Model Uncertainty Stochastic Mean-Field Control

Nacira Agram[1,2,3] and Bernt Øksendal[1,3]

1) Dept. of Mathematics, University of Oslo, Norway.
2) University Mohamed Khider of Biskra, Algeria.

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Introduction: Mean-Field Systems

Consider an SDE of mean-field type

$$\begin{cases} dX(t) &= b(t, X(t), \mathbb{E}[X(t)])dt + \sigma(t, X(t), \mathbb{E}[X(t)])dB(t), \\ X(0) &= x_0. \end{cases}$$

It can be regarded as a limit of large systems of interacting particles

$$\begin{cases} dX^{i,n}(t) &= b(t, X^{i,n}(t), \frac{1}{n}\sum_{j=1}^{n} X^{j,n}(t))dt \\ & \quad +\sigma(t, X^{j,n}(t), \frac{1}{n}\sum_{j=1}^{n} X^{j,n}(t))dB^i(t), \\ X(0) &= x_0, \end{cases}$$

as the number of particles n goes to infinity (assuming that B^i are independent).

- Optimal control of mean-field sde has been studied by several authors lately, including Anderson and Djehiche [3] by means of suitably modified stochastic maximum principles, which involve mean-field *backward* sde (mean-field bsde).
- Mean-field bsde also represent interesting models in finance, for example models of risk measures and recursive utilities.
- See Duffie and Epstein [27], Duffie and Zin [33], Kreps and Parteus [35], El Karoui *et al* [34], Ø. and Sulem [20] and Agram and Røse [1].

More general, SDEs of the form

$$\begin{cases} dX(t) = \sigma\left(t, X(t), \mathcal{L}(X(t))\right) dB(t) + b\left(t, X(t), \mathcal{L}(X(t))\right) dt \\ \qquad\qquad + \int_{\mathbb{R}_0} \gamma\left(t, X(t), \mathcal{L}(X(t)), \zeta\right) \tilde{N}(dt, d\zeta), \\ X(0) = x_0. \end{cases}$$

with dynamics depending on the probability law $\mathcal{L}(X(t))$ of the state $X(t)$ at time t, have been studied by several authors, including Lasry & Lions [36], Carmona & Delarue [13], [12].

Model Uncertainty

There are many ways of introducing model uncertainty. For example, in recent works of Ø. and Sulem [22], [21], [20], the underlaying probability measure is not given a priori and there can be a family of possible probability measures to choose from.
The aim of this paper is to study stochastic optimal control under model uncertainty of a mean-field related type SDE driven by Brownian motion and an independent Poisson random measure.
The model uncertainty is represented by ambiguity about the law $\mathcal{L}(X(t))$ of the state $X(t)$ at time t. For example, it could be the law $\mathcal{L}_{\mathbb{P}}(X(t))$ of $X(t)$ with respect to the given, underlying probability measure $\mathbb{P}$. This is the classical case when there is no model uncertainty. But it could also be the law $\mathcal{L}_{\mathbb{Q}}(X(t))$ with respect to some other probability measure $\mathbb{Q}$ or, more generally, any random measure $\mu(t)$ on $\mathbb{R}$ with total mass 1.

We represent this model uncertainty control problem as a *stochastic differential game* of a mean-field related type SDE with two players. The control of one of the players, representing the uncertainty of the law of the state, is a measure-valued stochastic process $\mu(t)$, and the control of the other player is a classical real-valued stochastic process $u(t)$. We penalize $\mu(t)$ for being far away from the law $\mathcal{L}_{\mathbb{P}}(X(t))$ with respect to the original probability measure $\mathbb{P}$. This leads to a new type of mean-field stochastic control problems in which the control is random measure-valued stochastic process $\mu(t)$ on $\mathbb{R}$.

By constructing a new Hilbert space $\mathcal{M}$ of measures, we obtain sufficient and necessary maximum principles for Nash equilibria for such games in the general nonzero-sum case, and saddle points for zero-sum games. As an application we find an explicit solution of the problem of optimal consumption under model uncertainty of a cash flow described by a mean-field related type SDE.

Mean-field games problems were first studied by Lasry and Lions [17], and Lions in [18] has proved the differentiability of functions of measures defined on a Wasserstein metric space $\mathcal{P}_2$ by using the lifting technics. Since then this type of problems has gained a lot attention, we can for example refer to Carmona *et al* [13], [12], Buckdahn *et al* [6], Bensoussan *et al* [9], Bayraktar *et al* [8], Corso and Pham [15], Djehiche and Hamadene [16], Pham and Wei [23] and Agram [2].

A weighted Sobolev space of random measures

In this section we, following Agram and Ø. [7], construct a Hilbert space $\mathcal{M}$ of random measures on $\mathbb{R}$. It is simpler to work with than the Wasserstein metric space that has been used by many authors previously. See e.g. Carmona *et al* [13], [12], Buckdahn *et al* [6] and the references therein.

Definition

(Weighted Sobolev spaces of measures)

For $k = 0, 1, 2, ...$ let $\tilde{\mathcal{M}}^{(k)}$ denote the set of random measures μ on $\mathbb{R}$ such that

$$(3.1) \qquad \mathbb{E}[\int_{\mathbb{R}} |\hat{\mu}(y)|^2 |y|^k e^{-y^2} dy] < \infty,$$

where

$$(3.2) \qquad \hat{\mu}(y) = \int_{\mathbb{R}} e^{ixy} d\mu(x)$$

is the Fourier transform of the measure μ.

If $\mu, \eta \in \tilde{\mathcal{M}}^{(k)}$ we define the inner product $\langle \mu, \eta \rangle_{\tilde{\mathcal{M}}^{(k)}}$ by

$$\langle \mu, \eta \rangle_{\tilde{\mathcal{M}}^{(k)}} = \mathbb{E}[\int_{\mathbb{R}} \mathrm{Re}(\overline{\hat{\mu}}(y)\hat{\eta}(y))|y|^k e^{-y^2} dy], \tag{3.3}$$

where, in general, $\mathrm{Re}(z)$ denotes the real part of the complex number z, and $\bar{z}$ denotes the complex conjugate of z. The norm $|| \cdot ||_{\tilde{\mathcal{M}}^{(k)}}$ associated to this inner product is given by

$$\|\mu\|^2_{\tilde{\mathcal{M}}^{(k)}} = \langle \mu, \mu \rangle_{\tilde{\mathcal{M}}^{(k)}} = \mathbb{E}[\int_{\mathbb{R}} |\hat{\mu}(y)|^2 |y|^k e^{-y^2} dy]. \tag{3.4}$$

The space $\tilde{\mathcal{M}}^{(k)}$ equipped with the inner product $\langle \mu, \eta \rangle_{\tilde{\mathcal{M}}^{(k)}}$ is a pre-Hilbert space. We let $\mathcal{M}^{(k)}$ denote the completion of this pre-Hilbert space. We denote by $\mathcal{M}_0^{(k)}$ the set of all deterministic elements of $\mathcal{M}^{(k)}$. For $k = 0$ we write $\mathcal{M}^{(0)} = \mathcal{M}$ and $\mathcal{M}_0^{(0)} = \mathcal{M}_0$.

There are several advantages with working with this Hilbert space $\mathcal{M}$, compared to the Wasserstein metric space:

- A Hilbert space has a useful stronger structure than a metric space.
- The Wasserstein metric space $\mathcal{P}_2$ deals only with probability measures with finite second moment, while our Hilbert space deals with any (random) measure satisfying (3.1).
- With this norm we have the following useful estimate:

Lemma

Let $X^{(1)}$ and $X^{(2)}$ be two random variables in $L^2(\mathbb{P})$. Then

$$\left\|\mathcal{L}(X^{(1)}) - \mathcal{L}(X^{(2)})\right\|^2_{\mathcal{M}_0} \leq \sqrt{\pi}\mathbb{E}[(X^{(1)} - X^{(2)})^2].$$

Let us give some examples of measures:

Example (Measures)

1. Suppose that $\mu = \delta_{x_0}$, the unit point mass at $x_0 \in \mathbb{R}$. Then $\delta_{x_0} \in \mathcal{M}_0$ and
$$\int_{\mathbb{R}} e^{ixy} d\mu(x) = e^{ix_0 y},$$
and hence
$$\|\mu\|^2_{\mathcal{M}_0} = \int_{\mathbb{R}} |e^{ix_0 y}|^2 e^{-y^2} dy < \infty.$$

2. Suppose $d\mu(x) = f(x)dx$, where $f \in L^1(\mathbb{R})$. Then $\mu \in \mathcal{M}_0$ and by Riemann-Lebesque lemma, $\hat{\mu}(y) \in C_0(\mathbb{R})$, i.e. $\hat{\mu}$ is continuous and $\hat{\mu}(y) \to 0$ when $|y| \to \infty$. In particular, $|\hat{\mu}|$ is bounded on $\mathbb{R}$ and hence
$$\|\mu\|^2_{\mathcal{M}_0} = \int_{\mathbb{R}} |\hat{\mu}(y)|^2 e^{-y^2} dy < \infty.$$

Example

1. Suppose that μ is any finite positive measure on $\mathbb{R}$. Then $\mu \in \mathcal{M}_0^{(k)}$ for all k, because

$$|\hat{\mu}(y)| \quad \leq \int_{\mathbb{R}} d\mu(y) = \mu(\mathbb{R}) \quad < \infty, \text{ for all } y,$$

and hence

$$\|\mu\|^2_{\mathcal{M}_0^{(k)}} \quad = \int_{\mathbb{R}} |\hat{\mu}(y)|^2 |y|^k e^{-y^2} dy \leq \mu^2(\mathbb{R}) \int_{\mathbb{R}} |y|^k e^{-y^2} dy \quad < \infty.$$

2. Next, suppose $x_0 = x_0(\omega)$ is random. Then $\delta_{x_0(\omega)}$ is a random measure in $\mathcal{M}$. Similarly, if $f(x) = f(x, \omega)$ is random, then $d\mu(x, \omega) = f(x, \omega)dx$ is a random measure in $\mathcal{M}$.

t-absolute continuity and t-derivative of the law process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ generated by a one-dimensional Brownian motion B and an independent Poisson random measure $N(dt, d\zeta)$. Let $\nu(d\zeta)dt$ denote the Lévy measure of N, and let $\tilde{N}(dt, d\zeta)$ denote the compensated Poisson random measure $N(dt, d\zeta) - \nu(d\zeta)dt$. Suppose that $X(t) = X_t$ is an Itô-Lévy process of the form

(3.5)
$$\begin{cases} dX_t = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,\zeta)\tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ X_0 = x \in \mathbb{R}, \end{cases}$$

where α, β and γ are bounded predictable processes.

Let $\varphi \in C^2$. Then by the Itô formula

$$\mathbb{E}[\varphi(X_{t+h})] - \mathbb{E}[\varphi(X_t)] = \mathbb{E}[\int_t^{t+h} A\varphi(X_s)ds], \tag{3.6}$$

where

$$\begin{aligned} A\varphi(X_s) &= \alpha(s)\varphi'(X_s) + \tfrac{1}{2}\beta^2(s)\varphi''(X_s) \\ &\quad + \int_{\mathbb{R}_0} \{\varphi(X_s + \gamma(s,\zeta)) - \varphi(X_s) - \varphi'(X_s)\gamma(s,\zeta)\}\nu(d\zeta). \end{aligned}$$

In particular, if

$$\varphi(x) = \varphi_y(x) := \exp(ixy); \quad y \in \mathbb{R},$$

then

$$\begin{aligned} A\varphi_y(X_s) &= (iy\alpha(s) - \tfrac{1}{2}\beta^2(s)y^2 \\ &\quad + \int_{\mathbb{R}_0} \{\exp(i\gamma(s,\zeta)y) - 1 - iy\gamma(s,\zeta)\}\, \nu(d\zeta))\varphi_y(X_s), \end{aligned}$$

for all $y \in \mathbb{R}$.

Definition (Law process)

From now on we use the notation

$$M_t := M(t) := \mathcal{L}(X_t); \quad 0 \leq t \leq T$$

for the law process $\mathcal{L}(X_t)$ of $X_t = X(t)$ with respect to $\mathbb{P}$.

Lemma

(i) The map $t \mapsto M_t : [0, T] \to \mathcal{M}_0$ is absolutely continuous, and the derivative

$$M'(t) := \frac{d}{dt} M(t)$$

exists for all t.

(ii) There exists a constant $C < \infty$ such that

(3.7)

$$||M'(t)||_{\mathcal{M}_0} \leq C ||M(t)||_{\mathcal{M}_0^{(4)}} \text{ for all } t \in [0, T]; M(t) \in \mathcal{M}_0^{(4)}.$$

Proof. (i) Let $0 \leq t < t + h \leq T$. Then by (3.2) and (3.4) we get

$$\begin{aligned} \|M_{t+h} - M_t\|^2_{\mathcal{M}_0} &= \int_{\mathbb{R}} |\hat{M}_{t+h}(y) - \hat{M}_t(y)|^2 e^{-y^2} dy \\ &= \int_{\mathbb{R}} |\int_{\mathbb{R}} e^{ixy} d\mathcal{L}(X_{t+h}) - \int_{\mathbb{R}} e^{ixy} d\mathcal{L}(X_t)(x)|^2 e^{-y^2} dy \\ &= \int_{\mathbb{R}} |\mathbb{E}[\varphi_y(X_{t+h})] - \mathbb{E}[\varphi_y(X_t)]|^2 e^{-y^2} dy. \qquad (3.8) \end{aligned}$$

The last equality holds by using that for any bounded function ψ we have

$$\mathbb{E}[\psi(X)] \quad = \quad \int_{\mathbb{R}} \psi(x) d\mathcal{L}(X)(x).$$

By (3.6), we obtain

$$\begin{aligned} \|M_{t+h} - M_t\|^2_{\mathcal{M}_0} &= \int_{\mathbb{R}} |\mathbb{E}[\int_t^{t+h} A\varphi_y(X(s))ds]|^2 e^{-y^2} dy \\ &\le \int_{\mathbb{R}} (\int_t^{t+h} \mathbb{E}[|A\varphi_y(X_s)|]ds)^2 e^{-y^2} dy \le C_1 \; h^2, \end{aligned} \tag{3.9}$$

for some constant C_1 which does not depend on t and h.

We have proved that for different t and $t+h$, $\|M_{t+h} - M_t\|^2_{\mathcal{M}_0} \le C \; h^2$ and it is easy to see that this holds for every finite disjoint partition of the interval $[0,T]$. Thus we get that $t \mapsto M(t)$ is absolutely continuous, and the derivative $M'(t) = \frac{d}{dt}M(t)$ exists for all t.

(ii) This follows from (3.9), using that the coefficients α, β, γ are bounded and that

$$\mathbb{E}[|A_{\varphi_y}(X_s)|] \le const.y^2 |\mathbb{E}[\exp(iyX_s)]| \le const.y^2 |\widehat{M}_s(y)|. \tag{3.10}$$

□

From the lemma above we conclude the following:

Lemma

If X_t is an Itô-Lévy process as in (3.5), then the derivative $M'_s := \frac{d}{ds} M_s$ exists in $\mathcal{M}_0$ for a.a. s, and we have

$$M_t = M_0 + \int_0^t M'_s ds; \quad t \geq 0.$$

In the following we will apply the results above to the solutions $X(t)$ of the mean-field related type SDEs.

Example

(a) Suppose that $X(t) = B(t)$ with $B(0) = 0$. Then

$$d\mathcal{L}(X(t))(x) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})dx,$$

i.e. $\mathcal{L}(X(t))$ has a density $\frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})$. Therefore $\frac{d}{dt}\mathcal{L}(X(t))$ is a measure with density

$$\frac{d}{dt}\frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t}) = (\frac{x^2-t}{2t^2})(\frac{1}{\sqrt{2\pi t}} \exp(-\frac{x^2}{2t})).$$

(b) Suppose $X(t) = N(t)$, a Poisson process with intensity $\bar{\lambda}$. Then for $k = 1, 2, \ldots$ we have

$$\mathbb{P}(N(t) = k) = \frac{e^{-\bar{\lambda}t}(\bar{\lambda}t)^k}{k!}$$

and hence

$$\frac{d}{dt}\mathbb{P}(N(t) = k) = \frac{1}{k!}(\bar{\lambda}e^{-\bar{\lambda}t}(\lambda t)^{k-1}\{k - \bar{\lambda}t\}).$$

Preliminaries

We will recall some concepts and spaces which will be used on the sequel.
The probability $\mathbb{P}$ is a reference probability measure. We introduce two smaller filtrations $\mathbb{G}^{(i)}=(\mathcal{G}_t^{(i)})_{t\geq 0}$ such that $\mathcal{G}_t^{(i)} \subseteq \mathcal{F}_t$, for $i = 1, 2$ and for all $t \geq 0$. These filtrations represent the information available to player number i at time t.

Some basic concepts from Banach space theory

Since we deal with functions defined on an Hilbert space $\mathcal{M}$ of measures, we need the Fréchet derivative to differentiate functions of measures.

Let $\mathcal{X}, \mathcal{Y}$ be two Banach spaces with norms $\|\cdot\|_{\mathcal{X}}, \|\cdot\|_{\mathcal{Y}}$, respectively, and let $F : \mathcal{X} \to \mathcal{Y}$.

- We say that F has a directional derivative (or Gâteaux derivative) at $v \in \mathcal{X}$ in the direction $w \in \mathcal{X}$ if

  $$D_w F(v) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F(v + \varepsilon w) - F(v))$$

  exists in $\mathcal{Y}$.

- We say that F is Fréchet differentiable at $v \in \mathcal{X}$ if there exists a continuous linear map $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\lim_{\substack{h \to 0 \\ h \in \mathcal{X}}} \frac{1}{\|h\|_{\mathcal{X}}} \|F(v+h) - F(v) - A(h)\|_{\mathcal{Y}} = 0.$$

  In this case we call A the *gradient* (or Fréchet derivative) of F at v and we write

$$A = \nabla_v F.$$

- If F is Fréchet differentiable at v with Fréchet derivative $\nabla_v F$, then F has a directional derivative in all directions $w \in \mathcal{X}$ and

$$D_w F(v) := \langle \nabla_v F, w \rangle = \nabla_v F(w) = \nabla_v F w.$$

In particular, note that if F is a linear operator, then $\nabla_v F = F$ for all v.

Spaces

Throughout this work, we will use the following spaces:

- $\mathcal{S}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted càdlàg processes $(X(t))_{t\in[0,T]}$ such that

$$\|X\|_{\mathcal{S}^2}^2 := \mathbb{E}[\sup_{t\in[0,T]} |X(t)|^2] \ < \ \infty \ ,$$

- $\mathbb{L}^2$ is the set of $\mathbb{R}$-valued $\mathbb{F}$-adapted predictable processes $(Q(t))_{t\in[0,T]}$ such that

$$\|Q\|_{\mathbb{L}^2}^2 := \mathbb{E}[\int_0^T |Q(t)|^2 dt] < \ \infty \ .$$

- $L^2(\mathcal{F}_t)$ is the set of $\mathbb{R}$-valued square integrable $\mathcal{F}_t$-measurable random variables.
- $\mathbb{L}_\nu^2$ is the set of $\mathbb{F}$-adapted predictable processes $R : [0, T] \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$ such that

$$||R||_{\mathbb{L}_\nu^2}^2 := \mathbb{E}[\int_{\mathbb{R}_0} |R(t,\zeta)|^2 \nu(d\zeta) dt] \ < \ \infty \ .$$

- In general, for any given filtration $\mathbb{H}$, we say that the measure-valued process $\mu(t) = \mu(t, \omega) : [0, T] \times \Omega \to \mathcal{M}$ is adapted to $\mathbb{H}$ if $\mu(t)(V)$ is $\mathbb{H}$-adapted for all Borel sets $V \subseteq \mathbb{R}$. Let $\mathbb{M}_{\mathbb{G}} = \mathbb{M}_{\mathbb{G}^1}$ be a given set of $\mathcal{M}$-valued, $\mathbb{G}^1 = (\mathcal{G}_t^1)_{t \geq 0}$-adapted, stochastic processes $\mu(t)$. We call $\mathbb{M}_{\mathbb{G}}$ the set of admissible measure-valued control processes $\mu(\cdot)$.
- Let $\mathcal{A}_{\mathbb{G}} = \mathcal{A}_{\mathbb{G}^2}$ be a given set of real-valued, $\mathbb{G}^2 = (\mathcal{G}_t^2)_{t \geq 0}$-adapted, stochastic processes $u(t)$ required to have values in a given convex subset $\mathcal{U}$ of $\mathbb{R}$. We call $\mathcal{A}_{\mathbb{G}}$ the set of admissible real-valued control processes $u(\cdot)$.
- $\mathcal{R}$ is the set of measurable functions $r : \mathbb{R}_0 \to \mathbb{R}$.
- $C_a([0, T], \mathcal{M}_0)$ denotes the set of absolutely continuous functions $m : [0, T] \to \mathcal{M}_0$.
- $\mathbb{K}$ is the set of bounded linear functionals $K : \mathcal{M}_0 \to \mathbb{R}$ equipped with the operator norm

$$||K||_{\mathbb{K}} := \sup_{m \in \mathcal{M}_0, ||m||_{\mathcal{M}_0} \leq 1} |K(m)|.$$

- $\mathcal{S}^2_{\mathbb{K}}$ is the set of $\mathbb{F}$-adapted càdlàg processes $p : [0, T] \times \Omega \mapsto \mathbb{K}$ such that

$$||p||^2_{\mathcal{S}_{\mathbb{K}}} := \mathbb{E}[\sup_{t\in[0,T]} ||p(t)||^2_{\mathbb{K}}] < \infty.$$

- $\mathbb{L}^2_{\mathbb{K}}$ is the set of $\mathbb{F}$-adapted predictable processes $q : [0, T] \times \Omega \mapsto \mathbb{K}$ such that

$$||q||^2_{\mathbb{L}^2_{\mathbb{K}}} := \mathbb{E}[\int_0^T ||q(t)||^2_{\mathbb{K}} dt] < \infty.$$

- $\mathbb{L}^2_{\nu,\mathbb{K}}$ is the set of $\mathbb{F}$-adapted predictable processes $r : [0, T] \times \mathbb{R}_0 \times \Omega \mapsto \mathbb{K}$ such that

$$||r||^2_{\mathbb{L}^2_{\nu,\mathbb{K}}} := \mathbb{E}[\int_0^T \int_{\mathbb{R}_0} ||r(t,\zeta)||^2_{\mathbb{K}} \nu(d\zeta) dt] < \infty.$$

- $\mathbb{M}_0$ is the set of t-differentiable $\mathcal{M}_0$-valued processes $m(t); t \in [0, T]$.
  If $m \in \mathbb{M}_0$ we put $m'(t) = \frac{d}{dt} m(t)$.

The model uncertainty stochastic optimal control problem

As pointed out in the Introduction, there are several ways to represent model uncertainty in a stochastic system. In this paper, we are interested in systems governed by controlled mean-field related type SDE $X^{\mu,u}(t) = X(t) \in \mathcal{S}^2$ on the form

(5.1)
$$\begin{cases} dX(t) & = & b(t, X(t), \mu(t), u(t))\, dt + \sigma(t, X(t), \mu(t), u(t))\, dB(t) \\ & & + \int_{\mathbb{R}_0} \gamma(t, X(t), \mu(t), u(t), \zeta)\, \tilde{N}(dt, d\zeta);\ \ t \in [0, T], \\ X(0) & = & x \in \mathbb{R}. \end{cases}$$

The functions

$$
\begin{aligned}
b(t,x,\mu,u) &= b(t,x,\mu,u,\omega) && : [0,T]\times\mathbb{R}\times\mathcal{M}\times\mathcal{U}\times\Omega \\
\sigma(t,x,\mu,u) &= \sigma(t,x,\mu,u,\omega) && : [0,T]\times\mathbb{R}\times\mathcal{M}\times\mathcal{U}\times\Omega \\
\gamma(t,x,\mu,u,\zeta) &= \gamma(t,x,\mu,u,\zeta,\omega) && : [0,T]\times\mathbb{R}\times\mathcal{M}\times\mathcal{U}\times\mathbb{R}_0\times\Omega
\end{aligned}
$$

are supposed to be Lipschitz on $x \in \mathbb{R}$, uniformly with respect to t and ω for given $u \in \mathcal{U}$ and $\mu \in \mathcal{M}$. Then by e.g. Theorem 1.19 in Ø. and Sulem [19], we have existence and uniqueness of the solution of $X(t)$.

We may regard (5.1) as a perturbed version of the mean-field equation

(5.2)

$$\begin{cases} dX(t) & = & b(t, X(t), \mathcal{L}(X(t)), u(t))\,dt + \sigma(t, X(t), \mathcal{L}(X(t)), u(t))\,d \\ & & + \int_{\mathbb{R}_0} \gamma(t, X(t), \mathcal{L}(X(t)), u(t), \zeta)\,\tilde{N}(dt, d\zeta);\ \ t \in [0, T], \\ X(0) & = & x \in \mathbb{R} \end{cases}$$

For example, we could have $\mu(t) = \mathcal{L}_{\mathbb{Q}}(X(t))$ for some probability measure $\mathbb{Q} \neq \mathbb{P}$.

Thus the model uncertainty is represented by an uncertainty about what law $\mu(t)$ is influencing the coefficients of the system, and we are penalising the laws that are far away from $\mathcal{L}(X(t))$. See the application in Section 5.

Let us consider a performance functional of the form

$$J(\mu, u)$$

(5.3)

$$= \mathbb{E}[g(X(T), M(T)) + \int_0^T \ell(s, X(s), M(s), \mu(s), u(s))\, ds],$$

where
$\ell(t, x, m, \mu, u) = \ell(t, x, m, \mu, u, \omega)$:
$[0, T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \Omega \to \mathbb{R}$
and $g : \mathbb{R} \times \mathcal{M}_0 \times \Omega \to \mathbb{R}$ are given functions.

For fixed x, m, μ, u we assume that $\ell(s, \cdot)$ is $\mathcal{F}_s$-measurable for all $s \in [0, T]$ and $g(\cdot, \cdot)$ is $\mathcal{F}_T$-measurable. We also assume the following integrability condition

$$\mathbb{E}[\,|g(X(T), M(T))|^2 + \int_0^T |\ell(s, X(s), M(s), \mu(s), u(s))|^2 \, ds] < \infty,$$

for all $\mu \in \mathbb{M}_{\mathbb{G}}$ and $u \in \mathcal{A}_{\mathbb{G}}$.

Note that the system (5.1) and the performance (31) are not Markovian. However, recently dynamic programming approaches to mean-field stochastic control problems have been introduced. See e.g. Bayraktar *et al* [8] and Pham and Wei [23]. In this paper we will use an approach based on a suitably modified stochastic maximum principle, which also works in partial information settings.

In the next section we study a stochastic differential game of two players, where one of the players is solving an optimal measure-valued control problem of the type described above, while the other player is solving a classical real-valued stochastic control problem. To the best of our knowledge this type of stochastic differential game has not been studied before.

Nonzero-sum games

We first consider a nonzero-sum mean-field game. We will establish a maximum principle for a Nash equilibrium of such games:

We consider the $\mathbb{R} \times \mathcal{M}_0$-valued process $(X(t), M(t))$ where $M(t) = \mathcal{L}(X(t))$, where $X(t)$ is given by (5.1) and

$$(5.4) \qquad dM(t) = \beta(M(t))dt; \quad M(0) \in \mathcal{M}_0 \text{ given },$$

where β is the operator on $\mathbb{M}_0$ given by

$$(5.5) \qquad \beta(m(t)) = m'(t).$$

The cost functionals are assumed to be on the form

(5.6)

$$\begin{aligned} J_i(\mu, u) &= \mathbb{E}[g_i(X(T), M(T)) \\ &\quad + \int_0^T \ell_i(s, X(s), M(s), \mu(s), u(s))\, ds]; \text{ for } i = 1, 2, \end{aligned}$$

where $M(s) := \mathcal{L}(X(s))$ and the functions

$$\begin{array}{lll} \ell_i(t, x, m, \mu, u) & = \ell_i(t, x, m, \mu, u, \omega) & : [0, T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \Omega \\ g_i(x, m) & = g_i(x, m, \omega) & : \mathbb{R} \times \mathcal{M}_0 \times \Omega \end{array}$$

are continuously differentiable with respect to x, u and admit Fréchet derivatives with respect to m and μ.

Problem

The general nonzero-sum stochastic game is to find $(\mu^*, u^*) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ *such that*

$$\begin{aligned} J_1(\mu, u^*) \le J_1(\mu^*, u^*), &\quad \textit{for all } \mu \in \mathbb{M}_{\mathbb{G}}, \\ J_2(\mu^*, u) \le J_2(\mu^*, u^*), &\quad \textit{for all } u \in \mathcal{A}_{\mathbb{G}}. \end{aligned}$$

The pair (μ^*, u^*) *is called a Nash equilibrium.*

Definition

(The Hamiltonian) For $i = 1, 2$ we define the Hamiltonian

$$H_i : [0, T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times C_a([0, T], \mathcal{M}_0) \to \mathbb{R}$$

by

$$\begin{aligned} & H_i(t, x, m, \mu, u, p_i^0, q_i^0, r_i^0(\cdot), p_i^1) \\ & = \ell_i(t, x, m, \mu, u) + p_i^0 b(t, x, \mu, u) + q_i^0 \sigma(t, x, \mu, u) \\ & + \int_{\mathbb{R}_0} r_i^0(\zeta) \gamma(t, x, \mu, u, \zeta) \, \nu(d\zeta) + \left\langle p_i^1, \beta(m) \right\rangle, \end{aligned} \tag{5.7}$$

where p_i^0, p_i^1 represent generic values of the adjoint processes defined below.

We assume that H_i is continuously differentiable with respect to x, u and admits Fréchet derivatives with respect to m and μ.

For $u \in \mathcal{A}_{\mathbb{G}}, \mu \in \mathbb{M}_{\mathbb{G}}$ with corresponding solution $X = X^{\mu,u}$, define $p_i^0 = p_i^{0,\mu,u}, q_i^0 = q_i^{0,\mu,u}$ and $r_i^0 = r_i^{0,\mu,u}$ and $p_i^1 = p_i^{1,\mu,u}, q_i^1 = q_i^{1,\mu,u}$ and $r_i^1 = r_i^{1,\mu,u}$ for $i = 1, 2$ by the following set of adjoint equations:

- The real-valued BSDE in the unknown $(p_i^0, q_i^0, r_i^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}_\nu^2$ is given by

$$
\begin{aligned}
dp_i^0(t) &= -\frac{\partial H_i}{\partial x}(t)dt + q_i^0(t)dB(t) \\
&\qquad + \int_{\mathbb{R}_0} r_i^0(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T] \\
p_i^0(T) &= \frac{\partial g_i}{\partial x}(X(T), M(T)),
\end{aligned}
\tag{5.8}
$$

and the operator-valued BSDE in the unknown $(p_i^1, q_i^1, r_i^1) \in \mathcal{S}_{\mathbb{K}}^2 \times \mathbb{L}_{\mathbb{K}}^2 \times \mathbb{L}_{\nu,\mathbb{K}}^2$ is given by

-

$$
\begin{aligned}
dp_i^1(t) &= -\nabla_m H_i(t)dt + q_i^1(t)dB(t) \\
&\quad + \int_{\mathbb{R}_0} r_i^1(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T], \\
p^1(T) &= \nabla_m g_i(X(T), M(T)),
\end{aligned} \tag{5.9}
$$

where
$H_i(t) = H_i(t, X(t), M(t), \mu(t), u(t), p_i^0(t), q_i^0(t), r_i^0(t,\cdot), p_i^1(t))$
etc.

We remark that the BSDEs (39) is linear, so whenever knowing the Hamiltonian H_i and the function g_i, we can get a solution explicitly. To remind the reader of this solution formula, let us consider the solution $(P, Q, R) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_\nu$ of the linear BSDE

$$\begin{aligned} dP(t) = -\Big[&\varphi(t) + \alpha(t)P(t) + \beta(t)Q(t) \\ &+ \int_{\mathbb{R}_0} \phi(t,\zeta)R(t,\zeta)\nu(d\zeta)\Big]dt \\ &+ Q(t)dB(t) + \int_{\mathbb{R}_0} R(t,\zeta)\tilde{N}(dt,d\zeta);\ \ t \in [0,T], \end{aligned} \tag{5.10}$$

$$P(T) = \theta \in L^2(\mathcal{F}_T). \tag{5.11}$$

Here φ, α, β and ϕ are bounded predictable processes with ϕ is assumed to be an $\mathbb{R}$-valued process defined on $[0, T] \times \mathbb{R}_0 \times \Omega$. Then it is well-known (see e.g. Theorem 1.7 in Ø. and Sulem [20]) that the component $P(t)$ of the solution of equation (41) can be written in closed form as follows:

$$(5.12) \qquad P(t) = \mathbb{E}[\theta \frac{\Gamma(T)}{\Gamma(t)} + \int_t^T \frac{\Gamma(s)}{\Gamma(t)} \varphi(s) | \mathcal{F}_t]; \quad t \in [0, T],$$

where $\Gamma(t) \in \mathcal{S}^2$ is the solution of the linear SDE with jumps

$$\begin{cases} d\Gamma(t) &= \Gamma(t^-)[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \phi(t,\zeta)\tilde{N}(dt, d\zeta)]; \\ \Gamma(0) &= 1. \end{cases}$$

For notational convenience, we will employ the following short hand notations

$$\hat{H}_1(t) = H_1(t, \hat{X}(t), \hat{M}(t), \hat{\mu}(t), \hat{u}(t), \hat{p}_1^0(t), \hat{q}_1^0(t), \hat{r}_1^0(t,\cdot), \hat{p}_1^1(t)), \tag{5.13}$$

$$\check{H}_1(t) = H_1(t, \hat{X}(t), \hat{M}(t), \mu(t), \hat{u}(t), \hat{p}_1^0(t), \hat{q}_1^0(t), \hat{r}_1^0(t,\cdot), \hat{p}_1^1(t)), \tag{5.14}$$

$$\bar{H}_2(t) = H_2(t, \hat{X}(t), \hat{M}(t), \hat{\mu}(t), \hat{u}(t), \hat{p}_2^0(t), \hat{q}_2^0(t), \hat{r}_2^0(t,\cdot), \hat{p}_2^1(t)), \tag{5.15}$$

$$\check{H}_2(t) = H_2(t, \hat{X}(t), \hat{M}(t), \hat{\mu}(t), u(t), \hat{p}_2^0(t), \hat{q}_2^0(t), \hat{r}_2^0(t,\cdot), \hat{p}_2^1(t)).$$

Similar notation is used for the derivatives of $H, \ell, g, b, \sigma, \gamma$ etc.

We now state a sufficient theorem for the nonzero-sum games.

Theorem (Sufficient nonzero-sum maximum principle)

Let $(\hat{\mu}, \hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ *with corresponding solutions* $\hat{X}$, (p_i^0, q_i^0, r_i^0) *and* (p_i^1, q_i^1, r_i^1) *of the forward and backward stochastic differential equations* (5.1), (39) *and* (5.9) *respectively. Suppose that*

1. *(Concavity) The functions*

$$\begin{array}{ll} (x, m, \mu) & \mapsto H_1(t) \\ (x, m, u) & \mapsto H_2(t) \\ (x, m) & \mapsto g_i(x, m), \text{ for } i = 1, 2, \end{array}$$

   are concave $\mathbb{P}$*.a.s for each* $t \in [0, T]$.

2. *(Maximum conditions)*

$$\mathbb{E}[\hat{H}_1(t)|\mathcal{G}_t^{(1)}] = \operatorname*{ess\ sup}_{\mu \in \mathbb{M}_{\mathbb{G}}} \mathbb{E}[\check{H}_1(t)|\mathcal{G}_t^{(1)}], \tag{5.16}$$

and

$$\mathbb{E}[\bar{H}_2(t)|\mathcal{G}_t^{(2)}] \quad = \quad \underset{u\in\mathcal{A}_{\mathbb{G}}}{ess\ \sup}\mathbb{E}[\breve{H}_2(t)|\mathcal{G}_t^{(2)}],$$

$\mathbb{P}$.a.s for each $t \in [0, T]$.

Then $(\hat{\mu}, \hat{u})$ is a Nash equilibrium for our problem.

Proof. Let us first prove that $J_1(\mu, \hat{u}) \leq J_1(\hat{\mu}, \hat{u})$.
By the definition of the cost functional (5.6) we have for fixed $\hat{u} \in \mathcal{A}_{\mathbb{G}}$ and arbitrary $\mu \in \mathbb{M}_{\mathbb{G}}$

$$J_1(\mu, \hat{u}) - J_1(\hat{\mu}, \hat{u}) = I_1 + I_2, \tag{5.17}$$

where

$$\begin{aligned} I_1 &= \mathbb{E}[\int_0^T \{\check{\ell}_1(t) - \hat{\ell}_1(t)\}dt], \\ I_2 &= \mathbb{E}[\check{g}_1(X(T), M(T)) - \hat{g}_1(\hat{X}(T), \hat{M}(T))]. \end{aligned}$$

By the definition of the Hamiltonian (10) we have

$$I_1 = \mathbb{E}[\int_0^T \check{H}_1(t) - \hat{H}_1(t) - \hat{p}_1^0(t)\tilde{b}(t) - \hat{q}_1^0(t)\tilde{\sigma}(t)$$

$$- \int_{\mathbb{R}_0} \hat{r}_1^0(t,\zeta)\tilde{\gamma}(t,\zeta)\nu(d\zeta) - \langle \hat{p}_1^1(t), \beta(m)(t)\rangle dt], \tag{5.18}$$

where $\tilde{b}(t) = \check{b}(t) - \hat{b}(t)$ etc.

By the concavity of g_1 and the terminal values of the BSDEs (39), (5.9), we have

$$I_2 \leq \mathbb{E}[\frac{\partial g_1}{\partial x}(T)\tilde{X}(T) + \langle \nabla_m g_1(T), \tilde{M}(T)\rangle]$$

$$= \mathbb{E}[\hat{p}_1^0(T)\tilde{X}(T) + \langle \hat{p}_1^1(T), \tilde{M}(T)\rangle]. \tag{5.19}$$

Applying the Itô formula to $\hat{p}_1^0(t)\tilde{X}(t)$ and $\langle \hat{p}_1^1(t), \tilde{M}(t)\rangle$, we get

$$I_2 \leq \mathbb{E}[\hat{p}_1^0(T)\tilde{X}(T) + \langle \hat{p}_1^1(T), \tilde{M}(T)\rangle]$$
$$= \mathbb{E}[\int_0^T \hat{p}_1^0(t)d\tilde{X}(t) + \int_0^T \tilde{X}(t)d\hat{p}_1^0(t) + \int_0^T \hat{q}_1^0(t)\tilde{\sigma}(t)dt + \int_0^T \int_{\mathbb{R}_0} \hat{r}_1^0(t,\zeta$$
$$+ \mathbb{E}[\int_0^T \langle \hat{p}_1^1(t), d\tilde{M}(t)\rangle + \int_0^T \tilde{M}(t)d\hat{p}_1^1(t)]$$
$$= \mathbb{E}[\int_0^T \hat{p}_1^0(t)\tilde{b}(t)dt - \int_0^T \frac{\partial \hat{H}_1}{\partial x}(t)\tilde{X}(t)dt + \int_0^T \hat{q}_1^0(t)\tilde{\sigma}(t)dt$$
$$+ \int_0^T \int_{\mathbb{R}_0} \hat{r}_1^0(t,\zeta)\tilde{\gamma}(t,\zeta)\nu(d\zeta)dt + \int_0^T \langle \hat{p}_1^1(t), \tilde{\beta}(t)\rangle dt$$

(5.20)

$$- \int_0^T \langle \nabla_m \hat{H}_1(t), \tilde{M}(t)\rangle dt],$$

where we have used that the $dB(t)$ and $\tilde{N}(dt, d\zeta)$ integrals with the necessary integrability property are martingales and then have mean zero.

Substituting (5.18) and (5.20) in (5.17) yields

$$\begin{aligned} & J_1(\mu, \hat{u}) - J_1(\hat{\mu}, \hat{u}) \\ (5.21) \\ & \leq \mathbb{E}[\int_0^T \{\check{H}_1(t) - \hat{H}_1(t) - \tfrac{\partial \hat{H}_1}{\partial x}(t)\tilde{X}(t) - \langle \nabla_m \hat{H}_1(t), \tilde{M}(t)\rangle\} dt]. \end{aligned}$$

By the concavity of H_1 and the fact that the process μ is $\mathcal{G}_t^{(1)}$-adapted, we obtain

$$\begin{aligned} J_1(\mu, \hat{u}) - J_1(\hat{\mu}, \hat{u}) &\leq \mathbb{E}[\int_0^T \frac{\partial \hat{H}_1}{\partial \mu}(t)\,(\mu(t) - \hat{\mu}(t))\,dt] \\ &= \mathbb{E}[\int_0^T \mathbb{E}(\frac{\partial \hat{H}_1}{\partial \mu}(t)\,(\mu(t) - \hat{\mu}(t))\,|\mathcal{G}_t^{(1)})dt] \\ &= \mathbb{E}[\int_0^T \mathbb{E}(\frac{\partial \hat{H}_1}{\partial \mu}(t)|\mathcal{G}_t^{(1)})\,(\mu(t) - \hat{\mu}(t))\,dt] \\ &\leq 0, \end{aligned}$$

where $\frac{\partial \hat{H}_1}{\partial \mu} = \nabla_\mu \hat{H}_1$. The last equality holds because of the maximum condition of $\hat{H}_1$ at $\mu = \hat{\mu}$.

Similar considerations apply to prove that $J_2(\hat{\mu}, u) \leq J_2(\hat{\mu}, \hat{u})$.

$\square$

A necessary maximum principle

We now state and prove a necessary version of the maximum principle. We assume the following:

- Whenever $\mu \in \mathbb{M}_{\mathbb{G}}$ $(u \in \mathcal{A}_{\mathbb{G}})$ and $\eta \in \mathbb{M}_{\mathbb{G}}$ $(\pi \in \mathcal{A}_{\mathbb{G}})$ are bounded, there exists $\epsilon > 0$ such that

$$\mu + \lambda\eta \in \mathbb{M}_{\mathbb{G}} \ (u + \lambda\pi \in \mathcal{A}_{\mathbb{G}}), \text{ for each } \lambda \in [-\epsilon, \epsilon] .$$

- For each $t_0 \in [0, T]$ and each bounded $\mathcal{G}^{(1)}_{t_0}$-measurable random measure α_1 and $\mathcal{G}^{(2)}_{t_0}$-measurable random variable α_2, the process

$$\eta(t) = \alpha_1 \mathbf{1}_{[t_0,T]}(t) \tag{5.22}$$

belongs to $\mathbb{M}_{\mathbb{G}}$ and the process

$$\pi(t) = \alpha_2 \mathbf{1}_{[t_0,T]}(t)$$

belongs to $\mathcal{A}_{\mathbb{G}}$.

Definition

In general, if $K^u(t)$ is a process depending on u, we define the differential operator D on K by

$$DK^u(t) := D^\pi K^u(t) = \frac{d}{d\lambda} K^{u+\lambda\pi}(t)|_{\lambda=0}$$

whenever the derivative exists.

The *derivative* of the state $X(t)$ defined by (5.1) is

$$DX^{\mu}(t) := \tfrac{d}{d\lambda} X^{\mu+\lambda\eta}|_{\lambda=0} = Z(t)$$

exists, and is given by

(5.23)

$$\begin{cases} dZ(t) &= [\frac{\partial b}{\partial x}(t) Z(t) + \frac{\partial b}{\partial \mu}(t)\eta(t)]dt + [\frac{\partial \sigma}{\partial x}(t) Z(t) + \frac{\partial \sigma}{\partial \mu}(t)\eta(t \\ &\quad + \int_{\mathbb{R}_0} [\frac{\partial \gamma}{\partial x}(t,\zeta) Z(t) + \frac{\partial \gamma}{\partial \mu}(t,\zeta)\eta(t)] \tilde{N}(dt, d\zeta); \qquad t \in [0, \\ Z(0) &= 0. \end{cases}$$

We remark that this derivative process is a linear SDE. Hence, assuming that b, σ and γ admit bounded partial derivatives with respect to x and μ, there is a unique solution $Z(t) \in \mathcal{S}^2$ of (5.23). It is easy to see that $Z(t)$ is exactly the derivative in $\mathbb{L}^2(\mathbb{P})$ of $X^{\mu+\lambda\eta}(t)$ with respect to λ at $\lambda = 0$. More precisely, we have the following:

Lemma

$$\mathbb{E}[\int_0^T (\tfrac{X^{\mu+\lambda\eta}(t) - X^{\mu}(t)}{\lambda} - Z(t))^2 dt] \to 0 \text{ as } \lambda \to 0. \quad (5.24)$$

Theorem

(Necessary nonzero-sum maximum principle) *Let* $(\hat{\mu}, \hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ *with corresponding solutions* $\hat{X}$*,* (p_i^0, q_i^0, r_i^0) *and* (p_i^1, q_i^1, r_i^1) *of the forward and backward stochastic differential equations* (5.1) *and* (39) $-$ (5.9)*, with the corresponding derivative process* $\hat{Z}$ *given by* (5.23)*. Then the following (i) and (ii) are equivalent:*

(i) *For all* $\eta \in \mathbb{M}_{\mathbb{G}}$ *and for all* $\pi \in \mathcal{A}_{\mathbb{G}}$

$$\frac{d}{d\lambda} J_1(\hat{\mu} + \lambda\eta, \hat{u})|_{\lambda=0} = \frac{d}{ds} J_2(\hat{\mu}, \hat{u} + s\pi)|_{s=0} = 0,$$

(ii)

$$\mathbb{E}[\frac{\partial H_1}{\partial \mu}(t)|\mathcal{G}_t^{(1)}] = \mathbb{E}[\frac{\partial H_2}{\partial u}(t)|\mathcal{G}_t^{(2)}] = 0.$$

Proof. First note that, by using the linearity of $\langle \cdot, \cdot \rangle$ and the fact that the Fréchet derivative of a linear operator is the same operator, we get, by interchanging the order of the derivatives $\frac{d}{dt}$ and ∇_m, that

$$\begin{aligned} \nabla_m \langle p_1^1(t), \frac{d}{dt} m \rangle &= \langle p_1^1(t), \nabla_m \frac{d}{dt} m \rangle = \langle p_1^1(t), \frac{d}{dt} \nabla_m (m) \rangle \\ &= \langle p_1^1(t), \frac{d}{dt} (\cdot) \rangle, \end{aligned} \tag{5.25}$$

and hence

$$\begin{aligned} \langle \nabla_m \langle p_1^1(t), \frac{d}{dt} m \rangle, DM(t) \rangle &= \langle p_1^1(t), \frac{d}{dt} DM(t) \rangle \\ &= \left\langle p_1^1(t), DM'(t) \right\rangle \end{aligned} \tag{5.26}$$

Also, note that

$$dDM(t) \quad = DM'(t) dt.$$

Assume that (i) holds. Using the definition (5.6) of J_1, we get

$$
\begin{aligned}
0 &= \tfrac{d}{d\lambda} J_1(\mu + \lambda\eta, u)|_{\lambda=0} \\
&= \mathbb{E}[\textstyle\int_0^T \{ \tfrac{\partial \ell_1}{\partial x}(t)\, Z(t) + \langle \nabla_m \ell_1(t), DM(t) \rangle + \tfrac{\partial \ell_1}{\partial \mu}(t)\, \eta(t) \} dt \\
&+ \tfrac{\partial g_1}{\partial x}(T)\, Z(T) + \langle \nabla_m g_1(T), DM(T) \rangle].
\end{aligned}
$$

Hence, by the definition (10) of H_1, we have

$$0 = \frac{d}{d\lambda} J_1(\mu + \lambda\eta, u)|_{\lambda=0}$$
$$= \mathbb{E}[\int_0^T \{ \frac{\partial H_1}{\partial x}(t) - p_1^0(t)\frac{\partial b}{\partial x}(t) - q_1^0(t)\frac{\partial \sigma}{\partial x}(t) - \int_{\mathbb{R}_0} r_1^0(t,\zeta)\frac{\partial \gamma}{\partial x}(t,\zeta)\,\nu(d\zeta)$$
$$+ \int_0^T \langle \nabla_m H_1(t), DM(t)\rangle dt + \int_0^T \langle \nabla_{m'} H_1(t), DM'(t)\rangle dt$$
$$- \int_0^T \langle p_1^1(t), DM'(t)\rangle dt + \int_0^T \frac{\partial H_1}{\partial \mu}(t) - p_1^0(t)\frac{\partial b}{\partial \mu}(t)$$
$$- q_1^0(t)\frac{\partial \sigma}{\partial \mu}(t) - \int_{\mathbb{R}_0} r_1^0(t,\zeta)\frac{\partial \gamma}{\partial \mu}(t,\zeta)\,\nu(d\zeta)\}\eta(t)dt + p_1^0(T)Z(T)$$

(5.27)

$$+ \langle p_1^1(T), DM(T)\rangle].$$

Applying now the Itô formula to both $p_1^0 Z$ and $\langle p_1^1, DM \rangle$, we get

$$\mathbb{E}[p_1^0(T)Z(T) + \langle p_1^1(T), DM(T)\rangle]$$
$$= \mathbb{E}[\int_0^T p_1^0(t)dZ(t) + \int_0^T Z(t)dp_1^0(t) + \int_0^T q_1^0(t)(\frac{\partial\sigma}{\partial x}(t)\,Z(t) + \frac{\partial\sigma}{\partial\mu}(t)\,\eta(t$$
$$+ \int_0^T \int_{\mathbb{R}_0} r_1^0(t,\zeta)(\frac{\partial\gamma}{\partial x}(t,\zeta)\,Z(t) + \frac{\partial\gamma}{\partial\mu}(t,\zeta)\,\eta(t))\nu(d\zeta)dt]$$
$$+ \mathbb{E}[\int_0^T \langle p_1^1(t), DM'(t)\rangle dt + \int_0^T DM(t)dp_1^1(t)]$$
$$= \mathbb{E}[\int_0^T p_1^0(t)(\frac{\partial b}{\partial x}(t)\,Z(t) + \frac{\partial b}{\partial\mu}(t)\,\eta(t))dt - \int_0^T \frac{\partial H_1}{\partial x}(t)Z(t)dt$$
$$+ \int_0^T q_1^0(t)(\frac{\partial\sigma}{\partial x}(t)\,Z(t) + \frac{\partial\sigma}{\partial\mu}(t)\,\eta(t))dt$$
$$+ \int_0^T \int_{\mathbb{R}_0} r_1^0(t,\zeta)(\frac{\partial\gamma}{\partial x}(t,\zeta)\,Z(t) + \frac{\partial\gamma}{\partial\mu}(t,\zeta)\,\eta(t))\nu(d\zeta)dt$$

(5.28)

$$+ \int_0^T \langle p_1^1(t), DM'(t)\rangle dt - \int_0^T \langle \nabla_m H_1(t), DM(t)\rangle dt].$$

Combining the above and recalling that η is of the form (5.22), we conclude that

$$0 = \mathbb{E}[\int_0^T \frac{\partial H_1}{\partial \mu}(t)\eta(t)dt] = \mathbb{E}[\int_s^T \frac{\partial H_1}{\partial \mu}(t)\alpha_1 dt];\ s \geq t_0.$$

Differentiating with respect to s we obtain

$$\begin{aligned} 0 &= \mathbb{E}[\frac{\partial H_1}{\partial \mu}(s)\alpha_1] \\ &= \mathbb{E}[\frac{\partial H_1}{\partial \mu}(t_0)|\mathcal{G}_{t_0}^{(1)}], \end{aligned}$$

because this holds for all α_1 and all $s \geq t_0$.
This argument can be reversed, to prove that (ii)$\Longrightarrow$(i). We omit the details.
In the same manner, we can get the equivalence between

$$\frac{d}{ds} J_2(\mu, u + s\pi)|_{s=0} = 0$$

and

$$\mathbb{E}[\frac{\partial H_2}{\partial u}(t)|\mathcal{G}_t^{(2)}] = 0.$$

□

Zero-sum game

In this section, we proceed to study the maximum principle for the zero-sum game case. Let us then define the performance functional as

$$J(\mu, u) = \mathbb{E}[g(X(T), M(T)) + \int_0^T \ell(s, X(s), M(s), \mu(s), u(s))\, ds],$$

where the state $X(t)$ is the solution of a SDE (5.1).
The functions

$$\ell(s, x, m, \mu, u) = \ell(s, x, m, \mu, u, \omega) : [0, T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \Omega \to \mathbb{R}$$

and

$$g(x, m) = g(x, m, \omega) : \mathbb{R} \times \mathcal{M}_0 \times \Omega \to \mathbb{R}$$

are supposed to satisfy the following conditions:

(a) ℓ and g are continuously differentiable with respect to x, u and admits Fréchet derivatives with respect to m and μ.

(b) Moreover, the function

$$\mathbb{R} \times \mathcal{M}_0 \ni (x, m) \mapsto g(x, m)$$

is required to be affine $\mathbb{P}$-a.s.

We consider the stochastic zero-sum game to find (μ^*, u^*) such that

$$\sup_{u \in \mathcal{A}_{\mathbb{G}}} \inf_{\mu \in \mathbb{M}_{\mathbb{G}}} J(\mu, u) = \inf_{\mu \in \mathbb{M}_{\mathbb{G}}} \sup_{u \in \mathcal{A}_{\mathbb{G}}} J(\mu, u) = J(\mu^*, u^*).$$

We call (μ^*, u^*) a *saddle point* for $J(\mu, u)$.
In this case the Hamiltonian is given by

$$H : [0, T] \times \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times C_a([0, T], \mathcal{M}_0) \to \mathbb{R}$$

be given by

$$\begin{aligned}
&H(t, x, m, \mu, p^0, q^0, r^0(\cdot), p^1) \\
&= \ell(t, x, m, \mu, u) + p^0 b(t, x, \mu, u) + q^0 \sigma(t, x, \mu, u) \\
&+ \int_{\mathbb{R}_0} r^0(\zeta) \gamma\,(t, x, \mu, u, \zeta)\, \nu(d\zeta) + \langle p^1, \beta(m) \rangle.
\end{aligned}$$

We assume the following:

(c) H is continuously differentiable with respect to x, u and admits Fréchet derivatives with respect to m and μ.

(d) The Hamiltonian function

$$\begin{aligned} & \mathbb{R} \times \mathcal{M}_0 \times \mathcal{M} \times \mathcal{U} \ni (x, m, \mu, u) \\ (5.29) \qquad & \mapsto H(t, x, m, p^0, q^0, r^0(\cdot), p^1) \end{aligned}$$

is *convex* with respect to (x, m, μ) and *concave* with respect to (x, m, u) $\mathbb{P}$.a.s and for each $t \in [0, T]$, $p^0, q^0, r^0(\cdot)$ and p^1.
For $u \in \mathcal{A}_{\mathbb{G}}, \mu \in \mathbb{M}_{\mathbb{G}}$ with corresponding solution $X = X^{\mu,u}$, define $p = p^{\mu,u}, q = q^{\mu,u}$ and $r = r^{\mu,u}$ by the adjoint equations: the real-BSDE in the unknown $(p^0, q^0, r^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_\nu$ has the following form

$$dp^0(t) = -\frac{\partial H}{\partial x}(t)\,dt + q^0(t)dB(t) + \int_{\mathbb{R}_0} r^0(t,\zeta)\tilde{N}(dt,d\zeta);\ t \in [0,T]\,,$$

(5.30)

$$p^0(T) = \frac{\partial g}{\partial x}(X(T), M(T)),$$

and the operator-valued BSDE for the unknown $(p^1, q^1, r^1) \in \mathcal{S}^2_{\mathbb{K}} \times \mathbb{L}^2_{\mathbb{K}} \times \mathbb{L}^2_{\nu,\mathbb{K}}$ is given by

$$\begin{aligned} dp^1(t) &= -\nabla_m H(t)dt + q^1(t)dB(t) \\ &\qquad + \int_{\mathbb{R}_0} r^1(t,\zeta)\tilde{N}(dt,d\zeta); t \in [0,T], \\ p^1(T) &= \nabla_m g(X(T), M(T)). \end{aligned} \tag{5.31}$$

Theorem (Sufficient zero-sum maximum principle)

Let $(\hat{\mu}, \hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ *with corresponding solutions* $\hat{X}$ *and* (p^0, q^0, r^0), (p^1, q^1, r^1) *of the forward and backward stochastic differential equations* $(5.1), (65) - (65)$, *respectively. Assume the following:*

- $$\mathbb{E}[\hat{H}(t)|\mathcal{G}_t^{(1)}] = ess \sup_{\mu \in \mathbb{M}_{\mathbb{G}}} \mathbb{E}[\check{H}(t)|\mathcal{G}_t^{(1)}],$$
- $$\mathbb{E}[\bar{H}(t)|\mathcal{G}_t^{(2)}] = ess \sup_{u \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[\check{H}(t)|\mathcal{G}_t^{(2)}],$$

$\mathbb{P}$- *a.s and for all* $t \in [0, T]$, *and that assumptions (a)-(d) hold.*

Then $(\hat{\mu}, \hat{u})$ *is a saddle point for* $J(\mu, u)$.

This result will be applied in the next section.

Theorem (Necessary zero-sum maximum principle)

Let $(\hat{\mu}, \hat{u}) \in \mathbb{M}_{\mathbb{G}} \times \mathcal{A}_{\mathbb{G}}$ *with corresponding solutions* $\hat{X}$, (p_i^0, q_i^0, r_i^0) *and* (p_i^1, q_i^1, r_i^1) *of the forward and the backward stochastic differential equations* (5.1) *and* (65) $-$ (65), *respectively, with corresponding derivative process* $\hat{Z}$ *given by* (5.23) *. Then we have equivalence between*

$$\frac{d}{d\lambda} J(\mu + \lambda\eta, u)|_{\lambda=0} = \frac{d}{ds} J(\mu, u + s\pi)|_{s=0} = 0,$$

and

$$\mathbb{E}[\frac{\partial H}{\partial \mu}(t)|\mathcal{G}_t^{(1)}] = \mathbb{E}[\frac{\partial H}{\partial u}(t)|\mathcal{G}_t^{(2)}] = 0.$$

Optimal consumption of a mean-field cash flow under uncertainty

Consider a net cash flow $X^{\mu,\rho} = X$ modelled by

$$\begin{cases} dX(t) = [\mu(t)(V) - \rho(t)]\, X(t)dt + \sigma(t)\, X(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,\zeta)\, X(t \\ X(0) = x > 0, \end{cases}$$

where $\rho(t) \geq 0$ is our *relative consumption rate* at time t, assumed to be a càdlàg, $\mathcal{G}_t^{(2)}$-adapted process.

Here V is a given Borel subset of $\mathbb{R}$. The value of $\mu(t)$ on V models the relative growth rate of the cash flow. The relative consumption rate $\rho(t)$ is our control process. We assume that $\int_0^T \rho(t)dt < \infty$ a.s. This implies that $X(t) > 0$ for all t, a.s. However, the measure-valued process $\mu(t)$ represents a kind of scenario uncertainty, and we want to maximise the total expected utility of the relative consumption rate ρ in the worst possible scenario μ. We penalize $\mu(\cdot)$ for being far away from the law process $\mathcal{L}(X(\cdot))$, in the sense that we introduce a quadratic cost rate $[(\mu(t) - M(t))(V)]^2$ in the performance functional.

Hence we consider the zero-sum game

$$\sup_{\rho} \inf_{\mu} \mathbb{E}[\int_0^T \{\log(\rho(t)X(t)) + [(\mu(t) - M(t))(V)]^2\} dt + \theta \log(X(T))],$$

where $\theta = \theta(\omega) > 0$ is a given bounded $\mathcal{F}_T$-measurable random variable, expressing the importance of the terminal value $X(T)$.
Here we have chosen a logarithmic utility because it is a central choice, and in many cases, as here, this leads to a nice explicit solution of the corresponding control problem.
The Hamiltonian for this zero-sum game takes the form

$$H(t) = \log(\rho x) + (\mu(V) - m(V))^2 + p^0[\mu(V)x - \rho x] + q^0\sigma(t)x \\ + \int_{\mathbb{R}_0} r^0(\zeta)\gamma(t,\zeta)x\nu(d\zeta) + \langle p^1, \beta(m)\rangle,$$

and the adjoint processes
$(p^0, q^0, r^0) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_{\nu}$,$(p^1, q^1, r^1) \in \mathcal{S}^2_{\mathbb{K}} \times \mathbb{L}^2_{\mathbb{K}} \times \mathbb{L}^2_{\nu,\mathbb{K}}$ are given by the BSDEs

-
$$\begin{cases} dp^0(t) & = & -[\frac{1}{X(t)} + p^0(t)[\mu(t)(V) - \rho(t)] + q^0(t)\sigma(t) \\ & + & \int_{\mathbb{R}_0} r^0(t,\zeta)\gamma(t,\zeta)\nu(d\zeta)]dt \\ & & +q^0(t)dB(t) + \int_{\mathbb{R}_0} r^0(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T], \\ p^0(T) & = & \frac{\theta}{X(T)}, \end{cases}$$

-
$$\begin{cases} dp^1(t) & = \{(p^1)^*(t) - 2[\hat{\mu}(t)(V) - \hat{M}(t)(V)]\chi_V(\cdot) \\ & + < p^1(t), \beta(\cdot) >\}dt \\ & +q^1(t)dB(t) \\ & + \int_{\mathbb{R}_0} r^1(t,\zeta)\tilde{N}(dt,d\zeta); \quad t \in [0,T], \\ p^1(T) & = 0, \end{cases}$$

where $\chi_V(\cdot)$ is the evaluation operator, evaluating a given measure at V, i.e. $\langle \chi_V, \lambda \rangle = \lambda(V)$ for all $\lambda \in \mathcal{M}_0$.

The first order condition for the optimal consumption rate $\hat{\rho}$ is

$$\mathbb{E}[\frac{1}{\hat{\rho}(t)} - \hat{p}^0(t)\hat{X}(t)|\mathcal{G}_t^{(2)}] = 0.$$

Since $\hat{\rho}(t)$ is $\mathcal{G}_t^{(2)}$-adapted, we have

$$\hat{\rho}(t) = \frac{1}{\mathbb{E}[\hat{p}^0(t)\hat{X}(t)|\mathcal{G}_t^{(2)}]}.$$

Now we use the minimum condition with respect to μ at $\mu = \hat{\mu}$ and get

$$\mathbb{E}[2[\hat{\mu}(t)(V) - \hat{M}(t)(V)]\lambda(V) + \hat{p}^0(t)\hat{X}(t)\lambda(V)|\mathcal{G}_t^{(1)}] = 0, \text{ for all } \lambda \in \mathcal{M}_0.$$

Using that $\hat{\mu}(t)$ is $\mathcal{G}_t^{(1)}$-adapted, we obtain

$$\hat{\mu}(t)(V) = \mathbb{E}[\hat{M}(t)(V) - \tfrac{1}{2}\hat{p}^0(t)\hat{X}(t)|\mathcal{G}_t^{(1)}].$$

It remains to find $\hat{p}^0(t)\hat{X}(t)$: We have by applying the Itô formula to $P(t) := \hat{p}^0(t)\hat{X}(t)$:

$$
\begin{aligned}
dP(t) &= \hat{p}^0(t)d\hat{X}(t) + \hat{X}(t)d\hat{p}^0(t) + d[\hat{p}^0, \hat{X}]_t \\
&= \hat{p}^0(t)([(\hat{\mu}(t)(V) - \rho(t))\,\hat{X}(t)]dt + \hat{\sigma}\,(t)\,\hat{X}(t)dB(t) \\
&+ \int_{\mathbb{R}_0} \hat{\gamma}\,(t,\zeta)\,\hat{X}(t)\tilde{N}(dt,d\zeta)) \\
&+ \hat{X}(t)[-\tfrac{1}{\hat{X}(t)} - \hat{p}^0(t)[\hat{\mu}(t)(V) - \rho(t)] - \hat{q}^{(0)}(t)\sigma(t) \\
&- \int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\nu(d\zeta)]dt \\
&+ \hat{q}^0(t)\hat{X}(t)dB(t) + \int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{X}(t)\tilde{N}(dt,d\zeta) + \hat{q}^0(t)\hat{\sigma}\,(t)\,\hat{X}(t)dt
\end{aligned}
$$

(6.1)

$$
+ \int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\hat{X}(t)N(dt,d\zeta).
$$

By definition
(6.2)

$$\begin{aligned}\int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\hat{X}(t)\tilde{N}(dt,d\zeta) &= \int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\hat{X}(t)N(dt,d\zeta) \\ &\quad - \int_{\mathbb{R}_0} \hat{r}^0(t,\zeta)\hat{\gamma}(t,\zeta)\hat{X}(t)\nu(d\zeta)dt.\end{aligned}$$

Substituting (6.2) in (6.1) yields

$$\begin{aligned}dP(t) &= -dt + [P(t)\hat{\sigma}(t) + \hat{q}^0(t)\hat{X}(t)]dB(t) \\ &\quad + \int_{\mathbb{R}_0} [P(t)\hat{\gamma}(t,\zeta) + \hat{r}^0(t,\zeta)\hat{X}(t)(1+\hat{\gamma}(t,\zeta))]\tilde{N}(dt,d\zeta).\end{aligned}$$

Hence, if we put

$$\begin{aligned}P(t) &:= \hat{p}^0(t)\hat{X}(t), \\ Q(t) &:= P(t)\hat{\sigma}(t) + \hat{X}(t)\hat{q}^0(t), \\ R(t,\zeta) &:= P(t)\hat{\gamma}(t,\zeta) + \hat{r}^0(t,\zeta)\hat{X}(t)(1+\hat{\gamma}(t,\zeta)).\end{aligned}$$

with $(P, Q, R) \in \mathcal{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_\nu$ satisfies the BSDE

$$\begin{cases} dP(t) = -dt + Q(t)dB(t) + \int_{\mathbb{R}_0} R(t,\zeta)\tilde{N}(dt, d\zeta); & t \in [0, T], \\ P(T) = \theta. \end{cases}$$

Solving this BSDE as in (5.12), we find the closed formula for $P(t)$ as

$$\begin{aligned} P(t) &= \mathbb{E}[\theta + \int_t^T ds | \mathcal{F}_t] \\ &= \mathbb{E}\left[\theta | \mathcal{F}_t\right] + T - t. \end{aligned}$$

Hence we have proved the following:

Theorem

The optimal consumption rate $\hat{\rho}(t)$ and the optimal model uncertainty law $\hat{\mu}(t)$ are given respectively in feed-back form by

$$\begin{aligned}\hat{\rho}(t) &= \frac{1}{T-t+\mathbb{E}[\theta|\mathcal{G}_t^{(2)}]},\\ \hat{\mu}(t)(V) &= \hat{M}(t)(V)+T-t-\frac{1}{2}\mathbb{E}[\theta|\mathcal{G}_t^{(1)}].\end{aligned}$$

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