Some Portfolio Optimization Problems for Models with Delays

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Financial and Economic Applications,
IMA, Minneapolis, MN
June 11-15, 2018

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1 Motivation and Background

- Is the stock market really efficient?
- Three forms of market efficiency hypothesis (EMH)
  - Weak-Form EMH: The market is efficient and reflecting all market information. Under this rule, trading rules based on technical analysis are not valid. In addition, daily returns should be independent and we can use a Markov process to model stock prices.
  - Semi-Strong EMH: The market is efficient, reflecting all publicly available information. An investor cannot benefit over and above the market by trading on new information.
  - Strong-form EMH implies that the market is efficient: it reflects all information both public and private information.

- Reflexivity theory by George Soros
- In reality, a stock with strong performance usually attract more buyers, and the stronger demand will push the price even higher.
- Moving average, exponential moving average.
- History (memory, delay) does matter!
- In this presentation, we will go over some portfolio optimization problems of Merton's type for systems with delays.

2 Portfolio Optimization for Systems with Delays

2.1 Finite Time Horizon with Bounded Memory

- Investors portfolio consists of a risky asset and a riskless asset. The riskless asset earns the investor a fixed interest rate $r > 0$.
- Let $K(t)$ be the amount invested in the risky asset. $L(t)$ is the amount invested on the riskless asset. Net wealth: $X(t) = K(t) + L(t)$.
- Assume that the performance of the risky asset depends on the following delay variables $Y(t)$ and $Z(t)$:

$$Y(t) = \int_{-h}^{0} e^{\lambda\theta} X(t+\theta) d\theta, \quad Z(t) = X(t-h). \tag{2.1}$$

- $K(t)$ and $L(t)$ follow the stochastic differential equations:

$$\begin{aligned} dK(t) &= [\mu_1 K(t) + \mu_2 Y(t) + \mu_3 Z(t) + I(t)]\, dt + \sigma K(t) dB(t), \\ dL(t) &= [rL(t) - C(t) - I(t)] dt, \end{aligned}$$

- Choose $c(t) = \frac{C(t)}{X(t)}$ and $k(t) = \frac{K(t)}{X(t)}$ as controls. Note that $L(t) = X(t)(1 - k(t))$. We obtain the equation for wealth process:

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t) + \mu_3 Z(t)]dt \\ &\quad + \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T]. \end{aligned} \tag{2.2}$$

- The initial condition is given by

$$X(t) = \varphi(t - s), \quad \forall t \in [s - h, s], \tag{2.3}$$

where $\varphi \in C[-h, 0]$ and $\varphi(\theta) > 0, \forall \theta \in [-h, 0]$.

- Let Π denote the admissible control space. We assume that a control policy $(k(t), c(t))$ in Π is $\mathcal{F}^t-$measurable for any $t \in [0, T]$ with $c(t) \geq 0, \forall t \in [0, T]$. Also we assume that

$$\begin{cases} |k(t)X(t)| \leq \Lambda_1 |X(t) + \mu_3 Y(t)|, \\ |c(t)X(t)| \leq \Lambda_2 |X(t) + \mu_3 Y(t)|, \end{cases} \tag{2.4}$$

where $\Lambda_1 > 0, \Lambda_2 > 0$ are constants.

- We can show that $X(t) > 0$, almost surely.

- Objective Functional:

$$J(s,\varphi,k,c) \equiv \mathbf{E}_{s,\varphi}\Big[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\Big].$$

- The value function is given by

$$V(s,\varphi) = \sup_{k,c\in\Pi} J(s,\varphi,k,c) \tag{2.5}$$

- Under certain conditions, we have

$$V(s,\varphi) = V(s,x,y,z), \tag{2.6}$$

where

$$x = x(\varphi) \equiv \varphi(0), \quad y = y(\varphi) \equiv \int_{-h}^0 e^{\lambda\theta}\varphi(\theta)d\theta, \quad z = z(\varphi) \equiv \varphi(-h). \tag{2.7}$$

- To derive the HJB equation, we will need that the value function V only depends on (s,x,y), i.e.

$$V(s,\varphi) = V(s,x,y,z) = V(s,x,y). \tag{2.8}$$

2.2 Functional Ito's Formula and HJB Equation

$$df(X_t) = \partial_t f(X_t)dt + \partial_x f(X_t)dX(t) + \frac{1}{2}\partial_{xx} f(X_t)d\langle X\rangle(t).$$

where

$$\partial_t f(X_t) = \lim_{\delta\to 0} \frac{f(X_{t,\delta}) - f(X_t)}{\delta}, \tag{2.9}$$

$$X_{t,\delta}(\theta) = \begin{cases} X_t(\delta+\theta), & \theta \in [-h, -\delta], \\ X_t(0), & \theta \in [-\delta, 0]; \end{cases} \tag{2.10}$$

$$\partial_x f(X_t) = \lim_{\delta\to 0} \frac{f(X_t^\delta) - f(X_t)}{\delta}, \tag{2.11}$$

$$X_t^\delta(\theta) = \begin{cases} X_t(\theta), & \theta \in [-h, 0), \\ X_t(0)+\delta, & \theta = 0, \end{cases} \tag{2.12}$$

$$\partial_{xx} f(X_t) = \lim_{\delta\to 0} \frac{\partial_x f(X_t^\delta) - \partial_x f(X_t)}{\delta}. \tag{2.13}$$

Now let us consider the following functional:

$$y(X_t) = \int_{-h}^{0} \phi(\theta) X_t(\theta) d\theta; \quad z(X_t) = X_t(-h), \tag{2.14}$$

where $\phi(\theta)$ is a smooth function with a continuous first order derivative $\phi'(\theta)$.

Lemma 2.1. *If* $y(X_t), z(X_t)$ *is given by* (2.14), *then we have*

$$dy(X_t) \; = \; \left[X_t(0)\phi(0) - \int_{-h}^{0} \phi'(\theta) X_t(\theta) d\theta - \phi(-h) z(X_t) \right] dt.$$

- Take $\phi(\theta) = e^{\lambda\theta}$. Then we can get

$$\partial_t y(X_t) = X_t(0) - \lambda y(X_t) - e^{-\lambda h} z(X_t).$$

Therefore, we can get

$$dy(X_t) = \partial_t y(X_t) dt = [X_t(0) - \lambda y(X_t) - e^{-\lambda h} z(X_t)] dt.$$

Lemma 2.2. *If the value function* V *depends on* z, *there is no way to write the HJB equation with respect to* (s, x, y, z).

Theorem 2.1 (HJB Equation)**.** *Assume that* (2.8) *holds and* $V(s,x,y) \in C^{1,2,1}([0,T] \times \mathbf{R} \times \mathbf{R})$. *Then the value function* $V(s,x,y)$ *given by* (2.5) *and* (2.8) *satisfies the following HJB equation*

$$\max_{k,c\geq 0}[\mathcal{L}^{k,c}V + U(cx)] - \beta V + (x - \lambda y - e^{-\lambda h}z)V_y = 0, \quad \forall z \in \mathbf{R} \tag{2.15}$$

where $\mathcal{L}^{k,c}$ *is defined by*

$$\mathcal{L}^{k,c}V \equiv V_s + \frac{\sigma^2 k^2 x^2 V_{xx}}{2} + (((\mu_1 - r)k - c + r)x + \mu_2 y + \mu_3 z)V_x. \tag{2.16}$$

and the boundary condition is

$$V(T,x,y) = \Psi(x,y). \tag{2.17}$$

Logarithmic utility function

Assume that

$$\mu_3 e^{\lambda h}(r + \mu_3 e^{\lambda h}) = \mu_2 - \mu_3 \lambda e^{\lambda h}. \tag{2.18}$$

The solution is

$$V(s, x, y) = Q(s) + \log(x + \mu_3 e^{\lambda h} y), \tag{2.19}$$

where

$$Q(s) = \frac{\Lambda_4}{\beta}(1 - e^{-\beta(T-s)}). \tag{2.20}$$

and

$$\Lambda_4 \equiv \frac{(\mu_1 - r)^2}{2\beta\sigma^2} + \log\beta - 1 + \frac{1}{\beta}(r + \mu_3 e^{\lambda h}). \tag{2.21}$$

Moreover, the optimal investment and consumption rates are given as

$$k^*(s) = \frac{(\mu_1 - r)(x + \mu_3 e^{\lambda h} y)}{\sigma^2 x}, \quad c^*(s) = \frac{\beta(x + \mu_3 e^{\lambda h} y)}{x}. \tag{2.22}$$

Exponential utility function given as $U(x) = 1 - e^{-\alpha x}, \quad \alpha > 0$. It is equivalent to consider the following $U(x) = -e^{-\alpha x}, \quad \alpha > 0$.

The explicit solution is given as

$$V(s, x, y) = -Q(s)e^{-\alpha(r+\mu_3 e^{\lambda h})(x+\mu_3 e^{\lambda h} y)}, \tag{2.23}$$

$$Q(s) = \exp\left(\frac{\Lambda_5}{r + \mu_3 e^{\lambda h}}\left(e^{-(r+\mu_3 e^{\lambda h})(T-s)} - 1\right) + e^{-(r+\mu_3 e^{\lambda h})(T-s)} \log \Lambda\right).$$

The optimal investment and consumption rates are given as

$$\begin{aligned} k^*(s) &= \frac{(\mu_1 - r)}{\alpha(r + \mu_3 e^{\lambda h})\sigma^2 x}, \\ c^*(s) &= -\frac{1}{x\alpha}\left[\log\{(r + \mu_3 e^{\lambda h})Q(s)\} - \alpha(r + \mu_3 e^{\lambda h})(x + \mu_3 e^{\lambda h} y)\right]. \end{aligned}$$

HARA Utility.

$$U(cX) = \frac{1}{\gamma}(cX)^{\gamma}.$$

The solution is given by

$$V(s,x,y) = \frac{1}{\gamma}Q(s)(x+\mu_3 e^{\lambda h}y)^{\gamma}. \tag{2.24}$$

where $Q(s)$ is given by

$$Q(s) \;=\; \left[\left(1-\frac{1-\gamma}{\Lambda}\right)e^{-\frac{\Lambda}{1-\gamma}(T-s)}+\frac{1-\gamma}{\Lambda}\right]^{1-\gamma}, \tag{2.25}$$

$$\Lambda \;\equiv\; \beta+\frac{(\mu_1-r)^2\gamma}{2\sigma^2(\gamma-1)}-\gamma(r+\mu_3 e^{\lambda h}). \tag{2.26}$$

Optimal investment ratio and the optimal consumption rate control are

$$k^*(s) \;=\; \frac{(\mu_1-r)(x+\mu_3 e^{\lambda h}y)}{(1-\gamma)\sigma^2 x}, \tag{2.27}$$

$$c^*(s) \;=\; \frac{x+\mu_3 e^{\lambda h}y}{x}Q(s)^{\frac{1}{\gamma-1}}. \tag{2.28}$$

2.3 Infinite Time Horizon with Bounded Memory

- We consider the following model on infinite time horizon.

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t) + \mu_3 Z(t)]dt \\ &\quad + \sigma k(t)X(t)dB(t), \quad \forall t \in [0,\infty). \end{aligned} \tag{2.29}$$

where

$$Y(t) = \int_{-h}^{0} e^{\lambda\theta} X(t+\theta)d\theta, \quad Z(t) = X(t-h), \quad \forall t \in [0,\infty), \tag{2.30}$$

Subject to the initial condition:

$$X(t) = \varphi(t), \quad \forall t \in [-h,0], \tag{2.31}$$

where $\varphi \in C[-h,0]$ and $\varphi(\theta) > 0, \forall\theta \in [-h,0]$.

- The value function:

$$V(\varphi) = \sup_{k,c\in\Pi} \mathbb{E}_\varphi \left[\int_0^\infty e^{-\beta t} U(c(t)X(t))dt \right]. \tag{2.32}$$

We assume that the value function V depends on the initial path φ only through the functionals $x(\varphi)$ and $y(\varphi)$.

$$V(\varphi) = V(x(\varphi), y(\varphi)) \equiv V(x, y). \tag{2.33}$$

The value function $V(x, y)$ given by (2.33) satisfies the following HJB equation

$$\beta V = \max_{k} \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r) k x V_x \right] + (rx + \mu_2 y + \mu_3 z) V_x + \max_{c \geq 0} \left[-cxV_x + U(cx) \right] + (x - \lambda y - e^{-\lambda h} z) V_y, \quad \forall z \in \mathbb{R} \tag{2.34}$$

Exponential Utility Function. The solution of the HJB equation (2.34) is given by

$$V(x,y) = -\frac{1}{r+\mu_3 e^{\lambda h}} \exp\left(1 - \frac{\beta + \frac{(\mu_1 - r)^2}{2\sigma^2}}{r+\mu_3 e^{\lambda h}}\right) e^{-\alpha(r+\mu_3 e^{\lambda h})(x+\mu_3 e^{\lambda h} y)} \tag{2.35}$$

and the optimal investment and the optimal consumption controls are

$$k^* = \frac{(\mu_1 - r)}{\alpha\sigma^2 x(r+\mu_3 e^{\lambda h})}, \tag{2.36}$$

$$c^* = \frac{1}{\alpha x}\left(\frac{\beta + \frac{(\mu_1 - r)^2}{2\sigma^2}}{r+\mu_3 e^{\lambda h}} - 1\right) + \frac{1}{x}(r+\mu_3 e^{\lambda h})\left(x + \mu_3 e^{\lambda h} y\right). \tag{2.37}$$

Logarithmic Utility Function. For this utility function the solution of the HJB equation (2.34) is

$$V(x,y) = \eta_2 + \frac{1}{\beta}\log\left(x + \mu_3 e^{\lambda h} y\right), \tag{2.38}$$

where

$$\eta_2 = \frac{1}{\beta^2}\left[\frac{(\mu_1 - r)^2}{2\sigma^2} + \beta\log\beta - \beta + \left(r + \mu_3 e^{\lambda h}\right)\right]. \tag{2.39}$$

The optimal investment and consumption controls are

$$k^* = \frac{(\mu_1 - r)}{\sigma^2 x}(x + \mu_3 e^{\lambda h} y), \quad c^* = \frac{\beta}{x}(x + \mu_3 e^{\lambda h} y). \tag{2.40}$$

HARA Utility Function. Consider HARA utility function

$$U(cX) = \frac{1}{\gamma}(cX)^{\gamma}. \tag{2.41}$$

Therefore the solution of the HJB equation (2.34) is given by

$$V(x,y) = \frac{1}{\gamma}\eta\left(x + \mu_3 e^{\lambda h} y\right)^{\gamma}, \tag{2.42}$$

where

$$\eta = \left(\frac{\gamma}{1-\gamma}\left(\frac{\beta}{\gamma} - \frac{1}{2}\frac{(\mu_1 - r)^2}{(1-\gamma)\sigma^2} - \left(r + \mu_3 e^{\lambda h}\right)\right)\right)^{\gamma-1}. \tag{2.43}$$

The optimal investment and consumption rates are

$$k^* = \frac{(\mu_1 - r)(x + \mu_3 e^{\lambda h} y)}{(1-\gamma)\sigma^2 x}, \quad c^* = \frac{1}{x}\eta^{\frac{1}{\gamma}}\left(x + \mu_3 e^{\lambda h} y\right). \tag{2.44}$$

2.4 A Stochastic Portfolio Model with Infinite Delay (Complete Memory)

We now consider a model with infinite delay:

$$Y(t) = \int_{-\infty}^{0} e^{\lambda\theta} X(t+\theta) d\theta. \tag{2.45}$$

It is equivalent to consider a delay starting from a fixed time point in the past (by assuming $X(\theta) = 0, \forall\theta \in (-\infty, -M)$):

$$Y(t) = \int_{-M}^{t} e^{\lambda(\theta-t)} X(\theta) d\theta = \int_{-M-t}^{0} e^{\lambda\theta} X(t+\theta) d\theta. \tag{2.46}$$

That is, we now consider an exponential average, not an exponential moving average.

- The system for this model is described by

$$
\begin{aligned}
dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t)]dt \\
&\quad + \sigma k(t)X(t)dB(t), \qquad (2.47) \\
dY(t) &= (X(t) - \lambda Y(t))dt, \quad \forall t \in [s, T], \qquad (2.48)
\end{aligned}
$$

- Initial conditions

$$
\begin{aligned}
X(t) &= \varphi(t), \quad t \in (-\infty, s], \qquad (2.49) \\
Y(0) &= y \equiv \int_{-\infty}^{0} e^{\lambda\theta}\varphi(s+\theta)d\theta. \qquad (2.50)
\end{aligned}
$$

where φ is a bounded function and $\varphi(\theta) > 0, \forall \theta \in (-\infty, s]$.

We assume that the initial path φ depends on the value function V only through the functionals $x(\varphi)$ and $y(\varphi)$. That is,

$$V(s,\varphi) = V(s, x(\varphi), y(\varphi)) \equiv V(s,x,y). \tag{2.51}$$

The HJB equation is given by

$$\begin{aligned} \beta V - V_s \;=\; & \max_k \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r)kxV_x\right] + (rx + \mu_2 y)V_x \\ & + \max_{c\geq 0}[-cxV_x + U(cx)] + (x - \lambda y)V_y, \end{aligned} \tag{2.52a}$$

with the boundary condition

$$V(T,x,y) = \Psi(x,y). \tag{2.52b}$$

Exponential Utility Function

- Exponential utility function: $U(cX) = -e^{-\alpha c X}, \quad \alpha > 0.$
- The HJB equations (2.52a)-(2.52b) have the solution

$$V(s,x,y) = -Q(s)e^{-\alpha(r+\eta)(x+\eta y)}, \tag{2.53}$$

where $\eta = \frac{1}{2}\left(\sqrt{(r+\lambda)^2 + 4\mu_2} - (r+\lambda)\right)$ and

$$Q(s) = \exp\left(-\frac{\Lambda_1}{r+\eta} + \left(\frac{\Lambda_1}{r+\eta} + \log\Lambda\right)e^{-(r+\eta)(T-s)}\right).$$

- The optimal controls are given as

$$k^*(s) = \frac{(\mu_1 - r)}{\alpha(r+\eta)\sigma^2 x}, \tag{2.54}$$

$$c^*(s) = -\frac{1}{\alpha x}\left[\log\{(r+\eta)Q(s)\} - \alpha(r+\eta)(x+\eta y)\right]. \tag{2.55}$$

Log Utility Function

- Log utility function: $U(cX) = \log(cX)$.
- The HJB equations (2.52a)-(2.52b) have the solution

$$V(s,x,y) = Q(s) + \log(x + \eta y), \tag{2.56}$$

where $\eta = \frac{1}{2}\left(\sqrt{(r+\lambda)^2 + 4\mu_2} - (r+\lambda)\right)$ and $Q(s) = \frac{\Lambda_2}{\beta}(1 - e^{-\beta(T-s)})$.

- If $\Lambda_2 > 0$, we have

$$0 \le Q(s) \le \Lambda_2, \quad \forall s \in [0,T]. \tag{2.57}$$

- The optimal investment and consumption rates are given as

$$k^*(s) = \frac{(\mu_1 - r)(x + \eta y)}{\sigma^2 x} \quad c^*(s) = \frac{\beta(x + \eta y)}{x}. \tag{2.58}$$

HARA Utility Function

- Consider HARA utility function $U(cX) = \frac{1}{\gamma}(cX)^{\gamma}, \quad \gamma < 1, \text{and } \gamma \neq 0.$
- The HJB equations (2.52a)-(2.52b) have the solution

$$V(s,x,y) = \frac{1}{\gamma} Q(s)(x+\eta y)^{\gamma}, \tag{2.59}$$

where $\eta = \frac{1}{2}\left(\sqrt{(r+\lambda)^2 + 4\mu_2} - (r+\lambda)\right)$ and

$$Q(s) = \left[\frac{1-\gamma}{\Lambda_3}\left(1 - e^{-\frac{\Lambda_3(T-s)}{1-\gamma}}\right) + \Lambda^{\frac{1}{1-\gamma}} e^{-\frac{\Lambda_3(T-s)}{1-\gamma}}\right]^{1-\gamma}.$$

- The optimal investment and consumption rates are given as

$$k^*(s) = -\frac{(\mu_1 - r)(x+\eta y)}{(1-\gamma)\sigma^2 x}, \quad c^*(s) = \frac{x+\eta y}{x} Q(s)^{\frac{1}{\gamma-1}}. \tag{2.60}$$

3 A More Realistic Model: Stock Price Model with Delays

Let $S(t)$ be the stock price and $Y(t) = \ln S(t)$ be the stock return process. Consider a model with delay:

$$dY(t) = (\mu_1 + \mu_2(Z(t)))dt + \sigma dB(t), \tag{3.1}$$

$$dZ(t) = \int_{-\infty}^{0} e^{\lambda\theta} Y(t+\theta) d\theta. \tag{3.2}$$

- The risky asset's return depends on $(Z(t))$.
- $\mu_2(\cdot)$ is assumed to be Bounded means that historical performance may only influence returns to a certain extent.
- $Z(t)$ satisfies

$$dZ(t) = [Y(t) - \lambda Z(t)]dt. \tag{3.3}$$

Let $X(t)$ be the total wealth, $\pi(t)$ be the portion invested on the stock (risky asset), $c(t)X(t)$ be the consumption rate, $r > 0$ be the risk-free interest rate. Then

$$\begin{aligned} dX(t) &= X(t)\left[r + (\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(Z(t))\pi(t) - c(t)\right] dt \\ &\quad + \sigma\pi(t)X(t)dB(t), \quad \forall t \in [0, \infty), \\ X(0) &= x. \end{aligned}$$

Consider a portfolio optimization problem over the infinite time horizon:

$$V(x, y, z) = \sup_{\pi, c} J(x, y, z, \pi, c), \tag{3.4}$$

where

$$J(x, y, z, \pi, c) = \mathbf{E}_{x,y,z}\left[\int_0^\infty e^{-\beta t} U(c(t)X(t))dt\right]. \tag{3.5}$$

The HJB equation is

$$\begin{aligned}\beta V &= \frac{1}{2}\sigma^2 V_{yy} + xrV_x + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z \\ &+ \sup_c \Big[U(cx) - cxV_x\Big] \\ &+ \sup_\pi \Big[\frac{1}{2}\sigma^2\pi^2x^2V_{xx} + \Big(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)\Big)x\pi V_x + \sigma^2 x\pi V_{xy}\Big]\end{aligned}$$

3.1 Log Utility Case

Consider log utility function $U(x) = \log x$ and assume that $V(x,y,z) = \frac{1}{\beta}\log x + W(y,z)$. Then $W(y,z)$ solves

$$\begin{aligned}\beta W \;=\; & \sup_{\pi}\left[\frac{1}{\beta}\left(\mu_1+\frac{1}{2}\sigma^2-r+\mu_2(z)\right)\pi-\frac{\sigma^2\pi^2}{2\beta}\right] \\ & +\frac{1}{2}\sigma^2 W_{yy}+\sup_{c}\left[\log c-\frac{c}{\beta}\right]+\frac{r}{\beta} \\ & +(\mu_1+\mu_2(z))W_y+(y-\lambda z)W_z.\end{aligned}$$

Candidates of the optimal control are

$$\pi^* = \frac{1}{2}+\frac{\mu_1-r+\mu_2(z)}{\sigma^2}, \quad c^*=\beta. \tag{3.6}$$

Now the HJB equation can be reduced to

$$\frac{\sigma^2}{2}W_{yy} + (\mu_1 + \mu_2(z))W_y + f(y,z)W_z - \beta W + g(z) = 0, \tag{3.7}$$

where

$$f(y,z) = (y - \lambda z), \quad g(z) = \frac{(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z))^2}{2\beta\sigma^2} + \log\beta + \frac{r}{\beta} - 1. \tag{3.8}$$

(3.7) is a degenerate elliptic partial differential equation with unbounded coefficient $f(y,z) = y - \lambda z$. No existing existence of solution result available.

The equation (3.7) is equivalent to

$$\frac{\sigma^2}{2(1+f^2(y,z))}W_{yy} + \frac{\mu_1 + \mu_2(z)}{1+f^2(y,z)}W_y + \frac{f(y,z)}{1+f^2(y,z)}W_z - \frac{\beta}{1+f^2(y,z)}W + \frac{g(z)}{1+f^2(y,z)} = 0$$

which is a linear equation with all coefficients bounded.

The existence results is not available due to the missing of W_{zz}. Here we use a perturbation approach.

For a small $\varepsilon > 0$, consider the following equation:

$$\begin{aligned}
&\frac{\sigma^2}{2(1+f^2(y,z))}W^\varepsilon_{yy} + \varepsilon W^\varepsilon_{zz} + \frac{\mu_1+\mu_2(z)}{1+f^2(y,z)}W^\varepsilon_y + \frac{f(y,z)}{1+f^2(y,z)}W^\varepsilon_z \\
&\quad - \frac{\beta}{1+f^2(y,z)}W^\varepsilon + \frac{g(z)}{1+f^2(y,z)} = 0 \qquad (3.9)
\end{aligned}$$

which is a linear second order elliptic equation.

Denote $a^2(y,z) = \frac{\sigma^2}{2(1+f^2(y,z))}$. Then $a^2(y,z) > 0$ and Equation (3.9) can be rewritten as

$$-a^2(y,z)W^\varepsilon_{yy} - \varepsilon W^\varepsilon_{zz} = H(y,z,W^\varepsilon,W^\varepsilon_y,W^\varepsilon_z), \qquad (3.10)$$

where

$$\begin{aligned}
&H(y,z,W^\varepsilon,W^\varepsilon_y,W^\varepsilon_z) \\
= \;&\frac{1}{(1+f^2(y,z))}\Big[(\mu_1+\mu_2(z))W^\varepsilon_y + f(y,z)W^e_z - \beta W^e + g(z)\Big].
\end{aligned}$$

Supersolutuons and Subsolutions

Definition 1 (Supersolution and Subsolution)**.** *We say that* $\bar{W}(\underline{W})$ *is a supersolution (subsolution) of* (3.10) *if*

$$-a^2(y,z)W_{yy} - \varepsilon W_{zz} \geq (\leq) H(y,z,W,W_y,W_z). \tag{3.11}$$

In addition, if $\underline{W}(y,z) \leq \bar{W}(y,z)$*, we say that* $\langle \underline{W}, \bar{W} \rangle$ *is an ordered pair of sub-super solutions.*

Similarly, we can define supersolution and subsolution for corresponding boundary problems on any bounded domain, such as a ball.

Lemma 3.3. *There exist two positive constants* K_1 *and* K_2 *such that* $K_1 < K_2$ *and* $\langle K_1, K_2 \rangle$ *is an ordered pair of sub-super solutions of* (3.9)*.*

Remark: It is easy to check that $\langle K_1, K_2 \rangle$ is also an ordered pair of sub-super solutions of (3.7), which is equivalent to

$$-a^2(y,z)W_{yy} = H(y,z,W,W_y,W_z). \tag{3.12}$$

Consider the boundary value problem:

$$\begin{cases} -a^2(y,z)W^\varepsilon_{yy} - \varepsilon W^\varepsilon_{zz} = H(y,z,W^\varepsilon,W^\varepsilon_y,W^\varepsilon_z), & (y,z)\in B_R, \\ W^\varepsilon(y,z) = K_1, & (y,z)\in \partial B_R, \end{cases} \tag{3.13}$$

where $B_R = \{(y,z) : y^2+z^2 \le R^2\}$ and K_1 is the subsolution of (3.7).

Lemma 3.4. (3.13) *has a unique solution* $W^\varepsilon_R(y,z) \in C^0(\bar{\mathbf{R}}) \cap C^{2,\alpha}(\mathbf{R})$.

Lemma 3.5. $W^\varepsilon_R(y,z)$ *is uniformly bounded:*

$$K_1 \le W^\varepsilon_R(y,z) \le K_2, \tag{3.14}$$

where K_1, K_2 *are positive and independent of* R *and* ε.

Lemma 3.6. $\nabla W^\varepsilon_R(y,z)$ *is uniformly bounded, and the bounds are independent of* R, ε.

Lemma 3.7. *For each fixed* $\varepsilon > 0$, $\lim_{R\to\infty} W_R^\varepsilon(y,z) = W^\varepsilon(y,z)$ *exists, and it solves* (3.10) *on the whole space.*

Lemma 3.8. *For each fixed* $R > 0$, $\lim_{\varepsilon\to 0} W_R^\varepsilon(y,z) = W_R(y,z)$ *exists, and it is a viscosity solution of*

$$\begin{cases} -a^2(y,z)W_{yy} = H(y,z,W,W_y,W_z), & (y,z) \in B_R, \\ W(y,z) = K_1, & (y,z) \in \partial B_R, \end{cases} \tag{3.15}$$

By the diagonal method, we can find a subsequence of W_R^ε to converge to $W(y,z)$ which is a viscosity solution of (3.7).

Theorem 3.1. *There is a unique viscosity solution* $W(y,z)$ *for equation* (3.7)*, which is equivalent to*

$$-a^2(y,z)W_{yy} = H(y,z,W,W_y,W_z). \tag{3.16}$$

Theorem 3.2 (Verification Theorem)**.** *Let* $W(y, z)$ *be the classical solution of* (3.7) *and*

$$V(x, y, z) = \log(x) + W(y, z). \tag{3.17}$$

Then $V(x; y; z)$ *equals to the value function. Moreover, the optimal control policy is given by*

$$\pi^* = \frac{1}{2} + \frac{\mu_1 - r + \mu_2(z)}{\sigma^2}, \quad c^* = \beta. \tag{3.18}$$

3.2 HARA Utility Case

HARA utility: $U(x) = \frac{1}{\gamma}x^\gamma, \gamma < 1, \gamma \neq 0$.

With the HARA utility, the HJB equation is

$$\begin{aligned}\beta V = &\frac{1}{2}\sigma^2 V_{yy} + xrV_x + (\mu_1 + \mu_2(z))V_y + (y - \lambda z)V_z \\ &+ \sup_c \left[\frac{1}{\gamma}(cx)^\gamma - cxV_x\right] \\ &+ \sup_\pi \left[\frac{1}{2}\sigma^2\pi^2x^2V_{xx} + \Big(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)\Big)x\pi V_x + \sigma^2 x\pi V_{xy}\right]\end{aligned}$$

Assume that $V(x, y, z) = x^\gamma W(y, z)$. Then $W(y, z)$ solves

$$\begin{aligned}\beta W \;=\; &\sup_\pi \left[\frac{1}{2}\sigma^2\pi^2\gamma(\gamma - 1)W + \gamma\left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)\right)\pi W + \gamma\sigma^2\pi W_y\right] \\ &+ \frac{1}{2}\sigma^2 W_{yy} + \sup_c \left[\frac{1}{\gamma}c^\gamma - \gamma cW\right] + \gamma rW \\ &+ (\mu_1 + \mu_2(z))W_y + (y - \lambda z)W_z.\end{aligned}$$

Candidates for optimal controls:

$$\pi^* = \frac{\left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)\right) W + \sigma^2 W_y}{\sigma^2(\gamma - 1)W}, \quad c^* = (\gamma W)^{\frac{1}{\gamma-1}}. \tag{3.19}$$

The equation for W can be written as

$$\begin{aligned}\beta W \;=\; & \frac{\gamma\left[\left(\mu_1 + \frac{1}{2}\sigma^2 - r + \mu_2(z)\right) W + \sigma^2 W_y\right]^2}{2\sigma^2(1-\gamma)W}, \\ & + \frac{1}{2}\sigma^2 W_{yy} + \left(1 - \frac{1}{\gamma}\right)(\gamma W)^{\frac{\gamma}{\gamma-1}} + \gamma r W \\ & + (\mu_1 + \mu_2(z))W_y + (y - \lambda z)W_z.\end{aligned}$$

The above equation can be written as

$$\frac{1}{2}\sigma^2 W_{yy} + H(y, z, W, W_y, W_z) = 0. \tag{3.20}$$

which is a semi-linear (degenerated) elliptic PDE. We can use similar idea as for log utility to establish the existence results and the verification result.

Thanks You!

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