematical Problem:

- $\{X(t); \quad t \in [0.\infty), \text{or} \quad t = 0, 1, 2, ...\}$ a random process
- running minimum: $m(t) = \min_{0 \leq s \leq t} X(s)$
- running maximum: $M(t) = \max_{0 \leq s \leq t} X(s)$
- τ a random variable
- we are interested in the distributions of $X(\tau)$ and $(X(\tau), M(\tau))$ or $(X(\tau), m(\tau))$.

Exponential stopping of Brownian motion

- $X(t) = \mu t + \sigma W(t)$
- $\{W(t)\}$: standard Wiener process
- $$f_{X(\tau)}(x) = \begin{cases} \kappa e^{-\alpha x}, & \text{if } x < 0, \\ \kappa e^{-\beta x}, & \text{if } x > 0. \end{cases}$$

  with $\kappa = \frac{2\lambda}{\sigma^2(\beta-\alpha)}$, $\alpha < 0$ and $\beta > 0$ solutions of the quadratic equation $\frac{\sigma^2}{2}\xi^2 + \mu\xi - \lambda = 0$
- $f_{X(\tau),M(\tau)}(x, m) = \frac{2\lambda}{\sigma^2} e^{\alpha(m-x)-\beta m}$,

  $-\infty < x \leq m, \; m \geq 0$

Erlang stopping of Brownian motion

$$f_\tau(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t}, \qquad t > 0,$$

Let $\widehat{f_{X(\tau)}}(z)$ denote the two-sided Laplace transform of $f_{X(\tau)}(x)$.

$$\widehat{f_{X(\tau)}}(z) \quad = \quad \mathrm{E}[\mathrm{E}[e^{-zX(\tau)}|\tau]] = \mathrm{E}[e^{Dz^2\tau - \mu z\tau}] = \hat{f}_\tau(-Dz^2 + \mu z)$$

where

$$\hat{f}_\tau(z) = \left(\frac{\lambda}{z+\lambda}\right)^n.$$

$$
\begin{aligned}
\widehat{f_{X(\tau)}}(z) &= \left(\frac{\lambda}{-Dz^2+\mu z+\lambda}\right)^n \\
&= \left(\frac{\lambda}{-D(z+\beta)(z+\alpha)}\right)^n \\
&= \kappa^n\left(\frac{1}{z+\beta}-\frac{1}{z+\alpha}\right)^n,
\end{aligned}
$$

for $-\beta < z < -\alpha$.

Lemma

Let a and b be elements with the property that $ab = a + b$. We have

$$(a+b)^n = P_n(a) + P_n(b),$$

where

$$P_n(x) = \sum_{k=0}^{n-1} \binom{n-1+k}{k} x^{n-k}.$$

If a and b have the property that $ab = \nu(a+b)$ for some number $\nu \neq 0$, then

$$(a+b)^n = \nu^n P_n(\frac{a}{\nu}) + \nu^n P_n(\frac{b}{\nu}).$$

Let

$$a = \frac{-1}{\alpha + z}, \quad b = \frac{1}{\beta + z}, \quad \nu = \frac{1}{\beta - \alpha}.$$

Thus

$$\begin{aligned}
\widehat{f_{X(\tau)}}(z) &= \kappa^n \nu^n \{P_n\left(\frac{a}{\nu}\right) + P_n\left(\frac{b}{\nu}\right)\} \\
&= \kappa^n \sum_{k=0}^{n-1} \binom{n-1+k}{k} \left(\frac{1}{\beta - \alpha}\right)^k \left(\frac{-1}{\alpha + z}\right)^{n-k} \\
&\quad + \kappa^n \sum_{k=0}^{n-1} \binom{n-1+k}{k} \left(\frac{1}{\beta - \alpha}\right)^k \left(\frac{1}{\beta + z}\right)^{n-k}.
\end{aligned}$$

Distribution of $X(\tau)$

Note that $(\frac{-1}{\alpha+z})^j$ is the two-sided Laplace transform of the function $\frac{(-x)^{j-1}}{(j-1)!}e^{-\alpha x}I_{(x<0)}$, and $(\frac{1}{\beta+z})^j$ is that of $\frac{x^{j-1}}{(j-1)!}e^{-\beta x}I_{(x>0)}$. Hence

$$f_{X(\tau)}(x) = \begin{cases} \kappa^n e^{-\alpha x} \sum_{j=1}^n \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}}(-x)^{j-1}, & \text{if} \quad x < 0, \\ \kappa^n e^{-\beta x} \sum_{j=1}^n \frac{\binom{2n-j-1}{n-j}}{(j-1)!(\beta-\alpha)^{n-j}}x^{j-1}, & \text{if} \quad x > 0, \end{cases}$$

Reflection principle of Brownian motion

If the drift μ of the Brownian motion $X(t)$ is zero, it follows from the reflection principle that

$$\Pr(X(t) \leq x, M(t) > y) = \Pr(X(t) \leq x - 2y), \quad y \geq \max(x, 0).$$

- By changing the probability measure, we can change the drift. If the drift μ is an arbitrary constant, then

$$\begin{aligned}\Pr(X(t) \le x, M(t) > y) &= e^{y\mu/D}\Pr(X(t) \le x - 2y), \\ &\quad y \ge \max(x, 0).\end{aligned}$$

The joint density function of $X(t)$ and $M(t)$ is

$$\begin{aligned}f_{X(t),M(t)}(x, y) &= -\frac{\partial^2}{\partial y \partial x}\Pr(X(t) \le x, M(t) > y) \\ &= -\frac{\partial}{\partial y}[e^{y\mu/D} f_{X(t)}(x - 2y)], \quad y \ge \max(x, 0).\end{aligned}$$

Distribution of $(X(\tau), M(\tau))$

- We can replace t by τ, then

$$f_{X(\tau),M(\tau)}(x, y) = -\frac{\partial}{\partial y}[e^{y\mu/D} f_{X(\tau)}(x - 2y)], \qquad y \geq \max(x, 0).$$

From this we obtain the distribution of $(X(\tau), M(\tau))$.

Exponential stopping of Lévy process

- moment generating function of $X(t)$ is

$$\mathsf{E}[e^{zX(t)}] = e^{t\Psi(z)},$$

where $\Psi(z)$ denotes the *Lévy exponent* of the process

- the mgf of $X(\tau)$:

$$\mathsf{E}[e^{zX(\tau)}] = \mathsf{E}[\mathsf{E}[e^{zX(\tau)}|\tau]] = \mathsf{E}[e^{\tau\Psi(z)}] = \frac{\lambda}{\lambda - \Psi(z)}.$$

Suppose that $\Psi(z)$ is a rational function and that the roots of

$$\Psi(z) = \lambda$$

are distinct. Let $\{\alpha_j\}$ and $\{\beta_k\}$ be the roots with negative and positive real part, respectively.

$$\frac{\lambda}{\lambda - \Psi(z)} = \sum_j \frac{\lambda}{\Psi'(\alpha_j)} \frac{1}{\alpha_j - z} + \sum_k \frac{\lambda}{\Psi'(\beta_k)} \frac{1}{\beta_k - z},$$

The pdf of $X(\tau)$ is

$$f_{X(\tau)}(x) = \begin{cases} \sum_j a_j e^{-\alpha_j x}, & \text{if} \quad x < 0, \\ \sum_k b_k e^{-\beta_k x}, & \text{if} \quad x \geq 0, \end{cases}$$

where

$$a_j = \frac{-\lambda}{\Psi'(\alpha_j)}$$

and

$$b_k = \frac{\lambda}{\Psi'(\beta_k)}.$$

Exponential stopping of jump diffusion

Let $\{X(t);\ t \geq 0\}$ be a Brownian motion (with drift and diffusion parameters μ and σ) extended by independent jumps in both directions. The downward jumps form an independent compound Poisson process; the frequency of these jumps is ν. Similarly, the upward jumps forms another independent compound Poisson process with Poisson parameter ω.

Assume that the pdf of each downward jump is

$$\sum_{j=1}^{m} A_j v_j e^{-v_j x}, \qquad x > 0,$$

with $\sum_{j=1}^{m} A_j = 1$ and $0 < v_1 < v_2 < ... < v_m$, and that the pdf of each upward jump is

$$\sum_{k=1}^{n} B_k w_k e^{-w_k x}, \qquad x > 0,$$

with $\sum_{k=1}^{n} B_k = 1$ and $0 < w_1 < w_2 < ... < w_n$.

Then

$$\Psi(z) = Dz^2 + \mu z - \nu \sum_{j=1}^{m} A_j \frac{z}{v_j + z} + \omega \sum_{k=1}^{n} B_k \frac{z}{w_k - z},$$

where

$$D = \sigma^2/2.$$

Under the assumption that the $m + n + 2$ solutions of the equation $\Psi(z) = \lambda$ are distinct, the density function of $X(\tau)$ is given above. In the case of mixtures (all $A_i's$ and $B_i's$ positive), the solutions are distinct and real as we have the following interlacing relationship:

$$\begin{aligned} -\infty &< \alpha_{m+1} < -v_m < \dots < -v_1 < \alpha_1 < 0 \\ &< \beta_1 < w_1 < \dots < w_n < \beta_{n+1} < \infty. \end{aligned}$$

For Lévy process, we have

$M(\tau)$ and $[M(\tau) - X(\tau)]$ are independent

$[X(\tau) - M(\tau)]$ and $m(\tau)$ have the same distribution

Therefore

$$\mathrm{E}[e^{zX(\tau)}] = \mathrm{E}[e^{zM(\tau)}]\mathrm{E}[e^{zm(\tau)}],$$

a version of the celebrated *Wiener-Hopf factorization*.

$$M_{X(\tau)}(z) = \frac{\lambda}{\lambda - \Psi(z)};$$

$M_{X(\tau)}(z)$ is the mgf $E[e^{zX(\tau)}]$ when the expectation exists. The zeros of the denominator are the poles of $M_{X(\tau)}(z)$.

Because $0 \leq M(\tau) < \infty$, the mgf $E[e^{zM(\tau)}]$ is an analytic function of z with negative real part and it has no negative zeros. Similarly, the mgf $E[e^{zm(\tau)}]$ is an analytic function of z with positive real part and it has no positive zeros.

Thus

$$
\begin{aligned}
\mathsf{E}[e^{zm(\tau)}] &\propto \left[\prod_{j=1}^{m}(z+v_j)\right]\left[\prod_{j=1}^{m+1}\frac{1}{z-\alpha_j}\right],\\
\mathsf{E}[e^{zM(\tau)}] &\propto \left[\prod_{k=1}^{n}(z-w_k)\right]\left[\prod_{k=1}^{n+1}\frac{1}{z-\beta_k}\right].
\end{aligned}
$$

As each mgf takes the value 1 when $z=0$, we have

$$
\begin{aligned}
\mathsf{E}[e^{zm(\tau)}] &= \left[\prod_{j=1}^{m}\frac{z+v_j}{v_j}\right]\left[\prod_{j=1}^{m+1}\frac{-\alpha_j}{z-\alpha_j}\right],\\
\mathsf{E}[e^{zM(\tau)}] &= \left[\prod_{k=1}^{n}\frac{w_k-z}{w_k}\right]\left[\prod_{k=1}^{n+1}\frac{\beta_k}{\beta_k-z}\right].
\end{aligned}
$$

Hence, the pdf of $M(\tau)$ is

$$f_{M(\tau)}(x) = \sum_{k=1}^{n+1} b_k^* e^{-\beta_k x}, \quad x > 0.$$

where

$$b_k^* = \left[\prod_{i=1}^{n} \frac{w_i - \beta_k}{w_i}\right]\left[\prod_{i=1, i \neq k}^{n+1} \frac{\beta_i}{\beta_i - \beta_k}\right]\beta_k.$$

Similarly, we obtain the pdf of $m(\tau)$,

$$f_{m(\tau)}(x) = \sum_{j=1}^{m+1} a_j^* e^{-\alpha_j x}, \quad x < 0,$$

where

$$a_j^* = \left[\prod_{i=1}^{m} \frac{\alpha_j + v_i}{v_i}\right]\left[\prod_{i=1, i\neq j}^{m+1} \frac{-\alpha_i}{\alpha_j - \alpha_i}\right](-\alpha_j).$$

Now

$$
\begin{aligned}
f_{X(\tau),M(\tau)}(x,y) &= f_{M(\tau),X(\tau)-M(\tau)}(y,x-y) \\
&= f_{M(\tau)}(y) f_{X(\tau)-M(\tau)}(x-y) \\
&= f_{M(\tau)}(y) f_{m(\tau)}(x-y) \\
&= \left[\sum_{k=1}^{n+1} b_k^* e^{-\beta_k y}\right] \left[\sum_{j=1}^{m+1} a_j^* e^{-\alpha_j (x-y)}\right] \\
&= \sum_{j=1}^{m+1} \sum_{k=1}^{n+1} a_j^* b_k^* e^{-(\beta_k-\alpha_j)y-\alpha_j x}.
\end{aligned}
$$

Link to ruin theory

For $t \geq 0$, let the *surplus* of a company at time t be

$$U(t) = u - X(t),$$

where u is a positive number representing the initial surplus. Let

$$T = \inf\{t : \; U(t) \leq 0\}$$

be the *time of ruin*.

Link to ruin theory

Then,

$$\Pr(M(\tau) \geq u) = \Pr(\tau > T) = \mathrm{E}[e^{-\lambda T}].$$

Thus, knowing the Laplace transform, with respect to λ, of the time of ruin random variable is equivalent to knowing the distribution of the $M(\tau)$.

Link to ruin theory

In particular, if $\{X(t)\}$ is a Lévy process with an upward jump pdf of the linear combination of exponential, we can use the results in Section 9 of Albrecher, Gerber and Yang (2010), with $w(x) \equiv 1$ and $w_0 = 1$, to obtain a closed-form expression for $\Pr(M(\tau) \geq u)$.

Geometric stopping of a random walk

Random walk

- $X(t) = X_1 + \cdots + X_t, \qquad X(0) = 0$
- $X_1, X_2, \ldots$ are i.i.d. r.v.'s, with $\Pr\{X_t = 1\} = p_1$, $\Pr\{X_t = 0\} = p_0$, $\Pr\{X_t = -1\} = p_{-1}$, $p_1 + p_0 + p_{-1} = 1$.

Geometric distribution

Let τ be an independent r.v. with a geometric distribution (say parameter π), such that

$$\Pr\{\tau = t\} = (1-\pi)\pi^t, \qquad t = 0, 1, 2, \ldots .$$

Its probability generating function (pgf) is

$$\mathrm{E}\left[z^\tau\right] = \frac{1-\pi}{1-\pi z}.$$

The pgf of $X(\tau)$

$$
\begin{aligned}
& \mathrm{E}\left[z^{X(\tau)}\right] = \mathrm{E}\left[\mathrm{E}\left[z^{X(\tau)} | \tau\right]\right] \\
& = \mathrm{E}\left[\mathrm{E}\left[z^{X(1)}\right]^{\tau}\right] \\
& = \mathrm{E}\left[(p_1 z + p_0 + p_{-1} z^{-1})^{\tau}\right] \\
& = \frac{1-\pi}{1-\pi(p_1 z + p_0 + p_{-1} z^{-1})}.
\end{aligned}
$$

This is a rational function of z, which we can expand by partial fractions. Let

$$0 < \alpha < 1 < \beta < \infty$$

denote the the solutions of the quadratic equation

$$\pi p_1 z^2 - (1 - \pi p_0) z + \pi p_{-1} = 0.$$

The pgf of $X(\tau)$

$$\mathrm{E}\left[z^{X(\tau)}\right] = C\frac{\alpha}{z-\alpha} - C\frac{\beta}{z-\beta},$$

with

$$C = \frac{1-\pi}{\pi}\frac{1}{p(\beta-\alpha)} = \frac{(1-\alpha)(\beta-1)}{\beta-\alpha}.$$

To identify the distribution of $X(\tau)$, we note that

$$\mathrm{E}\left[z^{X(\tau)}\right] = C\frac{\alpha/z}{1-\alpha/z} + C\frac{1}{1-z/\beta}.$$

The distribution of $X(\tau)$

$$
\begin{aligned}
\Pr\{X(\tau) = j\} &= C\alpha^{-j}, & j = -1, -2, ..., \\
\Pr\{X(\tau) = j\} &= C\beta^{-j}, & j = 0, 1, 2, ... \ .
\end{aligned}
$$

Thus $X(\tau)$ has a two-sided geometric distribution.

The record highs and lows of the random walk

- $$M(t) = \max\{0, X(1), ..., X(t)\}$$

  denote the running maximum and, similarly, $m(t)$ the running minimum after t steps.

Joint distribution in the trinomial tree model

Suppose $X(t) = X_1 + ... + X_t$,

where X_i takes three values: $-1, \ 0, \ 1$

and $P(X_i = 1) = P(X_i = -1) = p/2, \ P(X_i = 0) = q$ with $p + q = 1$.

We assume that $X_1, X_2, ...$ is an i.i.d. sequence. Since the random walk $X(t)$ is symmetric, the reflection principle is true (the proof is the same as that for simple symmetric random walk).

$$P(\{X(t) = j, M(t) \geq k\}) = P(X(t) = 2k - j)$$

Joint distribution in the trinomial tree model

Now we assume that the random walk is not symmetric,

In this case, we have

$$P(X(t) = j, M(t) \geq k) = (p_{-1}/p_1)^{k-j} P(X(t) = 2k - j).$$

Joint distribution in the trinomial tree model

Since τ is independent of $X(t)$, we have

$$P(X(\tau) = j, M(\tau) \geq k) = (p_{-1}/p_1)^{k-j} P(X(\tau) = 2k - j).$$

From this we can obtain the joint probability function of $X(\tau)$ and $M(\tau)$.

Distributions of $M(\tau)$ and $m(\tau)$

Both $M(\tau)$ and $m(\tau)$ have geometric distributions:

$$\begin{aligned}\Pr\{M(\tau) \geq k\} &= \beta^{-k}, \qquad k = 0, 1, 2, ..., \\ \Pr\{m(\tau) \leq k\} &= \alpha^{-k}, \qquad k = 0, -1, -2, ...,\end{aligned}$$

or,

$$\begin{aligned}\Pr\{M(\tau) = k\} &= (\beta - 1)\beta^{-k-1}, \qquad k = 0, 1, 2, ..., \\ \Pr\{m(\tau) = k\} &= (1 - \alpha)\alpha^{-k}, \qquad k = 0, -1, -2, ...\end{aligned}$$

We note that $M(\tau) \geq k$ is the event that $X(n)$ reaches level k before or at time τ. Similarly, $m(\tau) \leq k$ is the event that $X(n)$ falls to level k before or at time τ.

The distributions of $M(\tau)$ and $X(\tau) - m(\tau)$ are the same

We note that for each t, the r.v.'s $M(t)$ and $X(t) - m(t)$ have the same distribution. This follows from

$$\begin{aligned} M(t) &= \max\{0, X_1, X_1 + X_2, \ \dots, X_1 + X_2 + \dots + X_t\}, \\ X(t) - m(t) &= \max\{0, X_t, X_t + X_{t-1}, \ \dots, X_t + X_{t-1} + \dots + X_1\}. \end{aligned}$$

Hence, the distributions of $M(\tau)$ and $X(\tau) - m(\tau)$ are the same. Similarly, the distributions of $m(\tau)$ and $X(\tau) - M(\tau)$ are the same.

The r.v.'s $M(\tau)$ and $X(\tau) - M(\tau)$ are independent

Because the conditional distribution of $X(\tau) - M(\tau)$, given $M(\tau)$, is the conditional distribution of $X(\tau)$, given $X(n) \leq 0$ for $n = 1, ..., \tau$, and hence the same for all values of $M(\tau)$. To see this, consider the first time t when $X(t) = M(\tau)$; thus $\tau \geq t$ and $X(n) - X(t) \leq 0$ for $n = t, ..., \tau$. Then observe that the conditional distribution of $\tau - t$ does not depend on t.

The joint distribution of $X(\tau)$ and $M(\tau)$

$$
\begin{aligned}
&\Pr\{X(\tau) = j, M(\tau) = h\} \\
&= \Pr\{M(\tau) - X(\tau) = h - j,\ M(\tau) = h\} \\
&= \Pr\{M(\tau) - X(\tau) = h - j\}\Pr\{M(\tau) = h\} \\
&= \Pr\{m(\tau) = -(h - j)\}\Pr\{M(\tau) = h\} \\
&= (1 - \alpha)\alpha^{h-j}(\beta - 1)\beta^{-h-1}.
\end{aligned}
$$

Similarly, for $h = 0, -1, -2, ...$ and $j \geq h$, we have

$$
\begin{aligned}
&\Pr\{X(\tau) = j, m(\tau) = h\} \\
&= \Pr\{X(\tau) - m(\tau) = j - h,\ m(\tau) = h\} \\
&= \Pr\{X(\tau) - m(\tau) = j - h\}\Pr\{m(\tau) = h\} \\
&= \Pr\{M(\tau) = j - h\}\Pr\{m(\tau) = h\} \\
&= (\beta - 1)\beta^{-(j-h)-1}(1 - \alpha)\alpha^{-h}.
\end{aligned}
$$

For $k = 0, 1, 2, ...$ and $j \leq k$, we find that

$$\Pr\{X(\tau) = j, M(\tau) \geq k\} = C\alpha^{-j}(\frac{\alpha}{\beta})^k.$$

Of course for $j \geq k$,

$$\Pr\{X(\tau) = j, M(\tau) \geq k\} = \Pr\{X(\tau) = j\} = C\beta^{-j}.$$

Similarly, for $k = 0, -1, -2, ...$ and $j \geq k$ one shows that

$$\Pr\{X(\tau) = j, m(\tau) \leq k\} = C\beta^{-j}(\frac{\beta}{\alpha})^k.$$

Of course

$$\Pr\{X(\tau) = j, m(\tau) \leq k\} = \Pr\{X(\tau) = j\} = C\alpha^{-j}$$

for $j \leq k$.

Remark

The proofs that $M(\tau)$ and $X(\tau) - M(\tau)$ are independent, and that $X(\tau) - m(\tau)$ has the same distribution as $m(\tau)$, are valid for a general random walk. It follows that

$$\begin{aligned} P_{X(\tau)}(z) &= \mathrm{E}[z^{X(\tau)}] = \mathrm{E}[z^{M(\tau)+X(\tau)-M(\tau)}] \\ &= \mathrm{E}[z^{M(\tau)}] \times \mathrm{E}[z^{X(\tau)-M(\tau)}] \\ &= \mathrm{E}[z^{M(\tau)}] \times \mathrm{E}[z^{m(\tau)}] = P_{M(\tau)}(z) P_{m(\tau)}(z). \end{aligned}$$

This formula is a version of the *Wiener-Hopf factorization*.

Remark

If X_1 takes integer values from $-n$ to $+m$, then

$$\begin{aligned} P_{X(\tau)}(z) &= \frac{1-\pi}{1-\pi P_{X_1}(z)]} = \frac{1-\pi}{1-\pi\sum_{j=-n}^{m} p_j z^j} \\ &= \frac{(1-\pi)z^n}{g(z)}, \end{aligned}$$

where $g(z) = \left(1-\pi\sum_{j=-n}^{m} p_j z^j\right) z^n$ is a polynomial of degree $m+n$.

Remark

Because $1 > \pi = \pi \sum_{j=-n}^{m} p_j$, we have

$$|z^n| > |g(z) - z^n|, \qquad \text{for} \quad |z| = 1.$$

Then by Rouché's Theorem, $g(z)$ has the same number of zeros inside the complex disk of radius 1 as the function z^n. Denote these n zeros of $g(z)$ as $\alpha_1, ..., \alpha_n$. Denote the other zeros of $g(z)$, those with absolute value greater than 1, as $\beta_1, ..., \beta_m$. Then, the pgf $P_{X(\tau)}(z)$ is proportional to

$$\frac{z^n}{\left(\prod_{j=1}^{n}(z-\alpha_j)\right)\left(\prod_{j=1}^{m}(z-\beta_j)\right)}$$

$$= \frac{1}{\left(\prod_{j=1}^{n}(1-\alpha_j/z)\right)\left(\prod_{j=1}^{m}(z-\beta_j)\right)}.$$

Remark

Note that $P_{M(\tau)}(1) = 1$ and $P_{m(\tau)}(1) = 1$. Because $M(\tau) \geq 0$, $P_{M(\tau)}(z)$ exists for each z with $|z| < 1$. Similarly, $P_{m(\tau)}(z)$ exists for each z with $|z| > 1$. Therefore

$$P_{M(\tau)}(z) = \frac{\prod_{j=1}^{m}(\beta_j - 1)}{\prod_{j=1}^{m}(\beta_j - z)}, \quad P_{m(\tau)}(z) = \frac{\prod_{j=1}^{n}(1 - \alpha_j)}{\prod_{j=1}^{m}(1 - \alpha_j/z)}.$$

Remark

In the special case of a simple random walk, we have

$$P_{X(\tau)}(z) = C\frac{(\beta-\alpha)z}{(z-\alpha)(\beta-z)} = \frac{\beta-1}{\beta-z} \times \frac{(1-\alpha)z}{z-\alpha}.$$

Applications to Valuing Equity-linked Insurance Products

- To value guarantees and options in Variable Annuities
- Variable Annuities
  $=$ Investment Funds (Mutual Funds)
  $+$
  Rider(s) : Guaranteed Minimum Benefits

Some Examples

- Guaranteed Minimum Death Benefits
- Payoff: $\max\{S(T_x), K\}$

  where $S(t)$ denotes the price of a stock or stock index at time t, T_x is the future life time of policyholder aged x.
- Note that $\max\{S(T_x), K\} = (K - S(T_x))_+ + S(T_x)$.

Some Examples

- High water mark method or low water mark method
- Payoff: $\max\{M_S(T_x), K\}$

  where $M_S(t)$ denotes the running maximum of the the price of a stock or stock index up to time t,
- Note that $\max\{M_S(T_x), K\} = (M_S(T_x) - K)_+ + K$.

Mathematical problem

- How to calculate

  $E[e^{-\delta T_x} b(S(T_x), M_S(T_x))]$. (or
  $E[v^{(K_x+1)} b(S(K_x), M_S(K_x))]$.)

  where $b(.,.)$ is a benefit function, (v is the discount factor per unit time, K_x denotes the curtate future life time r.v.).

Index process

- $S(t)$ price of one unit of a fund at time t
- $S(t) = S(0)e^{X(t)}, \quad t \geq 0$ $X(t)$ Brownian motion, a Lévy process (in the discrete time case, a random walk)
- T_x (or K_x) is assumed to be independent of $S(t)$

The reduced problem

- Idea: the pdf of T_x can be approximated by

$$\sum_{i=1}^{n} A_i \lambda_i e^{-\lambda_i t}, \qquad t > 0$$

(the A_i's can be negative, as long as the sum is pdf)

The reduced problem

$$
\begin{aligned}
\mathrm{E}[e^{-\delta T_x} b(S(T_x))] &= \mathrm{E}[\mathrm{E}[e^{-\delta T_x} b(S(T_x)) | T_x]] \\
&= \int_0^\infty e^{-\delta t} f_{T_x}(t) \mathrm{E}[b(S(t)) | T_x = t] dt \\
&= \int_0^\infty e^{-\delta t} f_{T_x}(t) \mathrm{E}[b(S(t))] dt \\
&\simeq \int_0^\infty e^{-\delta t} \sum_{i=1}^{n} A_i \lambda_i e^{-\lambda_i t} \mathrm{E}[b(S(t))] dt \\
&= \sum_{i=1}^{n} A_i \int_0^\infty e^{-\delta t} \lambda_i e^{-\lambda_i t} \mathrm{E}[b(S(t))] dt.
\end{aligned}
$$

The reduced problem

So, it suffices that we know how to calculate

$$E[e^{-\delta\tau} b(S(\tau))] = \int_0^{\infty} e^{-\delta t} \lambda e^{-\lambda t} E[b(S(t))] dt$$

The reduced problem

δ can be eliminated

$$\int_0^\infty e^{-\delta t}\lambda e^{-\lambda t}\mathrm{E}[b(S(t))]dt = \frac{\lambda}{\lambda+\delta}\int_0^\infty (\lambda+\delta)e^{-(\lambda+\delta)t}\mathrm{E}[b(S(t))]dt$$

rule: do the calculation without discounting
but replace λ by $\lambda+\delta$ multiply by $\frac{\lambda}{\lambda+\delta}$

- Want to calculate

$$E[b(S(\tau), M_S(\tau))] = E[b(S(0)e^{X(\tau)}, S(0)e^{M(\tau)})]$$

where $M(t)$ is the running maximum of $X(s)$ up to time t and τ is an exponential random variable.

- so we need

$f_{X(\tau),M(\tau)}(x, y)$ the joint pdf of $(X(\tau), M(\tau))$

Examples

In the following we assume that $X(t)$ is a Brownian motion.

$$\begin{aligned}
&\mathrm{E}[e^{-\delta\tau} b(S(\tau))] = \mathrm{E}[b(S(\tau^*))] \\
&= \kappa \int_{-\infty}^{0} b(S(0)e^x) e^{-\alpha x} dx + \kappa \int_{0}^{\infty} b(S(0)e^x) e^{-\beta x} dx.
\end{aligned}$$

Examples

- (1) $b(s) = (K - s)_+, \; K < S(0)$ out-of-the-money put option

$$E[e^{-\delta\tau}(K - S(\tau))_+] \;=\; \frac{\kappa K}{-\alpha(1-\alpha)}\left[\frac{K}{S(0)}\right]^{-\alpha}$$

- (2) $b(s) = (s - K)_+, \; K > S(0)$ out-of-the-money call option

$$E[e^{-\delta\tau}(S(\tau) - K)_+] \;=\; \frac{\kappa K}{\beta(\beta-1)}\left[\frac{S(0)}{K}\right]^{\beta}$$

Examples

Lookback options

$$
\begin{aligned}
& \mathrm{E}[e^{-\delta\tau}[\max_{0\le t\le\tau} S(t) - K]_+] \\
& = \mathrm{E}[e^{-\delta\tau}[S(0)e^{M(\tau)} - K]_+]
\end{aligned}
$$

out-of-the-money

$$
= \frac{\lambda}{\lambda+\delta}\frac{K}{\beta-1}\left[\frac{S(0)}{K}\right]^{\beta}.
$$

Examples

Barrier options

Let L denote the barrier and $\ell = \ln[L/S(0)]$. Consider the up-and-in option $(S(0) < L)$, the value is given by

$$\Pr(M(\tau) \geq \ell)\mathcal{E}_b(L) = \left[\frac{S(0)}{L}\right]^{\beta}\mathcal{E}_b(L),$$

where

$$\mathcal{E}_b(s) = \mathrm{E}[b(S(\tau))|S(0) = s]$$

THANK YOU