Linear Quadratic Mean Field Games: asymptotic solvability and the fixed point approach

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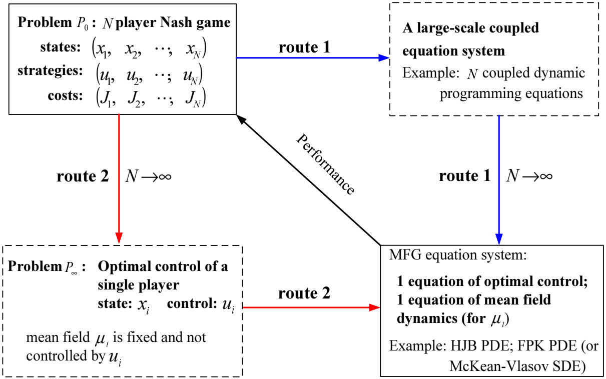
Outline of talk

- The N player LQ Nash game
- Mean field game (MFG) theory provides a very compact specification and solution for non-cooperative decision making with a large number of players
  - The **direct** approach and the **asymptotic solvability (AS) problem** with time horizon $[0, T]$.
  - The **fixed point** approach
- Relation of the two approaches
- Long time behavior in the AS problem

The fundamental diagram of MFG theory



- blue: direct approach red: fixed point approach (Huang, Caines, Malhamé, 03, 07), (Huang, Malhamé, Caines, 06), (Lasry and Lions, 06); see overview of the two approaches in (Caines, Huang, and Malhamé, 17)
- Basic questions. Relation of the two approaches? Their respective domains of applicability?

- The diagram links finite population games to an infinite population problem
- Also, the blue route provides a possible way to derive the (**infinite population**) mean field game via solving **finite population** models
  - Question: is the blue route always feasible? We will clarify.

There is a large literature on the relation between games of finite and infinite populations (traditionally for static models).

R. Aumann (1964), G. Carmona and K. Podczeck (2010), E. Green (194), A. Haurie and P. Marcotte (1985), A. Mas-Colell (1983), $\cdots$

Other references related to the direct approach:

- Huang (2003, thesis ch. 6) – mean field social optimization of N players; solve a large scale algebraic Riccati equation
- Papavassilopoulos (2014) – for LQ mean field game, analyze existence for weakly coupled algebraic Riccati equations using the implicit function theorem
- Herty, Pareschi and Steffensen (2015) – N agent mean field optimal control via a large Riccati equation
- Priuli (2015) – Convergence of HJB-FPKs of N players to a mean field limit; start with decentralized control
- Cardaliaguet, Delarue, Lasry and Lions (2015) – **Fully coupled** HJBs for N players; start with centralized info; always uniquely solvable due to special dynamics and costs; convergence to a master equation; no asymptotic solvability problem as we will face.

LQ Nash games are not always solvable on $[0, T]$. This is actually a useful feature for us to **distinguish the direct and fixed point approaches**.

By use of the diagram, we attempt to **"classify" models**. In the LQ setting, a model means a specification $(A, B, G, Q, Q_f, R, \Gamma, ...)$, not including the population size N.

Nash game of N players $\mathcal{A}_i$, $1 \le i \le N$

Dynamics of player $\mathcal{A}_i$:

$$dX_i(t) = \big(AX_i(t) + Bu_i(t) + GX^{(N)}(t)\big)dt + DdW_i(t), \quad 1 \le i \le N,$$

where the state $X_i \in \mathbb{R}^n$, control $u_i \in \mathbb{R}^{n_1}$, $X^{(N)} = \frac{1}{N}\sum_{k=1}^N X_k$, $W_i \in \mathbb{R}^{n_2}$: N independent Brownian motions (so, white noise).

Cost:

$$J_i = E\int_0^T \Big(|X_i(t) - \Gamma X^{(N)}(t) - \eta|_Q^2 + u_i^T(t)Ru_i(t)\Big)dt + E|X_i(T) - \Gamma_f X^{(N)}(T) - \eta_f|_{Q_f}^2.$$

The matrices (or vectors) A, B, G, D, Γ, Q, R, Γ_f, Q_f, η, η_f have compatible dimensions, and $Q \ge 0$, $R > 0$, $Q_f \ge 0$.

Notation:

$$X(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_N(t) \end{bmatrix} \in \mathbb{R}^{Nn}, \quad W(t) = \begin{bmatrix} W_1(t) \\ \vdots \\ W_N(t) \end{bmatrix} \in \mathbb{R}^{Nn_2},$$

$$\widehat{\boldsymbol{A}} = \mathrm{diag}[A, \cdots, A] + \mathbf{1}_{n\times n} \otimes \frac{G}{N} \in \mathbb{R}^{Nn\times Nn},$$

$$\widehat{\boldsymbol{D}} = \mathrm{diag}[D, \cdots, D] \in \mathbb{R}^{Nn\times Nn_2},$$

$$\boldsymbol{B}_k = e_k^N \otimes B \in \mathbb{R}^{Nn\times n_1}, \quad 1 \le k \le N.$$

Other matrices (Q_{if}, K_{if}, etc) appearing later can be determined from model parameters

Dynamics: $dX(t) = \Big(\widehat{\boldsymbol{A}}X(t) + \sum_{k=1}^{N} \boldsymbol{B}_k u_k(t)\Big)dt + \widehat{\boldsymbol{D}}dW(t).$

$V_i(t, \boldsymbol{x})$: value function of $\mathcal{A}_i$.

HJB equation system:

$$
\begin{aligned}
0 = & \frac{\partial V_i}{\partial t} + \frac{\partial^T V_i}{\partial \boldsymbol{x}}\Big(\widehat{\boldsymbol{A}}\boldsymbol{x} - \sum_{k=1}^{N} \frac{1}{2}\boldsymbol{B}_k R^{-1} \boldsymbol{B}_k^T \frac{\partial V_k}{\partial \boldsymbol{x}}\Big) \\
& + |x_i - \Gamma x^{(N)} - \eta|_Q^2 \\
& + \frac{1}{4}\frac{\partial^T V_i}{\partial \boldsymbol{x}} \boldsymbol{B}_i R^{-1} \boldsymbol{B}_i^T \frac{\partial V_i}{\partial \boldsymbol{x}} + \frac{1}{2}\mathrm{Tr}\big(\widehat{\boldsymbol{D}}^T (V_i)_{\boldsymbol{x}\boldsymbol{x}} \widehat{\boldsymbol{D}}\big).
\end{aligned}
$$

Terminal condition: $V_i(T, \boldsymbol{x}) = |x_i - \Gamma_f x^{(N)} - \eta_f|_{Q_f}^2$

Feedback Nash strategies:

$$
u_i = -\frac{1}{2}R^{-1}\boldsymbol{B}_i^T \frac{\partial V_i}{\partial \boldsymbol{x}}, \quad 1 \le i \le N.
$$

Write $V_i(t,\boldsymbol{x}) = \boldsymbol{x}^T P_i(t)\boldsymbol{x} + 2\boldsymbol{x}^T S_i(t) + r_i(t)$. We derive

N Riccati ODEs:
$$\begin{cases} \dot{\boldsymbol{P}}_i(t) = -\Big(\boldsymbol{P}_i(t)\widehat{\boldsymbol{A}} + \widehat{\boldsymbol{A}}^T\boldsymbol{P}_i(t)\Big) + \\ \qquad \Big(\boldsymbol{P}_i(t)\sum_{k=1}^N \boldsymbol{B}_k R^{-1}\boldsymbol{B}_k{}^T\boldsymbol{P}_k(t) \\ \qquad + \sum_{k=1}^N \boldsymbol{P}_k(t)\boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T\boldsymbol{P}_i(t)\Big) \\ \qquad -\boldsymbol{P}_i(t)\boldsymbol{B}_i R^{-1}\boldsymbol{B}_i^T\boldsymbol{P}_i(t) - \boldsymbol{Q}_i, \ 1\le i\le N, \\ \boldsymbol{P}_i(T) = \boldsymbol{Q}_{if}, \quad \text{(see e.g. Basar and Olsder'99)} \end{cases}$$

$$\begin{cases} \dot{\boldsymbol{S}}_i(t) = -\widehat{\boldsymbol{A}}^T\boldsymbol{S}_i(t) - \boldsymbol{P}_i(t)\boldsymbol{B}_i R^{-1}\boldsymbol{B}_i{}^T\boldsymbol{S}_i(t) \\ \qquad +\boldsymbol{P}_i(t)\sum_{k=1}^N \boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_k(t) \\ \qquad +\sum_{k=1}^N \boldsymbol{P}_k(t)\boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_i(t) + \boldsymbol{K}_i^T Q\eta, \\ \boldsymbol{S}_i(T) = -\boldsymbol{K}_{if}^T Q_f\eta_f, \end{cases}$$

$$\begin{cases} \dot{\boldsymbol{r}}_i(t) = 2\boldsymbol{S}_i{}^T(t)\sum_{k=1}^N \boldsymbol{B}_k R^{-1}\boldsymbol{B}_k^T\boldsymbol{S}_k(t) \\ \qquad -\boldsymbol{S}_i^T(t)\boldsymbol{B}_i R^{-1}\boldsymbol{B}_i^T\boldsymbol{S}_i(t) - \eta^T Q\eta - \mathrm{Tr}\big(\widehat{\boldsymbol{D}}^T\boldsymbol{P}_i(t)\widehat{\boldsymbol{D}}\big), \\ \boldsymbol{r}_i(T) = \eta_f^T Q_f\eta_f. \end{cases}$$

For an $l \times m$ real matrix $Z = (z_{ij})_{i \le l, j \le m}$, denote the l_1-norm $\|Z\|_{l_1} = \sum_{i,j} |z_{ij}|$.

Definition The sequence of N player Nash games with closed-loop perfect state information has **asymptotic solvability** if

- there exists N_0 such that for all $N \ge N_0$, $(\boldsymbol{P}_1, \cdots, \boldsymbol{P}_N)$ in the N coupled Riccati ODEs has a solution on $[0, T]$ and,
- $$\sup_{N \ge N_0} \sup_{1 \le i \le N, 0 \le t \le T} \|\boldsymbol{P}_i(t)\|_{l_1} < \infty.$$

The l_1 norm may be informally interpreted as the“total mass” of a large “pie” (the $Nn \times Nn$ matrix).

Theorem We assume that the Riccati ODE system has a solution $(\boldsymbol{P}_1(t), \cdots, \boldsymbol{P}_N(t))$ on $[0, T]$. Then the following holds.

i) $\boldsymbol{P}_1(t)$ has the representation ($N \times N$ blocks)

$$\boldsymbol{P}_1(t) = \begin{bmatrix} \Pi_1(t) & \Pi_2(t) & \Pi_2(t) & \cdots & \Pi_2(t) \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \end{bmatrix},$$

where Π_1 and Π_3 are $n \times n$ symmetric matrices.

ii) For $i > 1$, $\boldsymbol{P}_i(t) = J_{1i}^T \boldsymbol{P}_1(t) J_{1i}$ (i.e., use simultaneous row and column exchange).

We may write an ODE system of the form

$$\begin{bmatrix} \dot{\Pi}_1 \\ \dot{\Pi}_2 \\ \dot{\Pi}_3 \end{bmatrix} = \Psi_N(\Pi_1, \Pi_2, \Pi_3).$$

Main issue:

- As $N \to \infty$, Π_2 and Π_3 will vanish.
- Directly taking $N \to \infty$ in the ODE causes the loss of useful information; it's overkill.

Strategy: **re-scaling**

- Define $\Lambda_1^N = \Pi_1$, $\Lambda_2^N = N\Pi_2$ and $\Lambda_3^N = N^2\Pi_3$.
- We obtain 3 equations for $(\Lambda_1^N, \Lambda_2^N, \Lambda_3^N)$; see next page.
- Check the limit of the **new vector field**.

$$\begin{cases} \dot{\Lambda}_1^N = \Lambda_1^N M \Lambda_1^N - (\Lambda_1^N A + A^T \Lambda_1^N) - Q + g_1(1/N, \Lambda_1^N, \Lambda_2^N), \\ \Lambda_1^N(T) = (I - \frac{\Gamma_f^T}{N}) Q_f (I - \frac{\Gamma_f}{N}), \end{cases}$$

$$\begin{cases} \dot{\Lambda}_2^N = \Lambda_1^N M \Lambda_2^N + \Lambda_2^N M \Lambda_1^N + \Lambda_2^N M \Lambda_2^N \\ \qquad -(\Lambda_1^N G + \Lambda_2^N (G + A) + A^T \Lambda_2^N) + Q\Gamma + g_2(1/N, \Lambda_2^N, \Lambda_3^N), \\ \Lambda_2^N(T) = -(I - \frac{\Gamma_f^T}{N}) Q_f \Gamma_f, \end{cases}$$

$$\begin{cases} \dot{\Lambda}_3^N = (\Lambda_2^N)^T M \Lambda_2^N + \Lambda_3^N M \Lambda_1^N + \Lambda_1^N M \Lambda_3^N + \Lambda_3^N M \Lambda_2^N + (\Lambda_2^N)^T M \Lambda_3^N \\ \qquad -((\Lambda_2^N)^T G + G^T \Lambda_2^N + \Lambda_3^N G + G^T \Lambda_3 + \Lambda_3^N A + A^T \Lambda_3^N) \\ \qquad -\Gamma^T Q \Gamma + g_3(1/N, \Lambda_2^N, \Lambda_3^N), \\ \Lambda_3^N(T) = \Gamma_f^T Q_f \Gamma_f. \end{cases}$$

g_1, g_2, g_3 are "small" error terms.
Taking $N \to \infty$ leads to the construction $\longrightarrow$

Let $M = BR^{-1}B^T$.

The symmetric Riccati ODEs (always having a solution):

$$\begin{cases} \dot{\Lambda}_1 = \Lambda_1 M \Lambda_1 - (\Lambda_1 A + A^T \Lambda_1) - Q, \\ \Lambda_1(T) = Q_f, \end{cases}$$

The non-symmetric Riccati ODE:

$$\begin{cases} \dot{\Lambda}_2 = \Lambda_1 M \Lambda_2 + \Lambda_2 M \Lambda_1 + \Lambda_2 M \Lambda_2 \\ \qquad\qquad -(\Lambda_1 G + \Lambda_2(A+G) + A^T \Lambda_2) + Q\Gamma, \\ \Lambda_2(T) = -Q_f \Gamma_f. \end{cases}$$

Finally,

$$\begin{cases} \dot{\Lambda}_3 = \Lambda_2^T M \Lambda_2 + \Lambda_3 M \Lambda_1 + \Lambda_1 M \Lambda_3 + \Lambda_3 M \Lambda_2 + \Lambda_2^T M \Lambda_3 \\ \qquad - \left(\Lambda_2^T G + G^T \Lambda_2 + \Lambda_3(A+G) + (A^T + G^T)\Lambda_3\right) - \Gamma^T Q \Gamma, \\ \Lambda_3(T) = \Gamma_f^T Q_f \Gamma_f. \end{cases}$$

If Λ_2 exists on $[0, T]$, so does Λ_3. The second equation is crucial!

Theorem The sequence of N player Nash games, $N \geq 2$, has asymptotic solvability **if and only if** Λ_2 has a unique solution on $[0, T]$.

Proof. View the ODE of $(\Lambda_1^N, \Lambda_2^N, \Lambda_3^N)$ as a slightly perturbed version of the ODE of $(\Lambda_1, \Lambda_2, \Lambda_3)$; existence in the latter is determined by that of Λ_2.

Theorem If Λ_2 has a solution on $[0, T]$, then

$$\sup_{0 \leq t \leq T} (|\Pi_1 - \Lambda_1| + |N\Pi_2 - \Lambda_2| + |N^2\Pi_3 - \Lambda_3|) = O(1/N).$$

Recall:

$$\boldsymbol{P}_1(t) = \begin{bmatrix} \Pi_1(t) & \Pi_2(t) & \Pi_2(t) & \cdots & \Pi_2(t) \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Pi_2^T(t) & \Pi_3(t) & \Pi_3(t) & \cdots & \Pi_3(t) \end{bmatrix},$$

Consider

$$\dot{x} = f(t,x), \quad x(0) = z \in \mathbb{R}^K,$$
$$\dot{y} = f(t,y) + g(\epsilon, t, y), \quad y(0) = z_\epsilon \in \mathbb{R}^K, \quad 0 < \epsilon \leq 1.$$

Let $\phi(t,x) = f(t,x)$, or $f(t,x) + g(\epsilon,t,x)$.
A1) $\sup_{\epsilon, 0\leq t\leq T} |f(t,0)| + |g(\epsilon,t,0)| \leq C_1$.
A2) $\phi(\cdot, x)$ is Lebesgue measurable for each fixed $x \in \mathbb{R}^K$.
A3) For each $t \in [0,T]$, $\phi(t,x) : \mathbb{R}^K \to \mathbb{R}^K$ is locally Lipschitz in x, uniformly with respect to (t,ϵ), i.e., for any fixed $r > 0$, and $x, y \in B_r(0)$ which is the open ball of radius r centering 0,

$$|\phi(t,x) - \phi(t,y)| \leq \mathrm{Lip}(r)|x - y|,$$

where $\mathrm{Lip}(r)$ depends only on r, not on $\epsilon \in (0,1], t \in [0,T]$.
A4) For each fixed $r > 0$,

$$\lim_{\epsilon\to 0} \sup_{0\leq t\leq T, y\in B_r(0)} |g(\epsilon,t,y)| = 0, \quad \lim_{\epsilon\to 0} |z_\epsilon - z| = 0.$$

$$\dot{x} = f(t, x), \quad x(0) = z \in \mathbb{R}^K, \tag{1.1}$$

$$\dot{y} = f(t, y) + g(\epsilon, t, y), \quad y(0) = z_\epsilon \in \mathbb{R}^K. \tag{1.2}$$

If the solutions to (1.1) and (1.2), denoted by $x^z(t)$ and $y^\epsilon(t)$, exist on $[0, T]$, they are unique by the local Lipschitz condition; in this case denote $\delta_\epsilon = \int_0^T |g(\epsilon, \tau, x^z(\tau))| d\tau$, which converges to 0 as $\epsilon \to 0$ due to A4).

Theorem i) If (1.1) has a solution $x^z(t)$ on $[0, T]$, then there exists $0 < \bar{\epsilon} \leq 1$ such that for all $0 < \epsilon \leq \bar{\epsilon}$, (1.2) has a solution $y^\epsilon(t)$ on $[0, T]$ and

$$\sup_{0 \leq t \leq T} |y^\epsilon(t) - x^z(t)| = O(|z_\epsilon - z| + \delta_\epsilon).$$

ii) Suppose there exists a sequence $\{\epsilon_i, i \geq 1\}$ where $0 < \epsilon_i \leq 1$ and $\lim_{i\to\infty} \epsilon_i = 0$ such that (1.2) with $\epsilon = \epsilon_i$ has a solution y^{ϵ_i} on $[0, T]$ and $\sup_{i\geq 1, 0\leq t\leq T} |y^{\epsilon_i}(t)| \leq C_2$ for some constant C_2. Then (1.1) has a solution on $[0, T]$.

The fixed point approach

- Take a certain abstraction/approximation of the model from the beginning
- Then a typical agent places itself in a macroscopic environment of decision-making

Step 1. Assume $\overline{X} \in C([0,T], \mathbb{R}^n)$ were given to approximate $X^{(N)}$ in the N player game and consider the optimal control problem

$$dX_i^\infty(t) = (AX_i^\infty(t) + Bu_i(t) + G\overline{X}(t))dt + DdW_i, \quad X_i^\infty(0) = X_i(0)$$

$$\bar{J}_i(u_i) = E\int_0^T (|X_i^\infty - \Gamma\overline{X} - \eta|_Q^2 + u_i^T Ru_i)dt + E|X_i^\infty(T) - \Gamma_f\overline{X}(T) - \eta_f|_{Q_f}^2,$$

Optimal control law: $\hat{u}_i = -R^{-1}B^T(\Lambda_1 X_i^\infty(t) + s(t)),$

where Riccati ODE solution Λ_1 is the same as in the AS problem,

$$\begin{cases} \dot{s}(t) = -(A^T - \Lambda_1 M)s(t) - \Lambda_1 G\overline{X}(t) + Q(\Gamma\overline{X}(t) + \eta), \\ s(T) = -Q_f(\Gamma_f\overline{X}(T) + \eta_f). \end{cases}$$

Step 2. By the standard **consistency requirement** in MFG theory, set

$$\overline{X}(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} EX_i^{\infty}(t)$$

for $t \in [0, T]$, which amounts to specifying $\overline{X}$ as a **fixed point**.

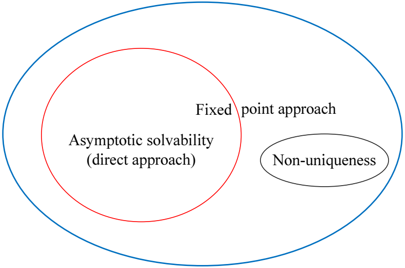
Combining the ODEs of s and the resulting $\overline{X}$ gives the MFG solution equation system

$$\begin{cases} \frac{d\overline{X}}{dt} = (A - M\Lambda_1 + G)\overline{X} - Ms, \\ \dot{s} = -(A^T - \Lambda_1 M)s - \Lambda_1 G\overline{X} + Q(\Gamma\overline{X} + \eta), \end{cases} \tag{2.1}$$

where $\overline{X}(0) = x_0$ and $s(T) = -Q_f(\Gamma_f\overline{X}(T) + \eta_f)$.

This is a two point boundary value (**TPBV**) problem.

Relation of the two approaches –



Theorem Asymptotic solvability implies that the TPBV problem in the fixed point approach has a unique solution.

example of no asymptotic solvability Take the parameters $A = 0.2$, $B = G = Q = R = 1$, $\Gamma = 1.2$, $\Gamma_f = 0$, $Q_f = 0$ and $T = 3$.

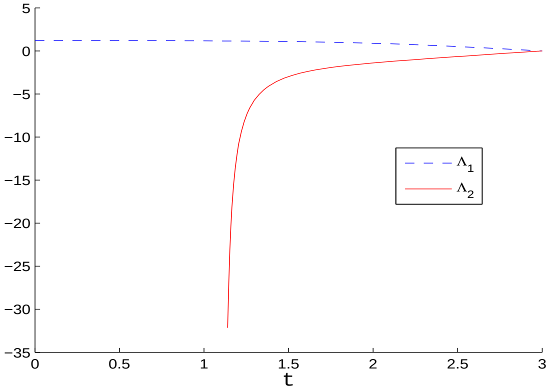


Figure : Λ_2 has a maximal existence interval small than $[0, T]$

Non-uniqueness example Consider the system with

$$A = -\frac{1}{4}, \quad G = \frac{4}{5}, \quad Q = \frac{1}{16}, \quad \Gamma = \frac{4}{3}, \quad \eta = \eta_f = 1.$$

Further take

$$\hat{T} = 33.587095, \quad \hat{x}_0 = -0.500426.$$

- **No asymptotic solvability** since Λ_2 has the maximal existence interval $(0, \hat{T}]$ (smaller than $[0, \hat{T}]$).
- However, the TPBV problem in the fixed point approach has **an infinite number of solutions**.

Long time behaviour

Corresponding to the standard Ricccati ODE

$$\begin{cases} \dot{\Lambda}_1 = \Lambda_1 M \Lambda_1 - (\Lambda_1 A + A^T \Lambda_1) - Q, \\ \Lambda_1(T) = Q_f, \end{cases}$$

we introduce the ARE

$$\Lambda_{1\infty} M \Lambda_{1\infty} - (\Lambda_{1\infty} A + A^T \Lambda_{1\infty}) - Q = 0.$$

There **exists a unique solution** $\Lambda_{1\infty} \geq 0$ under the standard stabilizability and detectability condition which now we assume.

Long time behaviour

Corresponding to the non-symmetric Riccati ODE

$$\begin{cases} \dot{\Lambda}_2 = \Lambda_1 M \Lambda_2 + \Lambda_2 M \Lambda_1 + \Lambda_2 M \Lambda_2 \\ \qquad\qquad -(\Lambda_1 G + \Lambda_2 (A+G) + A^T \Lambda_2) + Q\Gamma, \\ \Lambda_2(T) = -Q_f \Gamma_f, \end{cases}$$

we introduce the algebraic equation

$$\begin{aligned} 0 = &\Lambda_{1\infty} M \Lambda_{2\infty} + \Lambda_{2\infty} M \Lambda_{1\infty} + \Lambda_{2\infty} M \Lambda_{2\infty} \\ &- (\Lambda_{1\infty} G + \Lambda_{2\infty}(A+G) + A^T \Lambda_{2\infty}) + Q\Gamma, \end{aligned}$$

which is a non-symmetric algebraic Riccati equation (NARE).
Main issue now: If there is a real matrix solution, there may be multiple such solutions.

How to select a **solution of interest**?

Idea: impose a certain stability requirement; see e.g. Kremer and Stefan (2002)

Suppose $\Lambda_{2\infty} \in \mathbb{R}^{n\times n}$ is a solution to the NARE

$$\begin{aligned} 0 = &\Lambda_{1\infty} M \Lambda_{2\infty} + \Lambda_{2\infty} M \Lambda_{1\infty} + \Lambda_{2\infty} M \Lambda_{2\infty} \\ &- (\Lambda_{1\infty} G + \Lambda_{2\infty}(A+G) + A^T \Lambda_{2\infty}) + Q\Gamma, \end{aligned}$$

Denote

$$\begin{aligned} A_G &= A - M(\Lambda_{1\infty} + \Lambda_{2\infty}) + G, \\ A_M &= A - M(\Lambda_{1\infty} + \Lambda_{2\infty}^T). \end{aligned}$$

Definition $\Lambda_{2\infty} \in \mathbb{R}^{n\times n}$ is called a **stabilizing solution** of the NARE if it satisfies NARE and both A_G and A_M are Hurwitz.

Motivation of such a stability condition $\longrightarrow$

The AS problem determines –

1) Mean field state dynamics:

$$\frac{d\bar{X}}{dt} = (A - M(\Lambda_1 + \Lambda_2) + G)\,\bar{X} - M\chi_1(t),$$

where $\bar{X}(0) = x_0$ and $\chi_1(t)$ can be explicitly specified by an ODE.
i) At $(\Lambda_{1\infty}, \Lambda_{2\infty})$, want the **forward dynamics** to have stability on $[0, \infty)$

2) The Riccati ODE:

$$\begin{cases} \dot{\Lambda}_2 = \Lambda_1 M \Lambda_2 + \Lambda_2 M \Lambda_1 + \Lambda_2 M \Lambda_2 \\ \qquad\quad -(\Lambda_1 G + \Lambda_2(A + G) + A^T \Lambda_2) + Q\Gamma, \\ \Lambda_2(T) = -Q_f \Gamma_f. \end{cases}$$

ii) Linearizing the vector field at $(\Lambda_{1\infty}, \Lambda_{2\infty})$, want the **backward dynamics** to have stability

i) and ii) motivate the stability requirement in the definition

Denote

$$\mathbb{A}_\infty = \begin{bmatrix} A - M\Lambda_{1\infty} + G & -M \\ Q\Gamma - \Lambda_{1\infty} G & -A^T + \Lambda_{1\infty} M \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$

which may be viewed as a steady state form of $\mathbb{A}(t)$ (coefficient matrix in the TPBV problem in the fixed point approach).

H_g) The eigenvalues of $\mathbb{A}_\infty$ are strong (n, n) c-splitting (i.e., n eigenvalue in OLHP, and n eigenvalues in ORHP) and the associated n-dimensional stable invariant subspace is a **graph subspace** (i.e., spanned by columns of a matrix of the form $\begin{bmatrix} I_n \\ X \end{bmatrix}$).

Theorem

- The NARE has a stabilizing solution $\Lambda_{2\infty}$ **if and only if** H_g) holds.
- If H_g) holds, the NARE has a unique stabilizing solution.

example of a stabilizing solution to NARE We take

$$A = \begin{bmatrix} 1 & 1 \\ -0.5 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; \Gamma = \begin{bmatrix} 0.9 & 0.1 \\ 0 & 0.9 \end{bmatrix}, \; \eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and $G = Q = I_2$, $R = 1$. Then NARE has a stabilizing solution

$$\Lambda_{2\infty} = \begin{bmatrix} 16.238985 & 4.099679 \\ 4.132523 & 1.570208 \end{bmatrix}.$$

In fact, the columns of the matrix

$$\begin{bmatrix} -0.167388 & -0.161703 \\ 0.448957 & 0.742511 \\ -0.877636 & 0.418170 \\ 0.013220 & 0.497657 \end{bmatrix}.$$

span the stable invariant subspace of $\mathbb{A}_\infty$ as a graph subspace. $\mathbb{A}_\infty$ has the eigenvalues

$$-1.022350 \pm 0.730733i, \quad 2.022350 \pm 0.707903i.$$

Summary:

- There are two fundamental approaches for MFGs.
- We formulate an asymptotic solvability problem as an instance of the direct approach
- We examine their relation in the LQ case.
- The re-scaling method can be generalized to other LQ models; in progress.

Thank you!