Optimal Extraction and Taxation of Natural Resources: A Differential Game Approach

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Outline

Introduction

- The extraction and regulation of strategic natural resources such as oil, copper, uranium,..., etc, have always been of great interest to economists and policymakers as well.
- The earliest work on the extraction angle of this topic was done by Hotelling (1931), he derived an optimal extraction policy under the assumption that the commodity price is constant.
- The taxation and regulation of natural resources greatly influence the behavior of Multinational Mining Companies.

Introduction

- The mining industry is usually one the main beneficiaries of tax exonerations and incentives in various counties.
- There are not been many studies about the interplay between the taxation and the extraction of natural resources.
- In this project, we use the framework of differential game theory to analyze the competitive interests involved in mining strategic natural resources.

Problem Formulation

- Consider a long term Production Sharing Agreement between a mining company and the government of a resource-rich country.
- The mining company takes 100θ percents and the government takes $100(1-\theta)$ percents of the production profits, $\theta \in (0,1)$.
- The company and the government are the two players of our game, the company is Player 1 and the government is Player 2.
- Each player will act as a controller.
- The mining company will try to maximize its share of profits from the mining activities.
- The government will also try to maximize both its share of the profits from the sales of the mineral and the income tax it levies on the mining company.

Problem Formulation

- Given that the commodity price is subject to random fluctuations and global geopolitical shocks, we model it using Markov switching Lévy processes.
- Let $X(t)$ be the commodity price at time t.
- Let $\alpha(t) \in \mathcal{M} = \{0, 1, .., m\}$ be a Markov chain with generator $Q = (q_{ij})$, $\alpha(t)$ captures the states of the global commodity market.
- Let $(\eta_t)_t$ be a Lévy process with differential form $\bar{N}(dt, dz)$ and Lévy measure ν. The process η_t will capture the spikes and rapid changes in the commodity price $X(t)$.

Problem Formulation

- We assume that $X(t)$ satisfies the SDE

$$\begin{cases} dX(t) = [X(t)\mu(\alpha(t)) - \rho u_1]\mathrm{d}t + \sigma(\alpha(t))X(t)dW(t) \\ \qquad\qquad +X(t)\int_{\mathbb{R}} \gamma(\alpha(t))z\bar{N}(dt,dz), \\ X(0) = x \geq 0, \quad 0 \leq t \leq \infty, \end{cases} \tag{1}$$

where $\rho \in [0,1)$ captures the relative impact of the extracting activities, $u_1(t) \in [0,\bar{u}_1]$ is the extraction rate chosen by the company and $u_2(t) \in [0,\bar{u}_2]$ is the tax rate chosen by the government. The processes $u_1(t)$ and $u_2(t)$ are control variables, and $W(t)$ is the Wiener process defined on a probability space $(\Omega,\mathcal{F},P)$.
- For each state $i \in \mathcal{M}$, we assume that $\mu(i),\sigma(i)$ and $\gamma(i)$ are known constants.

Profit Rate Function

- Let $C(u_1)$ be the extraction cost function, we assume that this function depends only on the extraction rate u_1.
- We assume naturally that the extraction cost function is proportional to the production function. One of the popular production functions is the quadratic production function. This is due to its simplicity and the fact that it captures most stages of the production cycle.
- Therefore we will assume that our cost function $C(u_1)$ has the same form as a quadratic production function, we set

$$C(u_1) = au_1^2, \qquad a > 0.$$

- The total profit rate for operating the mine is

$$P(x, u_1) = xu_1 - C(u_1).$$

Profit Rate Function

- The mining company pre-tax profit rate function is $\theta P(x, u_1)$.
- The government profit rate without the tax revenue is $(1 - \theta)P(x, u_1)$.
- The total income tax the government levies on the mining company is $u_2 \theta P(x, u_1)$.
- The post-tax profit rate of the company is

$$L_1(x, u_1, u_2) = \theta P(x, u_1)(1 - u_2).$$

- The government profit rate function is

$$L_2(x, u_1, u_2) = (1 - \theta)P(x, u_1) + u_2 \theta P(x, u_1).$$

Differential Game

- The payoff of each player $j = 1, 2,$ is defined as follows;

$$\begin{aligned} & J_j(x, i; u_1, u_2) \\ & = E\left[\int_0^\infty e^{-rt} L_j(X(t), u_1(t), u_2(t), \alpha(t)) \mathrm{dt} \middle| \mathrm{X}(0) = \mathrm{x}, \alpha(0) = \mathrm{i} \right]. \end{aligned}$$

- The company will try to maximize its payoff by adjusting the extraction rate $u_1(\cdot)$.
- The government will maximize its payoff by changing the tax rate $u_2(\cdot)$.
- Our goal is to find a noncooperative Nash equilibrium (u_1^*, u_2^*) such that

$$J_1(x, i; u_1^*, u_2^*) \geq J_1(x, i; u_1, u_2^*), \quad \text{for all } u_1(\cdot) \in \mathcal{U}_1(x, i), \tag{2}$$

$$J_2(x, i; u_1^*, u_2^*) \geq J_2(x, i; u_1^*, u_2), \quad \text{for all } u_2(\cdot) \in \mathcal{U}_2(x, i). \tag{3}$$

Nash Equilibrium

Definition

Let (u_1^*, u_2^*) *be a Nash equilibrium of our differential game, the functions*

$$V_1(x,i) = \sup_{u_1 \in \mathcal{U}_1} J_1(x,i;u_1,u_2^*)$$
$$V_2(x,i) = \sup_{u_2 \in \mathcal{U}_2} J_2(x,i;u_1^*,u_2)$$

are called value functions of Player 1 and Player 2 respectively.

- In order to find the optimal strategies u_1^*, u_2^*, of the Nash equilibrium we first have to derive the value functions V_1 and V_2.

Nash Equilibrium

- Assuming that we have a Nash equilibrium u_1^*, u_2^*, let us define corresponding Hamiltonians:

$$H_1\left(x, i, V, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}\right)$$
$$= rV - \sup_{u_1 \in U_1} \left(\frac{1}{2} x^2 \sigma^2(i) \frac{\partial^2 V}{\partial x^2} + (x\mu(i) - \rho u_1) \frac{\partial V}{\partial x} \right.$$
$$+ \int_{\mathbb{R}} \left(V(x + \gamma(i)zx, i) - V(x, i) - \mathbf{1}_{\{|z|<1\}}(z) \frac{\partial V}{\partial x} \gamma(i)xz \right) \nu(dz)$$
$$\left. + L_1(x, u_1, u_2^*) + QV(x, \cdot)(i) \right), \tag{4}$$

and

$$H_2\left(x, i, V, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}\right)$$
$$= rV - \sup_{u_2 \in U_2} \left(\frac{1}{2} x^2 \sigma^2(i) \frac{\partial^2 V}{\partial x^2} + (x\mu(i) - \rho u_1^*) \frac{\partial V}{\partial x} \right.$$
$$+ \int_{\mathbb{R}} \left(V(x + \gamma(i)xz, i) - V(x, i) - \mathbf{1}_{\{|z|<1\}}(z) \frac{\partial V}{\partial x} \gamma(i)zx \right) \nu(dz)$$
$$\left. + L_2(x, u_1^*, u_2) + QV(x, \cdot)(i) \right). \tag{5}$$

Nash Equilibrium

- The corresponding Hamilton Jacobi Isaacs equations of this noncooperative game are

$$\begin{cases} H_1\left(x, i, V_1, \frac{\partial V_1}{\partial x}, \frac{\partial^2 V_1}{\partial x^2}\right) = 0 \\ H_2\left(x, i, V_2, \frac{\partial V_2}{\partial x}, \frac{\partial^2 V_2}{\partial x^2}\right) = 0. \end{cases} \tag{6}$$

- We define $\tilde{\mu}_1 := (x\mu(i) - \rho u_1)$, $\tilde{L}_1 := L_1(x, u_1, u_2^*)$, $\tilde{\mu}_2 := (x\mu(i) - \rho u_1^*)$, $\tilde{L}_2 := L_2(x, u_1^*, u_2)$, we have the following result.

Theorem

Assume that there exists $(u_1^*, u_2^*) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that the nonlinear Hamilton Jacobi Isaacs equations (6) have classical solutions $V_j(x, i)$, $j = 1, 2$,

$$u_j^* = \arg\max\left(\tilde{\mu}_j \frac{\partial V_j}{\partial x} + \tilde{L}_j\right), \quad j = 1, 2. \tag{7}$$

Then the pair (u_1^*, u_2^*) is a Nash equilibrium solution and $J_j(x, i; u_1^*, u_2^*) = V_j(x, i)$, $j = 1, 2$.

Closed-form Solutions

Theorem

Assume that there exists a Nash equilibrium (u_1^*, u_2^*) *such that* u_2^* *is an open loop control, in other terms,* u_2^* *does not depend on the variable* x*. Then the solutions of the Hamilton Jacobi Isaacs equations* (6) *are*

$$V_1(x,i) = A_1(i)x^2, \quad V_2 = A_2(i)x^2, \quad i = 1,2,...,m. \tag{8}$$

Player 1 optimal strategy is

$$u_1^*(x,i) = \left(\frac{1}{2a} - \frac{\rho A_1(i)}{a\theta(1-u_2^*(i))}\right)x, \tag{9}$$

thus $u_1^*(x,i)$ *can be expressed as* $u_1^*(x,i) = K(i)x$.
Player 2 optimal strategy is

$$u_2^*(i) = \begin{cases} \bar{u}_2 & \text{if} \quad K(i) - aK(i)^2 > 0, \\ 0 & \text{if} \quad K(i) - aK(i)^2 < 0. \end{cases} \tag{10}$$

Closed-form Solutions

Moreover, the coefficients $A_1(i), i = 1, 2, ..., m$*, satisfy the system of quadratic equations*

$$
\begin{aligned}
&A_1(i)^2 \frac{\rho^2}{a\theta(1-u_2^*(i))} + A_1(i)\Big(\sigma(i)^2 - r + 2\mu(i) - \sum_{j\neq i} q_{ij} - \frac{\rho}{a} \\
&+ \int_{\mathbb{R}} \big(2\gamma(i)z + \gamma(i)^2 z^2 - \mathbf{1}_{\{|z|<1\}}(z) 2\gamma(i)z\big)\nu(dz)\Big) \qquad (11) \\
&+ \frac{\theta(1-u_2^*(i))}{4a} + \sum_{j\neq i} q_{ij} A_1(j) = 0, \qquad i = 1, 2, ..., m.
\end{aligned}
$$

and the coefficients $A_2(i), i = 1, 2, ..., m$*, satisfy the following linear system*

$$
\begin{aligned}
0 \;=\; & A_2(i)\Big(\sigma(i)^2 - r + 2(\mu(i) - \rho K(i)) - \sum_{j\neq i} q_{ij} \\
& + \int_{\mathbb{R}} \big(2\gamma(i)z + \gamma(i)^2 z^2 - \mathbf{1}_{\{|z|<1\}}(z) 2\gamma(i)z\big)\nu(dz)\Big) \qquad (12) \\
& + (1 - \theta + \theta \bar{u}_2 \mathbf{1}_{\{K(i) - aK(i)^2 > 0\}})(K(i) - aK(i)^2) \\
& + \sum_{j\neq i} q_{ij} A_2(j), \qquad i = 1, 2, ..., m.
\end{aligned}
$$

Applications

- We will particular look at two cases depending on whether the Lévy measure has finite jumps activity or infinite activity.
- If the commodity price X_t has finite jumps activity, without loss of generality we will assume that the Lévy measure follows an exponential distribution and has the form

$$\nu(dz) = \begin{cases} \eta e^{-\eta z} dz & \text{if } z \geq 0, \\ 0 & \text{if } z < 0, \end{cases} \quad \text{for some} \quad \eta > 0.$$

- If the X_t has infinite jumps activity we assume that ν has the form

$$\nu(dz) = \begin{cases} \frac{e^{-|z|}}{|z|^2} dz & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases} \tag{13}$$

Applications: Finite Activity

- If the Lévy process has finite activity and the Lévy measure has the form $\nu(dz) = \eta e^{-\eta z} dz, z > 0$, for some $\eta > 0$, the value functions and the optimal policies obtained in Theorem 4.1 are such that the coefficients $A_1(i)$ satisfy the system of quadratic equations

$$
\begin{aligned}
& A_1(i)^2 \frac{\rho^2}{a\theta(1-u_2^*)} + A_1(i)\Big(\sigma(i)^2 - r + 2\mu(i) \\
& - \sum_{j\neq i} q_{ij} - \frac{\rho}{a} + 2\gamma(i)\frac{\gamma(i) + (1+\eta)\eta e^{-\eta}}{\eta^2}\Big) \qquad (14) \\
& + \frac{\theta(1-u_2^*)}{4a} + \sum_{j\neq i} q_{ij} A_1(j) = 0, \\
& \quad i = 1, 2, ..., m.
\end{aligned}
$$

Applications: Finite Activity

- The coefficients $A_2(i)$, in the particular case of a two-state Markov chain, are defined as follows

$$\begin{aligned} A_2(1) &= \frac{R(1)P(2) - q_{12}R(2)}{P(1)P(2) - q_{12}q_{21}}, \\ A_2(2) &= \frac{P(1)R(2) - q_{21}R(1)}{P(1)P(2) - q_{12}q_{21}}, \end{aligned}$$

where

$$\begin{aligned} P(i) &= \sigma(i)^2 - r + 2(\mu(i) - \rho K(i)) - \sum_{j \neq i} q_{ij} \\ &+2\gamma(i)\frac{\gamma(i) + (1+\eta)\eta e^{-\eta}}{\eta^2}, \;\; i = 1, 2, \\ R(i) &= -(1 - \theta + \theta \bar{u}_2 \mathbf{1}_{\{K(i) - aK(i)^2 > 0\}})(K(i) - aK(i)^2), \quad i = 1, 2. \end{aligned}$$

Applications: Finite Activity

- Consider a Profit Sharing Agreement between a medium gold producer and a multinational mining company where the company takes 30% and the country keeps 70% of the profits, thus $\theta = 0.3$.
- Because the country is a medium producer, the country production level will not influence the commodity price, thus $\rho = 0$ and the commodity price follows the following SDE

$$dX(t) = X(t)\left(\mu(\alpha(t))dt + \sigma(\alpha(t))dW_t + \int_{\mathbb{R}} \lambda(\alpha(t))z\bar{N}(dt, dz)\right).$$

- We assume that the gold market has two trends, $\mathcal{M} = \{1, 2\}$, $\alpha(t) = 1$ represents the uptrend and $\alpha(t) = 2$ represents the downtrend.
- Moreover we assume, $r = 0.02$, $\mu = (0.08, -0.1)$, $\sigma = (0.2, 0.3)$, $\gamma = (0.05, 0.09)$, and $\nu(dz) = 5e^{-5z}dz, z > 0$.

Applications: Finite Activity

- The generator of the Markov chain $\alpha(t)$ is $Q = \begin{pmatrix} -0.4 & 0.4 \\ 0.1 & -0.1 \end{pmatrix}$.
- The extraction cost function is $C(u) = 15u^2$, so $a = 15$. Note that u is in millions of ounces of gold per year, and the unit of the cost function $C(u)$ is millions of dollars per year.
- if $u_2^* \equiv \bar{u}_2 = 0.2$ then $V_1(x,i) = A_1(i)x^2, \;\; i = 1,2,$ such that $A_1(1)$ and $A_2(2)$ solve the system

$$\begin{cases} -0.2190A_1(1) + 0.4A_1(2) + 0.004 = 0 \\ 0.1A_1(1) - 0.2279A_1(2) + 0.004 = 0, \end{cases}$$

and the solutions are $A_1(1) = 0.2535$ and $A_1(2) = 0.1288$. and the optimal extraction rates are $u_1^*(x,1) = u_1^*(x,2) = \frac{1}{30}x$.

Applications: Finite Activity

- if $u_2^* \equiv 0$ then $A_1(1)$ and $A_1(2)$ solve a slightly different system

$$\begin{cases} -0.2190A_1(1) + 0.4A_1(2) + 0.0050 = 0 \\ 0.1A_1(1) - 0.2279A_1(2) + 0.0050 = 0, \end{cases}$$

  and $A_1(1) = 0.3169$ and $A_1(2) = 0.1610$.
- Given that $u_1^*(x,1) = u_1^*(x,2) = \frac{1}{2a}x$, thus $K(i) = \frac{1}{2a}$.
- It is clear that $K(i) - aK(i)^2 = \frac{1}{2a} - \frac{1}{4a} > 0$, thus $u_2(1) = u_2(2) = \bar{u}_2 = 0.2$.
- In sum the Nash Equilibrium is

$$u_1^*(x,i) = \frac{1}{2a}x, \quad \text{and} \quad u_2^*(x,i) = \bar{u}_2 = 0.2.$$

Applications: Infinite Activity

- It is well known through empirical evidences that commodity prices follow Lévy processes with infinite jumps activity, Ait-Sahalia et al. (2011).
- If the Lévy process has infinite activity and the Lévy measure has the form

$$\nu(dz) = \frac{e^{-|z|}}{|z|^2} dz, z \neq 0,$$

  the value functions and the optimal policies obtained in Theorem 4.1 are such that the coefficients $A_1(i)$ satisfy the system of quadratic equations

$$A_1(i)^2 \frac{\rho^2}{a\theta(1-u_2^*)} + A_1(i)\Big(\sigma(i)^2 - r + 2\mu(i) - \sum_{j\neq i} q_{ij} - \frac{\rho}{a}$$

$$+2\gamma(i)^2\Big) + \frac{\theta(1-u_2^*)}{4a} + \sum_{j\neq i} q_{ij} A_1(j) = 0, \quad i = 1, 2.$$

Applications: Infinite Activity

- And the coefficients $A_2(i)$ are defined as follows

$$A_2(1) = \frac{R(1)P(2) - q_{12}R(2)}{P(1)P(2) - q_{12}q_{21}}, \tag{15}$$

$$A_2(2) = \frac{P(1)R(2) - q_{21}R(1)}{P(1)P(2) - q_{12}q_{21}} \tag{16}$$

with

$$P(i) = \sigma(i)^2 - r + 2(\mu(i) - \rho K(i)) - \sum_{j \neq i} q_{ij} + 2\gamma(i)^2, \quad i = 1, 2,$$

$$R(i) = -(1 - \theta + \theta \bar{u}_2 \mathbf{1}_{\{K(i) - aK(i)^2 > 0\}})(K(i) - aK(i)^2), \quad i = 1, 2.$$

Applications: Infinite Activity

- Consider an oil company with an extraction lease of an oil field with a known reserve of $M = 10$ billion barrels. We assume that the profit sharing agreement between the oil company and the government is such that the oil company takes 20% of profits and the government takes 80%, so $\theta = 0.2$.
- The yearly discount rate $r = 0.02$, the yearly return vector is $\mu = (0.02, -0.1)$, the yearly volatility vector is $\sigma = (0.2, 0.3)$, the yearly intensity vector is $\gamma = (0.022, 0.03)$, and the generator of the Markov chain is

$$Q = \begin{pmatrix} -0.3 & 0.3 \\ 0.5 & -0.5 \end{pmatrix}.$$

Applications: Infinite Activity

- The country is a large oil producer, thus $\rho \neq 0$, we assume $\rho = 0.001$.
- The extraction cost function is $C(u) = 2u^2$, so $a = 2$. Note that, in the cost function $C(u)$, the argument u is in millions of barrels per year, and the unit of the cost function $C(u)$ is millions of dollars per year.
- If $u_2^* \equiv \bar{u}_2 = 0.2$ then

$$V_1(x, i) = A_1(i)x^2$$

where $A_1(i), i = 1, 2$ solve the quadratic system

$$\begin{cases} 3.125 \times 10^{-6} A_1(1)^2 - 0.239532 A_1(1) + 0.3 A_1(2) + 0.02 = 0 \\ 3.125 \times 10^{-6} A_1(2)^2 - 0.6287 A_1(2) + 0.5 A_1(1) + 0.02 = 0. \end{cases}$$

Applications: Infinite Activity

- The acceptable solutions are $A_1(1) = 37.2674$ and $A_1(2) = 29.6747$. Therefore we have

$$V_1(x,1) = 37.2674x^2, \quad V_1(x,2) = 29.6747x^2,$$

$$u_1^*(x,1) = 0.133539x, \qquad u_1^*(x,2) = 0.157267x.$$

- It is worth noting that the value function $V(x,i)$ is given in millions of dollars and the extraction rate $u^*(x,i)$ is given in millions of barrels per year. In fact the daily optimal extraction rate is

$$u_1^*(x,i) = \frac{1}{365}\left(\frac{1}{2a} - \frac{\rho A_1(i)}{a\theta(1-u_2^*(i))}\right)x,$$

$$u_1^*(x,1) = 0.00036x \quad \text{millions of barrels per day,}$$

$$u_1^*(x,2) = 0.00043x \quad \text{millions of barrels per day.}$$

Applications: Infinite Activity

- If $u_2^* \equiv 0$, then the coefficients $A_1(1)$ and $A_1(2)$ solve the system

$$\begin{cases} 2.5 \times 10^{-6} A_1(1)^2 - 0.239532 A_1(1) + 0.3 A_1(2) + 0.025 = 0 \\ 2.5 \times 10^{-6} A_1(2)^2 - 0.6287 A_1(2) + 0.5 A_1(1) + 0.025 = 0. \end{cases}$$

Then the acceptable solution is $A_1(1) = 46.5843$ and $A_1(2) = 37.0934$. The value function and optimal extraction rate are

$$V_1(x,1) = 46.5843 x^2 \quad V_1(x,2) = 37.0934 x^2$$

$$u_1^*(x,1) = 0.133539x, \quad u_1^*(x,2) = 0.157267x.$$

The daily optimal extraction rates are

$$u_1(x,1) = 0.00036x \quad \text{millions of barrels per day,}$$

$$u_1(x,2) = 0.00043x \quad \text{millions of barrels per day.}$$

Applications: Infinite Activity

- Likewise, if $u_1^*(x,i) = K(i)x$ with $K(1) = 0.133539$ and $K(2) = 0.157267$, then the coefficient $A_2(1)$ and $A_2(2)$ obtained from (15) and (16) are

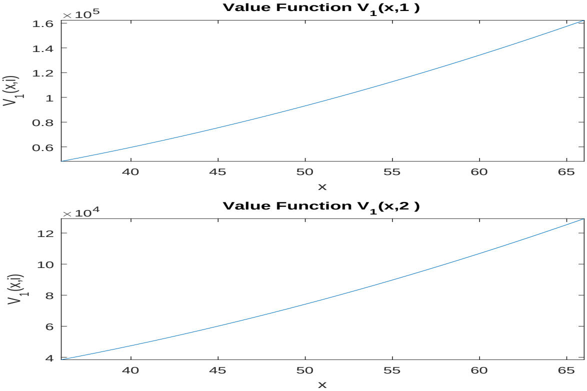
$$A_2(1) = 195.654 \quad \text{and} \quad A_2(2) = 155.792.$$

Thus the value function is

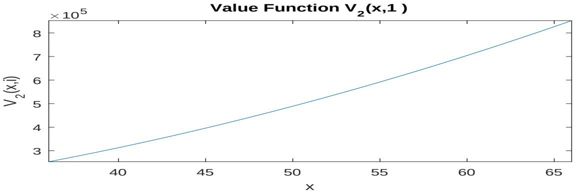
$$V_2(x,1) = 195.654x^2 \quad \text{and } V_2(x,2) = 155.792x^2.$$

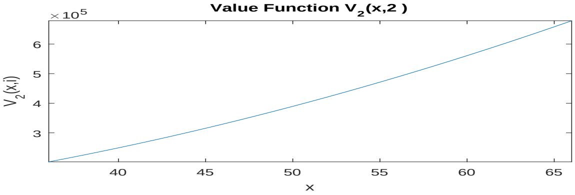
The optimal tax rate is $u_2^*(x,1) = u_2^*(x,2) = \bar{u}_2 = 20\%$ because $K(1) - aK(1)^2 = 0.0978738 > 0$ and $K(2) - aK(2)^2 = 0.107801 > 0$.

Applications: Infinite Activity



Applications: Infinite Activity





- Thank You!

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