Optimal Asset Allocation with Stochastic Interest Rates in Regime-switching Models

Ruihua Liu

Department of Mathematics
University of Dayton, Ohio

Joint Work With Cheng Ye and Dan Ren
To appear in International Journal of Theoretical and Applied Finance (IJTAF)

IMA Workshop on Financial and Economic Applications
University of Minnesota, Minneapolis, June 11 - 15, 2018

Outline of Presentation

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Problem Formulation

- Regime-switching is modeled by a continuous-time Markov chain $\alpha(t) \in \mathcal{M} := \{1, \dots, m_0\}$ with $m_0 > 0$ fixed.
- The intensity matrix of $\alpha(t)$, $Q = (q_{ij})_{m_0 \times m_0}$ is given.
- The interest rate follows a regime-switching Vasicek model:

$$dr(t) = [a(\alpha(t)) - b(t)r(t)]dt + \sigma_r(\alpha(t))dW^b(t). \tag{1}$$

- The risky asset (stock) follows a regime-switching geometric Brownian motion (GBM) model:

$$dS(t) = S(t)\Big([r(t) + \lambda_s(\alpha(t))]dt + \sigma_s(\alpha(t))dW^s(t)\Big), \tag{2}$$

  where $\lambda_s(\alpha(t))$ is the risk premium of the stock.
- A savings account follows $dB(t) = B(t)r(t)dt,\ B(0) = 1$.
- $W^s(t)$ and $W^b(t)$ are standard Brownian motions. $dW^s(t)dW^b(t) = \rho dt$.
- We assume that $\alpha(t)$ is independent of $W^s(t)$ and $W^b(t)$.

A Stock Portfolio Problem

- Optimal allocation of wealth between $S(t)$ and $B(t)$.
- Let $\pi_s(t)$ be the percentage of wealth in the stock, then the percentage in the savings account is $1 - \pi_s(t)$.
- Let $X(t)$ denote the wealth at time t. Then

$$dX(t) = X(t)\pi_s(t)\lambda_s(\alpha(t))dt + X(t)r(t)dt + X(t)\pi_s(t)\sigma_s(\alpha(t))dW^s(t). \tag{3}$$

- Given $0 \le t < T$, $(x(t), r(t), \alpha(t)) = (x, r, i)$, an admissible control $u(\cdot) := \pi_s(\cdot)$, the objective is

$$J(t, x, r, i; u(\cdot)) = E^{txri}\left[U(X^u(T), \alpha(T))\right], \tag{4}$$

  where $U(x, i)$ is a regime-dependent utility function.
- Let $\mathcal{A}_{txri}$ be the collection of admissible controls w.r.t $(x(t), r(t), \alpha(t)) = (x, r, i)$. The value function is

$$V(t, x, r, i) = \sup_{u(\cdot) \in \mathcal{A}_{txri}} J(t, x, r, i; u(\cdot)). \tag{5}$$

A Stock and Bond Portfolio Problem

- Optimal allocation of wealth among the stock $S(t)$, the saving account $B(t)$, and a bond $P(t)$ given by :

$$dP(t) = P(t)\left([r(t) + \lambda_b(t, \alpha(t))]dt + \sigma_b(t, \alpha(t))dW^b(t)\right), \tag{6}$$

  where $\lambda_b(t, \alpha(t))$ is the risk premium of the bond price.
- Let $\pi_s(t)$ be the percentage of wealth in the stock, $\pi_b(t)$ the percentage in the bond, then the percentage in the savings account is $1 - \pi_s(t) - \pi_b(t)$.
- Two-dimensional control process $u(\cdot) = (\pi_s(\cdot), \pi_b(\cdot))^T$.
- The wealth $X(t)$ follows:

$$\begin{aligned} dX(t) =& X(t)[r(t) + \pi_s(t)\lambda_s(\alpha(t)) + \pi_b(t)\lambda_b(t, \alpha(t))]dt \\ & + X(t)[\pi_s(t)\sigma_s(\alpha(t))dW^s(t) + \pi_b(t)\sigma_b(t, \alpha(t))dW^b(t)]. \end{aligned} \tag{7}$$

- Consider the same objective (4) and value function (5).

Existing Result and Our Contribution

- Korn and Kraft (SICON, 2001) considered the optimal asset allocation problems with stochastic interest rate. However, regime-switching was not incorporated in their models.
- Due to the presence of the unbounded interest rate process, the wealth equations (3) and (7) do not satisfy the usual Lipschitz continuity conditions as assumed in the classical verification theorems of stochastic optimal control (e.g, Fleming and Soner, 2006, Springer).
- By exploring the special structure of the wealth equation with stochastic interest rate, Korn and Kraft modified the standard verification arguments from Fleming and Soner and provided a verification theorem for the optimal control problem under their consideration.
- In this work we study the same problems using regime-switching models. Our results extend Korn and Kraft to the more complicated regime-switching cases.

A Stock Portfolio Problem - Solution

The HJB is given by a system of m_0 PDEs:

$$V_t(t,x,r,i) + xrV_x(t,x,r,i) + [a(i) - b(t)r]V_r(t,x,r,i) + \frac{1}{2}\sigma_r^2(i)V_{rr}(t,x,r,i) +$$
$$\sup_{\pi_s \in \mathbb{R}} \left\{ x\pi_s\lambda_s(i)V_x(t,x,r,i) + \frac{1}{2}x^2\pi_s^2\sigma_s^2(i)V_{xx}(t,x,r,i) + \rho x\pi_s\sigma_s(i)\sigma_r(i)V_{xr}(t,x,r,i) \right\}$$
$$+ \sum_{j \neq i} q_{ij}\left[V(t,x,r,j) - V(t,x,r,i)\right] = 0, \ i = 1, \dots, m_0, \tag{8}$$

with the boundary condition:

$$V(T,x,r,i) = U(x,i), \ i = 1, \dots, m_0. \tag{9}$$

The maximizer of (8) is given by:

$$\pi_s{}^*(t,i) = -\frac{\lambda_s(i)V_x}{x\sigma_s^2(i)V_{xx}} - \frac{\rho\sigma_r(i)V_{xr}}{x\sigma_s(i)V_{xx}}. \tag{10}$$

A Stock Portfolio Problem - Solution

Consider the power utility $U(x, i) = \lambda(i)x^\gamma$. We have

$$V(t, x, r, i) = g(t, i)x^\gamma e^{\beta(t)r}, \; i = 1, \ldots, m_0, \tag{11}$$

where

$$\beta'(t) - b(t)\beta(t) + \gamma = 0, \; \beta(T) = 0, \tag{12}$$

and

$$g_t(t, i) + h(t, i)g(t, i) + \sum_{j \neq i} q_{ij}[g(t, j) - g(t, i)] = 0, \tag{13}$$

with $g(T, i) = \lambda(i)$ for $i = 1, \ldots, m_0$, where

$$h(t, i) = a(i)\beta(t) + \frac{1}{2}\sigma_r^2(i)\beta^2(t) + \frac{\gamma}{2(1-\gamma)}\left[\frac{\lambda_s(i)}{\sigma_s(i)} + \rho\sigma_r(i)\beta(t)\right]^2. \tag{14}$$

A Stock Portfolio Problem - Solution

- Using (11), the maximizer (10) becomes

$$\pi_s^*(t,i) = \frac{\lambda_s(i)}{(1-\gamma)\sigma_s^2(i)} + \frac{\rho\sigma_r(i)}{(1-\gamma)\sigma_s(i)}\beta(t). \qquad (15)$$

- The verification arguments show that $\pi_s^*(t) := \pi_s^*(t,\alpha(t))$ given in (15) is an optimal control of the considered optimization problem.

A Stock and Bond Portfolio Problem - Solution

The HJB:

$$
\begin{aligned}
&V_t(t,x,r,i) + xrV_x(t,x,r,i) + [a(i) - b(t)r]V_r(t,x,r,i) + \frac{1}{2}\sigma_r^2(i)V_{rr}(t,x,r,i) \\
&+ \sup_{(\pi_s,\pi_b)} \Big\{ \frac{1}{2}x^2\sigma_s^2(i)V_{xx}(t,x,r,i)\pi_s^2 + x[\lambda_s(i)V_x(t,x,r,i) + \rho\sigma_s(i)\sigma_r(i)V_{xr}(t,x,r,i)]\pi_s \\
&+ \frac{1}{2}x^2\sigma_b^2(t,i)V_{xx}(t,x,r,i)\pi_b^2 + x[\lambda_b(t,i)V_x(t,x,r,i) + \sigma_b(t,i)\sigma_r(i)V_{xr}(t,x,r,i)]\pi_b \\
&+ x^2\rho\sigma_s(i)\sigma_b(t,i)V_{xx}(t,x,r,i)\pi_s\pi_b \Big\} + \sum_{j\neq i} q_{ij}[V(t,x,r,j) - V(t,x,r,i)] = 0,
\end{aligned} \tag{16}
$$

for $i = 1, \ldots, m_0$. The maximizer of (16) is given by:

$$
\begin{aligned}
\pi_s^*(t,i) &= -\left[\frac{\lambda_s(i)}{(1-\rho^2)\sigma_s^2(i)} - \frac{\rho\lambda_b(t,i)}{(1-\rho^2)\sigma_b(t,i)\sigma_s(i)}\right]\frac{V_x}{xV_{xx}}, \\
\pi_b^*(t,i) &= -\frac{\sigma_r(i)}{\sigma_b(t,i)}\frac{V_{xr}}{xV_{xx}} - \left[\frac{\lambda_b(t,i)}{(1-\rho^2)\sigma_b^2(t,i)} - \frac{\rho\lambda_s(i)}{(1-\rho^2)\sigma_s(i)\sigma_b(t,i)}\right]\frac{V_x}{xV_{xx}},
\end{aligned} \tag{17}
$$

A Stock and Bond Portfolio Problem - Solution

For the power utility $U(x, i) = \lambda(i)x^{\gamma}$,

$$V(t, x, r, i) = g(t, i)x^{\gamma}e^{\beta(t)r}, \ i = 1, \ldots, m_0, \tag{18}$$

where

$$g_t(t, i) + h(t, i)g(t, i) + \sum_{j \neq i} q_{ij}[g(t, j) - g(t, i)] = 0, \ i = 1, \ldots, m_0, \tag{19}$$

where

$$h(t, i) = a(i)\beta(t) + \frac{1}{2}\sigma_r^2(i)\beta^2(t) + \frac{\gamma}{2(1-\gamma)}\left[\left(\frac{\lambda_b(t, i)}{\sigma_b(t, i)} + \sigma_r(i)\beta(t)\right)^2 + \frac{1}{1-\rho^2}\left(\frac{\lambda_s(i)}{\sigma_s(i)} - \rho\frac{\lambda_b(t, i)}{\sigma_b(t, i)}\right)^2\right]. \tag{20}$$

A Stock and Bond Portfolio Problem - Solution

- The maximizer (17) is:

$$\pi_s^*(t,i) = \frac{1}{(1-\rho^2)(1-\gamma)\sigma_s(i)}\left(\frac{\lambda_s(i)}{\sigma_s(i)} - \rho\frac{\lambda_b(t,i)}{\sigma_b(t,i)}\right),$$
$$\pi_b^*(t,i) = \frac{1}{(1-\rho^2)(1-\gamma)\sigma_b(t,i)}\left(\frac{\lambda_b(t,i)}{\sigma_b(t,i)} - \rho\frac{\lambda_s(i)}{\sigma_s(i)} + (1-\rho^2)\sigma_r(i)\beta(t)\right). \tag{21}$$

- The verification arguments show that $(\pi_s^*(t), \pi_b^*(t)) := (\pi_s^*(t,\alpha(t)), \pi_b^*(t,\alpha(t)))$ as given in (21) is an optimal control for the stock and bond portfolio problem.

Theoretical Results

- A class of stochastic optimal control problems with Markovian regime-switching is formulated.
- A verification theorem is presented.
- Our results extend Korn and Kraft to the regime-switching models.
- The theory is applied to verify the optimality of the two portfolio problems considered in this work.

A Control Problem with Regime-Switching

- We consider a controlled process $\{Y(t) = (Y_1(t), \ldots, Y_n(t))^T \in \mathbb{R}^n, \ t \geq 0\}$ that depends on another process $\{z(t) \in \mathbb{R}, \ t \geq 0\}$ which is uncontrollable.
- The control process is $\{u(t) = (u_1(t), \ldots, u_d(t))^T \in U \subset \mathbb{R}^d, \ t \geq 0\}$, where U is a closed subset of $\mathbb{R}^d$.
- The dynamic of $(Y(t), z(t))$ is given by:

$$dY(t) = \mu(t, Y(t), z(t), u(t), \alpha(t))dt + \sigma(t, Y(t), z(t), u(t), \alpha(t))dW(t), \quad (22)$$

$$dz(t) = \mu_z(t, z(t), \alpha(t))dt + \sigma_z(t, z(t), \alpha(t))dW^z(t), \quad (23)$$

  where $W(t) = (W_1(t), \ldots, W_m(t))^T \in \mathbb{R}^m$ is an m-dimensional BM, $W^z(t) \in \mathbb{R}$ is an one-dimensional BM, and for each $j \in \{1, \ldots, m\}$, $dW_j(t)dW^z(t) = \rho_j dt$.
- We assume that the Markov chain $\alpha(t)$ is independent of the Brownian motions $W(t)$ and $W^z(t)$.

A Control Problem with Regime-Switching

- Suppose that the SDE (23) admits a unique strong solution $z(t)$ that satisfies the condition:

$$E\left[\sup_{0\leq t\leq T}|z(t)|^k\right] < \infty \text{ for all } k \in \mathbb{N}, \tag{24}$$

  where $T > 0$ is the fixed time-horizon for the optimal control problem.

- In addition, we assume that σ_z satisfies the condition:

$$E\left(\int_0^T \sigma_z^2(t, z(t), \alpha(t))dt\right) < \infty. \tag{25}$$

- A special case. μ_z and σ_z in (23) satisfy the Lipschitz continuous and linear growth conditions.

A Control Problem with Regime-Switching

The coefficients μ and σ in (22) take the form:

$$\begin{aligned}\mu(t, Y(t), z(t), u(t), \alpha(t)) &= Y(t)[A_1^T(t, z(t), \alpha(t))u(t) + A_2(t, z(t), \alpha(t))],\\ \sigma(t, Y(t), z(t), u(t), \alpha(t)) &= Y(t)[B_1(t, z(t), \alpha(t))u(t) + B_2(t, z(t), \alpha(t))]^T,\end{aligned} \tag{26}$$

where

$$\begin{aligned}&A_1(t, z(t), \alpha(t)) = \left(A_1^{(1)}(t, z(t), \alpha(t)), \ldots, A_1^{(d)}(t, z(t), \alpha(t))\right)^T \in \mathbb{R}^d,\\ &A_2(t, z(t), \alpha(t)) \in \mathbb{R},\\ &B_1(t, z(t), \alpha(t)) = \left(B_1^{(i,j)}(t, z(t), \alpha(t))\right)_{m\times d} \in \mathbb{R}^{m\times d},\\ &B_2(t, z(t), \alpha(t)) = \left(B_2^{(1)}(t, z(t), \alpha(t)), \ldots, B_2^{(m)}(t, z(t), \alpha(t))\right)^T \in \mathbb{R}^m,\end{aligned}$$

and $A_1^{(j)},\ B_1^{(i,j)},\ B_2^{(i)}$ $(i = 1, \ldots, m,\ j = 1, \ldots, d)$, and A_2 are progressively measurable processes satisfying the following integrability conditions:

A Control Problem with Regime-Switching

$$
\begin{aligned}
&\int_0^T |A_2(t, z(t), \alpha(t))| dt < \infty \quad a.s., \\
&\int_0^T \Big[\sum_{j=1}^d (A_1^{(j)}(t, z(t), \alpha(t)))^2 + \sum_{i=1}^m (B_2^{(i)}(t, z(t), \alpha(t)))^2 \Big] dt < \infty \quad a.s., \\
&\int_0^T \sum_{i=1}^m \sum_{j=1}^d (B_1^{(i,j)}(t, z(t), \alpha(t)))^4 dt < \infty \quad a.s..
\end{aligned}
\tag{27}
$$

- Korn and Kraft used "linear controlled SDE" for such systems (without regime-switching).
- We may call $Y(t)$ defined by the SDE (22) with coefficients (26) a linear controlled regime-switching diffusion process.
- For each control process $u(\cdot)$ satisfying (28) given below, the SDE (22) with (26) admits a Lebesgue$\otimes P$ unique solution.

Admissible Control

A process $u(\cdot) = \{u(t), 0 \leq t_0 \leq t \leq T\}$ is admissible w.r.t. the initial data $Y(t_0) = y_0$, $z(t_0) = z_0$, and $\alpha(t_0) = i$ if $u(\cdot)$ is progressively measurable and satisfies the following conditions:

1. $$E^{t_0 y_0 z_0 i}\left[\int_{t_0}^{T} |u(t)|^k \, dt\right] < \infty \text{ for all } k \in \mathbb{N}; \qquad (28)$$
2. The corresponding state process $Y^u(\cdot)$ satisfies

$$E^{t_0 y_0 z_0 i}\left[\sup_{t_0 \leq t \leq T} |Y^u(t)|^k\right] < \infty \text{ for all } k \in \mathbb{N}. \qquad (29)$$

Let $\mathcal{A}_{t_0 y_0 z_0 i}$ denote the collection of admissible controls w.r.t. the initial data $Y(t_0) = y_0$, $z(t_0) = z_0$, and $\alpha(t_0) = i$.

A Control Problem with Regime-Switching

- Given an open set $O \subset \mathbb{R}^{n+1}$. Let $Q = [t_0, T) \times O$. For $(t, y, z, i) \in Q \times \mathcal{M}$, let

$$\tau = \inf\{s \geq t : (s, Y(s), z(s)) \notin Q\} \tag{30}$$

  be the first exit time of $(s, Y(s), z(s))$ from Q

- Consider functions $f : \overline{Q} \times U \times \mathcal{M} \to \mathbb{R}$ and $g : \overline{Q} \times \mathcal{M} \to \mathbb{R}$. Assume that f and g are continuous functions for each $i \in \mathcal{M}$ and satisfy the following polynomial growth conditions:

$$\begin{aligned} &|f(t, y, z, u, i)| \leq C[1 + |y|^k + |z|^k + |u|^k], \ (t, y, z, u) \in \overline{Q} \times U, \\ &|g(t, y, z, i)| \leq C[1 + |y|^k + |z|^k], \ (t, y, z) \in \overline{Q}, \end{aligned} \tag{31}$$

  for some constant $C > 0$ and some integer $k \in \mathbb{N}$.

A Control Problem with Regime-Switching

- Given $(t, y, z, i) \in Q \times \mathcal{M}$, $u(\cdot) \in \mathcal{A}_{tyzi}$.
- Define the objective functional by

$$J(t, y, z, i; u(\cdot)) = E^{tyzi}\left[\int_t^{\tau} f(s, Y(s), z(s), u(s), \alpha(s))\, ds + g(\tau, Y(\tau), z(\tau), \alpha(\tau))\right], \tag{32}$$

where τ is defined by (30).

- The value function is defined by

$$V(t, y, z, i) = \sup_{u(\cdot) \in \mathcal{A}_{tyzi}} J(t, y, z, i; u(\cdot)). \tag{33}$$

- In addition, the boundary condition for V is :

$$V(t, y, z, i) = g(t, y, z, i) \text{ for } (t, z, y) \in \partial^* Q \text{ and } i \in \mathcal{M}, \tag{34}$$

where

$$\partial^* Q = ([t_0, T] \times \partial O) \cup (\{T\} \times O). \tag{35}$$

Hamilton-Jacobi-Bellman (HJB) equation

- Given $v \in C^{1,2}(Q \times \mathcal{M})$ and $u \in U$, define the operator $\mathcal{L}^u$ by

$$\begin{aligned}\mathcal{L}^u v(t,y,z,i) &= v_t(t,y,z,i) + \sum_{j=1}^{n} \mu_j(t,y,z,u,i) v_{y_j}(t,y,z,i) + \mu_z(t,z,i) v_z(t,y,z,i) \\ &+ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} (\sigma\sigma^T)_{jk}(t,y,z,u,i) v_{y_j y_k}(t,y,z,i) + \frac{1}{2}\sigma_z^2(t,z,i) v_{zz}(t,y,z,i) \\ &+ \sum_{j=1}^{n}\left(\sum_{k=1}^{m} \rho_k \sigma_{jk}(t,y,z,u,i)\right) v_{y_j z}(t,y,z,i) + \sum_{j\neq i} q_{ij}[v(t,y,z,j) - v(t,y,z,i)].\end{aligned} \tag{36}$$

- The HJB equation is a system of m_0 coupled PDEs:

$$\sup_{u\in U}\left\{\mathcal{L}^u v(t,y,z,i) + f(t,y,z,u,i)\right\} = 0, \quad (t,y,z,i) \in Q \times \mathcal{M} \tag{37}$$

with the boundary condition

$$v(t,y,z,i) = g(t,y,z,i), \quad \text{for } (t,y,z) \in \partial^* Q \text{ and } i \in \mathcal{M}. \tag{38}$$

Verification Theorem

Under the assumption (24), (25), (27) and (31), let $v \in C^{1,2}(Q \times \mathcal{M}) \bigcap C(\overline{Q} \times \mathcal{M})$ be a solution of the HJB equation (37) with the boundary condition (38). In addition, assume that for all $(t, y, z, i) \in Q \times \mathcal{M}$ and all admissible controls $u(\cdot) \in \mathcal{A}_{tyzi}$,

$$E^{tyzi}\left[\sup_{s\in[t,T]} |v(s, Y(s), z(s), \alpha(s))|\right] < \infty. \qquad (39)$$

Then we have the following results:

(a) $v(t, y, z, i) \geq J(t, y, z, i; u(\cdot))$ for any initial data $(t, y, z, i) \in Q \times \mathcal{M}$ and any admissible control $u(\cdot) \in \mathcal{A}_{tyzi}$.

(b) For $(t, y, z, i) \in Q \times \mathcal{M}$, if there exists an admissible control $u^*(\cdot) \in A_{tyzi}$ such that

Verification Theorem — Cont.

$$
\begin{aligned}
u^*(s) \in \operatorname*{argmax}_{u \in U} \Bigg[& \sum_{j=1}^{n} \mu_j(s, Y^*(s), z(s), u, \alpha(s)) v_{y_j}(s, Y^*(s), z(s), \alpha(s)) \\
& + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (\sigma\sigma^T)_{jk}(s, Y^*(s), z(s), u, \alpha(s)) v_{y_j y_k}(s, Y^*(s), z(s), \alpha(s)) \\
& \sum_{j=1}^{n} \left(\sum_{k=1}^{m} \rho_k \sigma_{jk}(s, Y^*(s), z(s), u, \alpha(s)) \right) v_{y_j z}(s, Y^*(s), z(s), \alpha(s)) \\
& + f(s, Y^*(s), z(s), u, \alpha(s)) \Bigg]
\end{aligned}
\tag{40}
$$

for Lebesgue $\bigotimes P$ almost all $(s, \omega) \in [t, \tau^*(\omega)] \times \Omega$, then $v(t, y, z, i) = V(t, y, z, i) = J(t, y, z, i; u^*(\cdot))$. Here $Y^*(s)$ is the unique solution of the SDE (22) when the control $u^*(\cdot)$ is being used, with $Y^*(t) = y$, $\alpha(t) = i$, $z(s)$ is the unique solution of (23) with $z(t) = z$, $\alpha(t) = i$, and τ^* is the first exit time of $(s, Y^*(s), z(s))$ form Q as defined in (30).

Numerical Results

- We consider a market with two regimes ($m_0 = 2$).
- The generator of $\alpha(\cdot)$ is given by

$$Q = \begin{pmatrix} -q_{12} & q_{12} \\ q_{21} & -q_{21} \end{pmatrix},$$

  where q_{12} is the switching rate from regime 1 to regime 2 and q_{21} is the switching rate from regime 2 to regime 1.
- We set $q_{12} = 3$ and $q_{21} = 4$. That implies, on average the market switches three times per year from regime 1 to regime 2 and four times from regime 2 to regime 1.
- Moreover, the stationary distribution of $\alpha(\cdot)$ is $p = (\frac{4}{7}, \frac{3}{7})$.

Numerical Results

- The model parameters used in the numerical study are:
- For the stock price model (2), $\lambda_s(1) = 0.04$, $\lambda_s(2) = 0.07$, $\sigma_s(1) = 0.3$, $\sigma_s(2) = 0.5$. Note that $0 < \lambda_s(1) < \lambda_s(2)$, $0 < \sigma_s(1) < \sigma_s(2)$, and $\frac{\lambda_s(2)}{\sigma_s^2(2)} < \frac{\lambda_s(1)}{\sigma_s^2(1)}$. So we may consider regime 1 as a bull market and regime 2 a bear market.
- For the interest rate model (1), $a(1) = 0.16$, $a(2) = 0.08$, $b = 2$, $\sigma_r(1) = 0.03$, $\sigma_r(2) = 0.05$.
- For the bond price model (6), $\lambda_b(t, \alpha(t)) = \lambda_b(\alpha(t))$ where $\lambda_b(1) = 0.006$, $\lambda_b(2) = 0.015$, $\sigma_b(t, \alpha(t)) = \sigma_b(\alpha(t))$ where $\sigma_b(1) = 0.1$, $\sigma_b(2) = 0.15$.
- The utility functions for the two regimes are $U(x, 1) = 6x^{0.5}$ and $U(x, 2) = 2x^{0.5}$. The correlation coefficient between the stock and the bond is $\rho = 0.3$ and the investment horizon is $T = 1$(year).

Numerical Results

- For comparison, we consider the averaged problems for which the parameters are replaced by their probabilistic averages over all regimes.
- Let $p = (p_1, \dots p_{m_0})$ denote the stationary distribution of $\alpha(\cdot)$ which is specified by the unique solution of the equation $pQ = 0$, $\sum_{i=1}^{m_0} p_i = 1$, $p_i > 0$ for $i = 1, \dots m_0$.
- Let $\bar{a} = \sum_{i=1}^{m_0} p_i a(i)$, $\bar{\sigma}_r = \sqrt{\sum_{i=1}^{m_0} p_i \sigma_r^2(i)}$, $\bar{\lambda}_s = \sum_{i=1}^{m_0} p_i \lambda_s(i)$, $\bar{\sigma}_s = \sqrt{\sum_{i=1}^{m_0} p_i \sigma_s^2(i)}$, $\bar{\lambda}_b = \sum_{i=1}^{m_0} p_i \lambda_b(i)$, and $\bar{\sigma}_b = \sqrt{\sum_{i=1}^{m_0} p_i \sigma_b^2(i)}$.
- Replacing $a(\alpha(t))$, $\sigma_r(\alpha(t))$, $\lambda_s(\alpha(t))$, $\sigma_s(\alpha(t))$, $\lambda_b(t, \alpha(t))$, $\sigma_b(t, \alpha(t))$ with $\bar{a}$, $\bar{\sigma}_r$, $\bar{\lambda}_s$, $\bar{\sigma}_s$, $\bar{\lambda}_b$, $\bar{\sigma}_b$, respectively in (1), (2), and (6), and using an averaged utility function $\bar{U}(x) = \frac{\bar{\lambda}}{\gamma} x^\gamma$ where $\bar{\lambda} = \sum_{i=1}^{m_0} p_i \lambda(i)$, then the optimization problems reduce to the single-regime problems studied in Korn and Kraft (SICON, 2001).

Numerical Results

- Solutions of the two averaged problems.
- A stock portfolio problem. The optimal percentage invested in stock, denoted by $\bar{\pi}_s^*$, is given by

$$\bar{\pi}_s^*(t) = \frac{\bar{\lambda}_s}{(1-\gamma)\bar{\sigma}_s^2} + \frac{\rho\bar{\sigma}_r}{(1-\gamma)\bar{\sigma}_s}\beta(t), \tag{41}$$

and the value function, denoted by $\bar{V}$, is given by $\bar{V}(t,x,r) = \bar{g}(t)x^\gamma e^{\beta(t)r}$, where $\beta(t)$ is given by (12), and $\bar{g}(t)$ is given by

$$\bar{g}(t) = \frac{1}{\gamma}\bar{\delta}e^{\int_t^T \bar{h}(s)ds}, \tag{42}$$

where

$$\bar{h}(t) = \bar{a}\beta(t) + \frac{1}{2}\bar{\sigma}_r^2\beta^2(t) + \frac{\gamma}{2(1-\gamma)}\left[\frac{\bar{\lambda}_s}{\bar{\sigma}_s} + \rho\bar{\sigma}_r\beta(t)\right]^2. \tag{43}$$

Numerical Results

- A stock and bond portfolio problem. The optimal control $(\bar{\pi}_s^*, \bar{\pi}_b^*)$ is given by

$$\bar{\pi}_s^* = \frac{1}{(1-\rho^2)(1-\gamma)\bar{\sigma}_s} \left(\frac{\bar{\lambda}_s}{\bar{\sigma}_s} - \rho \frac{\bar{\lambda}_b}{\bar{\sigma}_b} \right), \tag{44}$$

$$\bar{\pi}_b^*(t) = \frac{1}{(1-\rho^2)(1-\gamma)\bar{\sigma}_b} \left(\frac{\bar{\lambda}_b}{\bar{\sigma}_b} - \rho \frac{\bar{\lambda}_s}{\bar{\sigma}_s} + (1-\rho^2)\bar{\sigma}_r \beta(t) \right). \tag{45}$$

- The value function is $\bar{V}(t,x,r) = \bar{g}(t) x^\gamma e^{\beta(t) r}$, where $\bar{g}(t)$ is given in (42) with a new $\bar{h}(t)$ defined by

$$\begin{aligned} \bar{h}(t) &= \bar{a}\beta(t) + \frac{1}{2}\bar{\sigma}_r^2 \beta^2(t) \\ &\quad + \frac{\gamma}{2(1-\gamma)} \left[\left(\frac{\bar{\lambda}_b}{\bar{\sigma}_b} + \bar{\sigma}_r \beta(t) \right)^2 + \frac{1}{1-\rho^2} \left(\frac{\bar{\lambda}_s}{\bar{\sigma}_s} - \rho \frac{\bar{\lambda}_b}{\bar{\sigma}_b} \right)^2 \right]. \end{aligned} \tag{46}$$

Numerical Results

- Numerical results for the stock portfolio problem.
- Fig. 1 shows the optimal stock percentage $\pi_s^*(t, 1)$ and $\pi_s^*(t, 2)$ for the two regimes, together with the stock percentage $\bar{\pi}_s^*(t)$ for the averaged problem.
- Fig. 2 (upper panel) displays the surfaces of the value functions $V(t, x, r, 1)$, $V(t, x, r, 2)$, and the averaged value function $\bar{V}(t, x, r)$ at a fixed initial wealth $x = 1$. Fig. 3 (upper panel) displays the three value functions $V(t, x, r, 1)$, $V(t, x, r, 2)$, and $\bar{V}(t, x, r)$ at a fixed initial interest rate $r = 0.05$. We can clearly see that the value functions are different in different market regimes.

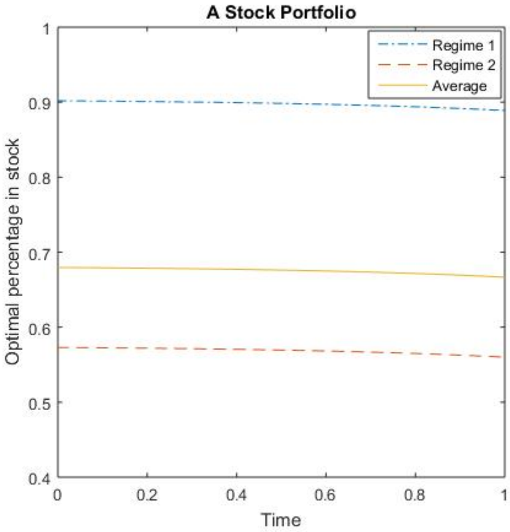

Figure 1: Optimal percentages of wealth in stock (two regimes).

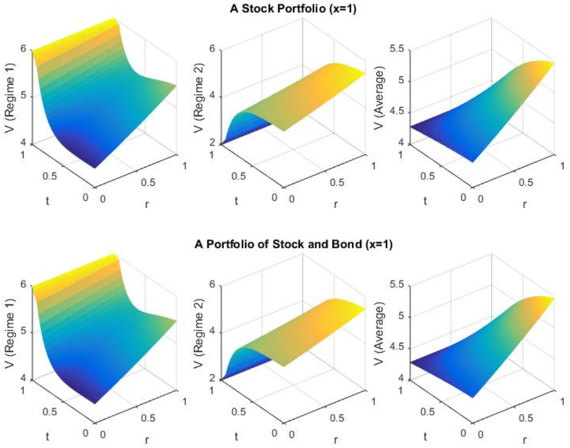

Figure 2: Value functions for fixed wealth x (two regimes).

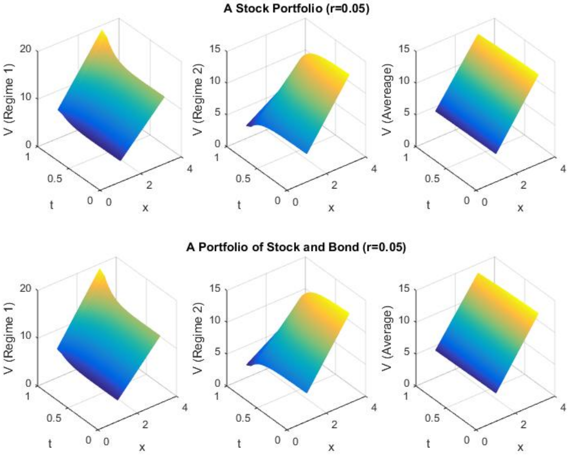

Figure 3: Value functions for fixed interest rate r (two regimes).

Numerical Results

- Numerical results for the stock and bond portfolio problem.
- Fig. 4 shows the optimal control for the mixed stock and bond problem. The stock percentages for regime 1, regime 2 and the averaged problem are displayed in the left panel of Fig. 4, while the optimal percentages in bond are shown in the right panel of Fig. 4.
- Similarly, we plot in Fig. 2 (lower panel) the value functions $V(t,x,r,1)$, $V(t,x,r,2)$, and the averaged value function $\bar{V}(t,x,r)$ at a fixed initial wealth $x = 1$, and in Fig. 3 (lower panel) the three value functions at a fixed initial interest rate $r = 0.05$.

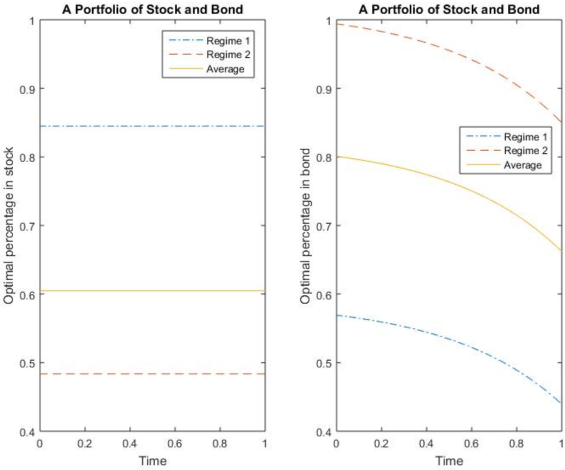

Figure 4: Optimal percentages of wealth in stock and in bond (two regimes).

On-going Work

- Consider other interest rate models, e.g., Using a regime-switching CIR model for the stochastic interest rate:

$$dr(t) = [a(\alpha(t)) - b(t)r(t)]dt + \sigma_r(\alpha(t))\sqrt{r(t)}dW^b(t). \quad (47)$$

- We have $V(t,x,r,i) = x^\gamma f(t,r,i)$, where

$$\begin{aligned} f(t,r,i)f_t(t,r,i) &+ A(t,r,i)f^2(t,r,i) + B(t,r,i)f(t,r,i)f_r(t,r,i) \\ &+ C(t,r,i)f(t,r,i)f_{rr}(t,r,i) + D(t,r,i)f_r^2(t,r,i) \\ &+ f(t,r,i)\sum_{j\neq i} q_{ij}[f(t,r,j) - f(t,r,i)] = 0, \\ f(T,r,i) &= \lambda(i), \quad i = 1,\ldots,m_0, \end{aligned} \quad (48)$$

  where A, B, C and D are deterministic functions.
- Study the the PDE system (48).

Thank You!