Ergodic properties of Lévy-driven SDEs arising from multiclass many-server networks, and ergodic control of a class of jump diffusions

Ari Arapostathis

The University of Texas at Austin

Institute of Mathematics and Its Applications
Queueing and Networks
May 17, 2018

(Collaborators: Luis Caffarelli, Guodong Pang, Nikola Sandrić, and Yi Zheng)

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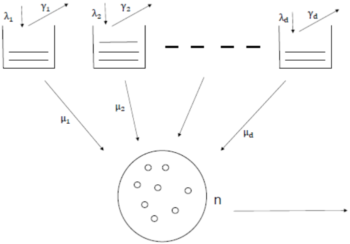
Multiclass queueing networks



- A. Arapostathis, A. Biswas, and G. Pang, "Ergodic control of multi-class $M/M/N+M$ queues in the Halfin-Whitt regime," *Ann. Appl. Probab.* **25** (2015), pp. 3511–3570.

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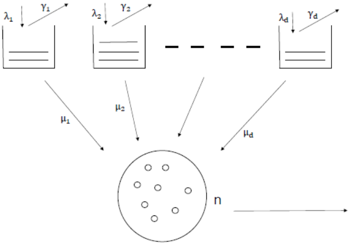

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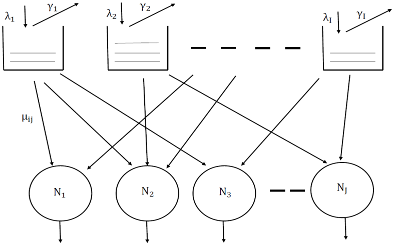
- A. Arapostathis, G. Pang, and N. Sandrić, "Ergodicity of Lévy-driven SDEs arising from multiclass many-server queues," arXiv: 1707.09674.
- A. Arapostathis, L. Caffarelli, G. Pang, and Y. Zheng "Ergodic control of a class of jump diffusions with finite Lévy measures and rough kernels," arXiv: 1801.07669.

Ergodic control of multiclass multi-pool (Halfin–Whitt regime)



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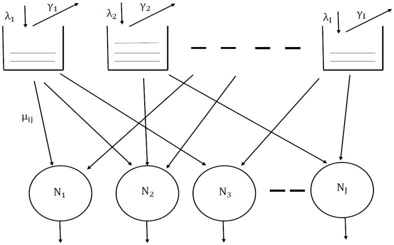

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- A. Arapostathis and G. Pang, "Infinite horizon average optimality of the N-network in the Halfin–Whitt regime," *Math. Oper. Res.* (2018) (articles in advance).
- A. Arapostathis and G. Pang, "Infinite horizon asymptotic average optimality for large-scale parallel server networks," *Stochastic Process. Appl.* (2018) (in press).

Multiclass many server networks with busty arrivals

- $G/M/n+M$ queues with d classes of customers, exponential service and abandonment rates, μ_i and γ_i.
- Arrival Process: A_i^n, $i = 1, \dots, d$, with arrival rate λ_i^n, and mutually independent.

Define $\widehat{A}_i^n := n^{-1/\alpha}(A_i^n - \lambda_i^n \varpi)$, where $\varpi(t) \equiv t$ for each $t \geq 0$, and $\alpha \in (1, 2]$. Assume that the arrival processes satisfy an FCLT

$$\widehat{A}^n \;\Rightarrow\; A = (A_1, \dots, A_d)' \qquad \text{in } (D^d, M_1), \text{ as } n \to \infty,$$

with A_i mutually independent symmetric α-stable processes with $A_i(0) \equiv 0$,

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(Modified) Halfin-Whitt regime:

$$\lambda_i^n/n \ \to \ \lambda_i \ > \ 0, \quad \text{and} \quad \hat{\ell}_i^n \ := \ n^{-1/\alpha}(\lambda_i^n - n\lambda_i) \ \to \ \ell_i \in \mathbb{R}\,,$$

With $\rho^n := \sum_{i=1}^d \frac{\lambda_i^n}{n\mu_i}$ the aggregate traffic intensity, we have

$$n^{1-1/\alpha}(1 - \rho^n) \ \to \ \rho \ = \ -\sum_{i=1}^{d} \frac{\ell_i}{\mu_i} \quad \text{as } n \to \infty\,,$$

where $\rho^n := \sum_{i=1}^d \frac{\lambda_i^n}{n\mu_i}$ is the aggregate traffic intensity.

Multiclass many server networks with busty arrivals (cont.)

- $X_i^n(t)$: the number of class-i customers
- Diffusion-scaled process: $\widehat{X}_i^n := n^{-1/\alpha}(X_i^n - \rho_i n)$

Theorem (SDE limit)

Under a fixed constant scheduling control $v \in \Delta$ (= set of probability vectors of dimension d), provided there exists $X(0)$ such that $\widehat{X}^n(0) \Rightarrow X(0)$ as $n \to \infty$, then we have

$$\widehat{X}^n \Rightarrow X \qquad \text{in} \quad (D^d, M_1) \quad \text{as } n \to \infty,$$

where the limit process X is a unique strong solution to the SDE

$$\mathrm{d}X(t) = b(X(t), v)\,\mathrm{d}t + \mathrm{d}A(t) - \sigma_\alpha\,\mathrm{d}W(t),$$

with an initial condition $X(0)$. The drift $b(x, v)\colon \mathbb{R}^d \times \Delta \to \mathbb{R}^d$ takes the form

$$b(x, v) = \ell - R(x - \langle e, x\rangle^+ v) - \langle e, x\rangle^+ \Gamma v,$$

with $e = (1, \dots, 1)' \in \mathbb{R}^d$, $R = \mathrm{diag}(\mu_1, \dots, \mu_d)$, and $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_d)$. Also, A is the limit of the arrival process, W is a standard d-dimensional Brownian motion, independent of $\widehat{A}$, and $\sigma_\alpha \sigma_\alpha' = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$ if $\alpha = 2$, while $\sigma_\alpha = 0$ if $\alpha \in (1, 2)$.

Multiclass many server networks with service interruptions

- Multiclass $G/M/n+M$ queues in the same renewal alternating (up-down, or on-off) random environment, where all the classes of customers are affected simultaneously.
- Assume that the system functions normally during up time periods, and all servers stop functioning during down periods, while customers continue entering the system and may abandon while waiting in queue and those that have started service will wait for the system to resume.

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Let $\{(u_k^n, d_k^n)\colon k \in \mathbb{N}\}$ be a sequence of i.i.d. positive random vectors representing the up-down cycles. Assume that

$$\big\{(u_k^n, n^{1/\alpha} d_k^n) : k \in \mathbb{N}\big\} \;\Rightarrow\; \big\{(u_k, d_k) : k \in \mathbb{N}\big\} \qquad \text{in } (\mathbb{R}^2)^\infty \text{ as } n \to \infty\,,$$

where (u_k, d_k), $k \in \mathbb{N}$, are i.i.d. positive random vectors and $\alpha \in (1,2]$.
Let $N(t) := \max\{k \geq 0\colon T_k \leq t\}$, $t \geq 0$, with $T_k := \sum_{i=1}^k u_i$ for $k \in \mathbb{N}$, and $T_0 \equiv 0$.
Assume that the process $\{N(t)\}_{t\geq 0}$ is Poisson.

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Similarly, we obtain a limit

$$\mathrm{d}X(t) \;=\; b(X(t), v)\,\mathrm{d}t + \mathrm{d}A(t) - \sigma_\alpha\,\mathrm{d}W(t) + c\,\mathrm{d}J(t)\,.$$

Also, $c = (\lambda_1, \ldots, \lambda_d)'$, and the process J is a compound Poisson process, defined by

$$J(t) \;:=\; \sum_{k=1}^{N(t)} d_k\,, \qquad t \geq 0\,.$$

A class of piecewise linear Lévy-driven SDEs

Consider a d-dimensional stochastic differential equation (SDE) of the form

$$\mathrm{d}X(t) \,=\, b(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t) + \mathrm{d}L(t), \qquad X(0) = x \in \mathbb{R}^d\,,$$

where

- the function $b\colon \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$b(x) \,=\, \ell - M(x - \langle e, x\rangle^+ v) - \langle e, x\rangle^+ \Gamma v \,=\, \begin{cases} \ell - \bigl(M + (\Gamma - M)ve'\bigr)x\,, & e'x > 0\,,\\ \ell - Mx\,, & e'x \le 0\,,\end{cases}$$

  Here, $M \in \mathbb{R}^{d\times d}$ is a nonsingular M-matrix[1] such that the vector $e'M$ has nonnegative components.
- $\{W(t)\}_{t\ge 0}$ is a standard n-dimensional Brownian motion, and the covariance function $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times n}$ is locally Lipschitz and satisfies, for some constant $\kappa > 0$,

$$\|\sigma(x)\|^2 \le \kappa(1 + |x|^2)\,, \qquad x \in \mathbb{R}^d\,;$$

- $\{L(t)\}_{t\ge 0}$ is a d-dimensional pure-jump Lévy process determined by a drift $\vartheta \in \mathbb{R}^d$ and Lévy measure $\nu(\mathrm{d}y)$.
- We also consider Markov controls $v\colon \mathbb{R}^d \to \Delta$.

[1] i.e., $M = s\mathbb{I} - N$, with $\sigma(N) < s$, N nonnegative

1. Heaviness of the tail of the Lévy measure:

$$\Theta_c \;:=\; \Bigl\{\theta > 0 : \int_{\mathcal{B}^c} |y|^{\theta} \nu(\mathrm{d}y) \;<\; \infty\Bigr\}, \quad \textit{and} \quad \theta_c \;:=\; \sup\,\bigl\{\theta \in \Theta_c\bigr\}\,.$$

For α-stable, $\Theta_c = (0, \alpha)$, and $\theta_c = \alpha$.

Two important parameters

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2. Effective spare capacity: $\tilde{\varrho} := -\langle e, M^{-1}\tilde{\ell}\rangle$, where

$$\tilde{\ell} := \begin{cases} \ell + \vartheta + \int_{\mathcal{B}^c} y\nu(\mathrm{d}y), & \text{if } \int_{\mathcal{B}^c} |y|\nu(\mathrm{d}y) < \infty \\ \ell + \vartheta, & \text{otherwise.} \end{cases}$$

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Theorem (Necessary and sufficient conditions)

Under no abandonment, or more generally, if the control assigns higher priority to queues that do not abandon ($\Gamma v = 0$), the conditions $\tilde{\varrho} > 0$ and $1 \in \Theta_c$ are necessary and sufficient for the process $\{X(t)\}_{t\geq 0}$ to have an invariant probability measure π under some Markov control. Moreover, if $\tilde{\varrho} < 0$, then $\{X(t)\}_{t\geq 0}$ is always transient.[a] *In addition,*

$$\tilde{\rho} = \int_{\mathbb{R}^d} \langle e, x\rangle^- \pi(\mathrm{d}x).$$

[a] For some reason this is hard to prove for α-stable.

The fractional Laplacian

A naive interpretation: Start with a symmetric density $\varphi(y) \propto |y|^{-(d+\alpha)}$ on $\mathbb{R}^d$, $\alpha \in (0,2)$, and consider a 'long-jump' random walk on the lattice $\mathbb{Z}^n$ dictated by this density. Scaling the space as $\varepsilon\mathbb{Z}^n$, and the time interval $\tau = \varepsilon^\alpha$, and noting that $\frac{\varphi(k)}{\tau} = \varepsilon^d \varphi(\epsilon k)$, $k \in \mathbb{Z}^d$, end letting $\epsilon \searrow 0$, we end up with the limit for the transition density

$$\partial_t p(t,x) = \int_{\mathbb{R}^d} \frac{p(t,x+y) - p(t,x)}{|y|^{d+\alpha}} \, \mathrm{d}y \,.$$

This transition density behaves as

$$p(t,x,y) = p(t,y-x) \sim \frac{t}{|y-x|^{d+\alpha}} \wedge t^{-d/\alpha} \,.$$

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In this manner we obtain the fractional Laplacian operator

$$-(-\Delta)^{\alpha/2} u(x) := \lim_{\epsilon \searrow 0} C(d,\alpha) \int_{\mathbb{R}^d \setminus \mathcal{B}_\epsilon} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}} \, \mathrm{d}y$$

$$= C(d,\alpha) \int_{\mathbb{R}^d} \frac{1}{2} \, \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+\alpha}} \, \mathrm{d}y \,,$$

expressed as the principal value of a singular integral, or as an integral of a second order incremental quotient.

The fractional Laplacian (cont.)

- Note that for $f \in \mathcal{C}^2(\mathbb{R}^d)$, this can also be computed as

$$-(-\Delta)^{\alpha/2} u(x) \;=\; C(d,\alpha) \int_{\mathbb{R}^d \setminus \{0\}} \bigl(f(x+y) - f(x) - \mathbb{1}_{\mathcal{B}}(y)\langle y, \nabla f(x)\rangle\bigr)\,\mathrm{d}y$$

- The normalization constant $C(d,\alpha) \approx d(2-\alpha)$ is such that

$$-(-\Delta)^{\alpha/2} u(x) \xrightarrow[\alpha \nearrow 2]{} \Delta u(x)\,.$$

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- Roughly speaking the fractional Laplacian acts like a derivative of order α.
- As a result, when $\alpha > 1$ (subcritical) it is a higher order operator compared to the drift, while if $\alpha < 1$ (supercritical) the opposite is the case.

Anisotropic α-stable kernel

For a system under heavy-tailed arrivals, the driving process is an anisotropic Lévy process with independent one-dimensional symmetric α-stable components, i.e., generated by

$$\mathcal{L}f(x) = (2-\alpha)\sum_{i=1}^{d}\int_{\mathbb{R}_*}\mathfrak{d}_1 f(x;y_i e_i)\,\frac{\eta_i\,\mathrm{d}y_i}{|y_i|^{1+\alpha}} = \int_{\mathbb{R}^d_*}\mathfrak{d}_1 f(x;y)\,\nu(\mathrm{d}y)\,,$$

where η_i's are constants, and

$$\mathfrak{d}_1 f(x;y) := f(x+y) - f(x) - \mathbb{1}_{\mathcal{B}}(y)\,\langle y, \nabla f(x)\rangle\,, \qquad f \in C^2(\mathbb{R}^d)\,.$$

As a result, the Lévy measure is highly singular, and lacks the regularity of the isotropic α-stable. In particular, as shown in Bass & Chen (2006, 2010), the Harnack inequality fails for this operator.

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Nevertheless, we have shown the following.

Theorem (irreducibility)

Suppose $\{L(t)\}_{t\ge0}$ is of the form $L(t)=L_1(t)+L_2(t)$, $t\ge0$, where $\{L_1(t)\}_{t\ge0}$ and $\{L_2(t)\}_{t\ge0}$ are independent d-dimensional pure-jump Lévy processes, such that $\{L_1(t)\}_{t\ge0}$ is an anisotropic Lévy process with independent symmetric one-dimensional α-stable components for $\alpha\in(0,2)$, and $\{L_2(t)\}_{t\ge0}$ is a compound Poisson process. Then the solution $\{X(t)\}_{t\ge0}$ of the SDE is open-set irreducible and aperiodic.

Extended generator

In summary, the extended generator of $\{X(t)\}_{t\geq 0}$ takes the form

$$\mathcal{A}f(x) = \frac{1}{2}\operatorname{Tr}\bigl(a(x)\nabla^2 f(x)\bigr) + \bigl\langle b(x) + \vartheta, \nabla f(x)\bigr\rangle + \int_{\mathbb{R}^d_*} \mathfrak{d}_1 f(x;y)\nu(\mathrm{d}y)\,,$$

with ∇^2 denoting the Hessian of f.

- Here ν is the Lévy measure $\nu(\mathrm{d}y)$. This is a σ-finite measure on $\mathbb{R}^d_* := \mathbb{R}^d \setminus \{0\}$ satisfying $\int_{\mathbb{R}^d_*}(1 \wedge |y|^2)\,\nu(\mathrm{d}y) < \infty$.

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- Recall that

$$\mathfrak{d}_1 f(x;y) := f(x+y) - f(x) - \mathbb{1}_{\mathcal{B}}(y)\bigl\langle y, \nabla f(x)\bigr\rangle\,, \qquad f\in C^2(\mathbb{R}^d)\,.$$

and define

$$\mathfrak{d} f(x;y) := f(x+y) - f(x) - \bigl\langle y, \nabla f(x)\bigr\rangle\,.$$

Write

$$\int_{\mathbb{R}^d_*} \mathfrak{d}_1 f(x;y)\nu(\mathrm{d}y) = \underbrace{\int_{\mathbb{R}^d_*} \mathfrak{d} f(x;y)\,\nu(\mathrm{d}y)}_{\mathfrak{J}_\nu f(x)} + \Biggl\langle \underbrace{\biggl(\int_{\mathcal{B}^c} y\nu(\mathrm{d}y)\biggr)}_{\text{we absorb this in the drift}}, \nabla f(x)\Biggr\rangle,$$

assuming from now on that $1\in\Theta_c$.

Extended generator (cont.)

- After this rearrangement, the constant term in the drift is $\tilde{\ell}$ (we rename the drift as $\tilde{b}$). Recall that $\tilde{\varrho} = -\langle e, M^{-1}\tilde{\ell}\rangle$ is the (effective) spare capacity.

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Some important properties of the operator $\mathfrak{J}_\nu$.
Two useful expansions of the singular integral over a domain $D \subset \mathbb{R}^d$:

$$\begin{aligned}\int_{D\setminus\{0\}} \mathfrak{d}f(x;y)\,\nu(\mathrm{d}y) &= \int_{D\setminus\{0\}} \left(\int_0^1 (1-t)\langle y, \nabla^2 f(x+ty)y\rangle\,\mathrm{d}t\right)\nu(\mathrm{d}y)\\ &= \int_{D\setminus\{0\}} \left(\int_0^1 \langle y, \nabla f(x+ty) - \nabla f(x)\rangle\,\mathrm{d}t\right)\nu(\mathrm{d}y)\,.\end{aligned}$$

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- Recall the definition

$$\Theta_c := \left\{\theta > 0 : \int_{\mathcal{B}^c} |y|^\theta \nu(\mathrm{d}y) < \infty\right\}, \quad \textit{and} \quad \theta_c := \sup\left\{\theta \in \Theta_c\right\}.$$

  So if $\theta_c \in (0,\infty)$, then $\Theta_c = (0,\theta_c)$, or $\Theta_c = (0,\theta_c]$.
- From the above expansion of the integral we have
  1. $\mathfrak{J}_\nu$ acts like a derivative of order θ_c (slightly better than that).
  2. If f is convex then $\mathfrak{J}_\nu f \geq 0$.

Extended generator (cont.)

More precisely:

If $\theta \in \Theta_c$, and $f \in C^2(\mathbb{R}^d)$ satisfies

$$\sup_{|x|\geq 1} |x|^{1-\theta} \max\left(|\nabla f(x)|, |x|\,\|\nabla^2 f(x)\|\right) \;<\; \infty\,,$$

then the function $\mathfrak{J}_\nu f(x)$ vanishes at infinity when $\theta \in [1,2)$, and $x \mapsto (1+|x|)^{2-\theta}\, \mathfrak{J}_\nu f(x)$ is bounded when $\theta \geq 2$.

Common Lyapunov functions

Recall that if $\Gamma v = 0$, then

$$\tilde{b}(x) = \begin{cases} \tilde{\ell} - M(\mathbb{I} - ve')x\,, & e'x > 0\,,\\ \tilde{\ell} - Mx\,, & e'x \le 0\,, \end{cases}$$

Dieker & Gao (2013) show that $\exists$ a positive definite matrix Q such that

$$QM + M'Q \succ 0\,, \qquad \text{and} \qquad QM(\mathbb{I} - ve') + (\mathbb{I} - ev')M'Q \succeq 0\,.$$

Using this, they assert positive recurrence in the Brownian case, but do not study the rate of convergence (which in the Brownian case it turns out to be exponential).

Common Lyapunov functions

Recall that if $\Gamma v = 0$, then

$$\tilde{b}(x) = \begin{cases} \tilde{\ell} - M(\mathbb{I} - ve')x\,, & e'x > 0\,, \\ \tilde{\ell} - Mx\,, & e'x \le 0\,, \end{cases}$$

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- Employing $V_{Q,\theta} := \langle x, Qx\rangle^{\theta/2}$, with $\theta \ge 1$, as Lyapunov function, after some careful calculations, we obtain

$$\bigl\langle \tilde{b}(x), \nabla V_{Q,\theta}(x)\bigr\rangle \le \kappa_0 \mathbb{1}_{\mathcal{B}}(x) - \kappa_1 V_{Q,\theta}(x)\,\mathbb{1}_{\mathcal{K}_\delta^c}(x) - \kappa_1 V_{Q,\theta-1}(x)\,\mathbb{1}_{\mathcal{K}_\delta}(x)\,.$$

  Here, $\mathcal{K}_\delta$ is the convex cone $\mathcal{K}_\delta := \bigl\{x \in \mathbb{R}^d : \langle e, x\rangle > \delta|x|\bigr\}$, for some $\delta > 0$.

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- We have already seen that $\mathfrak{J}_\nu V_{Q,\theta}$ vanishes at infinity, if $\theta \in \Theta_c$.
- Assuming that $a(x)$ has sublinear growth, then $\mathrm{Tr}\bigl(a(x)\nabla^2 f(x)\bigr)$ grows slower than $|x|^{\theta-1}$, so it is small compared to $\bigl\langle \tilde{b}(x), \nabla V_{Q,\theta}(x)\bigr\rangle$.
- Combining these we have

$$\mathcal{A} V_{Q,\theta}(x) \le c_0(\theta) - c_1 V_{Q,\theta}(x)\mathbb{1}_{\mathcal{K}^c_\delta}(x) - c_1 V_{Q,\theta-1}(x)\mathbb{1}_{\mathcal{K}_\delta}(x)\,.$$

  Here, $c_0(\theta)$, c_1, and δ are constants.

Brief review of subgeometric ergodicity

- Early work: Stone & Waigner (1967), Orey (1971), Lindvall (1979), Ney (1981), Nummelin & Tuominen (1983), Tuominen & Tweedie (1994)
- Foster–Lyapunov criteria: Connor & Fort (2009), Douc, Fort, and Moulines (2004, 2009), Fort & Roberts (2005), Locherbach (2016)
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The following result is obtained via Foster–Lyapunov criteria.

A simple upper bound

Assuming irreducibility for some skeleton chain, then the Foster–Lyapunov equation

$$\mathcal{A}V(x) \;\le\; \kappa \mathbb{1}_K(x) - V^{\rho}(x)\,,$$

where $\rho \in (0,1)$, κ is a constant, and K is a closed petite set, implies that

$$\big\|P_t(x,\cdot\,) - \pi(\,\cdot\,)\big\|_{\mathrm{TV}} \;\le\; C\, t^{\frac{-\rho}{1-\rho}}\, V(x)\,.$$

Upper bounds on the rate of convergence

Theorem (upper bound)

For any $v \in \Delta$ (with $\Gamma v = 0$), the process $\{X(t)\}_{t \geq 0}$ admits a unique invariant probability measure $\pi \in \mathcal{P}(\mathbb{R}^d)$, and

$$\left\| P_t(x, \cdot\,) - \pi(\,\cdot\,) \right\|_{\mathrm{TV}} \;\leq\; \tilde{C}_2(\epsilon)\,(t \vee 1)^{1+\epsilon-\theta_c}\,|x|^{\theta_c - \epsilon}\,, \qquad \epsilon \in (0, \theta_c - 1)\,.$$

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- If $\Gamma v \neq 0$, and M is diagonal, then we obtain the Foster–Lyapunov equation

$$\mathcal{A} V_{Q,\theta}(x) \;\leq\; c_0(\theta) - c_1\,V_{Q,\theta}(x)\,,$$

which implies exponential ergodicity.

Turning the page

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- The initial reaction was: This is easy! All that one needs is to show that the recurrence times to an open ball do not have exponential moments, i.e. $\mathbb{E}[\mathrm{e}^{\epsilon\tau}] = \infty$, for all $\epsilon > 0$. And indeed, this can be shown.

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- But then we changed the question to: What is the **tightest** lower bound one can get?

Lower bounds on the rate of convergence

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Theorem (Hairer)

Let X_t be a Markov process on a Polish space $\mathcal{X}$ with invariant measure π, and let $G\colon \mathcal{X} \to [1,\infty)$ be such that

- $\exists\, f\colon [1,\infty) \to [0,1]$ *such that* $s\,f(s) \nearrow \infty$ *as* $s \to \infty$, *and* $\pi(\{x\colon\ G(x) \ge s\}) \ge f(s)$ *for all* s.
- $\exists\, g(x,t)\colon\ \mathcal{X} \times \mathbb{R}_+ \to [1,\infty)$, *increasing in its second argument, such that* $\mathbb{E}_x\bigl[G(X_t) \mid X_0 = x_0\bigr] \le g(x_0,t)$.

Then, with $y(t)$ *a solution of* $y(t)f\bigl(y(t)\bigr) = 2g(x_0,t)$, *we have*

$$\bigl\|P_t(x_0,\,\cdot\,) - \pi\bigr\|_{\mathrm{TV}} \;\ge\; \frac{1}{2}f\bigl(y(t)\bigr)\,.$$

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- The proof is only one line long, using the definition of the total variation distance and Markov's inequality. So one might be inclined to discount it as giving very conservative estimates.

Another look at the Foster–Lyapunov equation

$$\mathcal{A}|x|^{\theta} \;\le\; c_0(\theta) - c_1|x|^{\theta}\mathbb{1}_{\mathcal{K}_\delta^c}(x) - c_1|x|^{\theta-1}\mathbb{1}_{\mathcal{K}_\delta}(x)\,.$$

1. With $G(x) \sim |x|^{\theta}$ for $\theta < \alpha$, we can estimate $\mathbb{E}_x[G(X(t)]$.
2. The difficult part is to obtain an estimate for the tail of π.
3. The answer to this: Move in the direction of the spare capacity: i.e., choose instead $G(x) = \langle e, M^{-1}x\rangle^{\alpha-\epsilon}$.

Lower bounds for the Lévy driven SDE

Theorem (lower bounds)

The diffusion limit process $\{X(t)\}_{t\geq 0}$ *is polynomially ergodic, and its rate of convergence is* $r(t) \approx t^{\theta_c - 1}$*. In particular, in the case of an* α*-stable process (isotropic or not), we obtain the following quantitative bounds. There exist positive constants* $\tilde{C}_1$*, and* $\tilde{C}_2(\epsilon)$ *such that for all* $\epsilon \in (0, \alpha - 1)$*, we have*

$$\tilde{C}_1 \Big(\frac{t}{\epsilon} + |x|^{\alpha-\epsilon}\Big)^{\frac{1-\alpha}{1-\epsilon}} \leq \big\| P_t(x, \cdot\,) - \pi(\,\cdot\,) \big\|_{\mathsf{TV}} \leq \tilde{C}_2(\epsilon)(t \vee 1)^{1+\epsilon-\alpha}|x|^{\alpha-\epsilon}$$

for all $t > 0$*, and all* $x \in \mathbb{R}^d$*.*

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for all $t > 0$, and all $x \in \mathbb{R}^d$.

In the case of the Lévy process, there exists a positive constant $\tilde{C}_3(\epsilon)$ such that for all $\epsilon \in (0, \frac{1}{3})$, and all $x \in \mathbb{R}^d$, we have

$$\tilde{C}_3(\epsilon) \left(t_n + |x|^{\theta_c - \epsilon}\right)^{-\frac{\theta_c - 1 + 2\epsilon}{1 - 3\epsilon}} \leq \left\| P_t(x, \cdot\,) - \pi(\,\cdot\,) \right\|_{\mathsf{TV}} \leq \tilde{C}_2(\epsilon)(t \vee 1)^{1+\epsilon-\theta_c} |x|^{\theta_c - \epsilon}.$$

for some sequence $\{t_n\}_{n\in\mathbb{N}} \subset [0, \infty)$, $t_n \to \infty$, depending on x.

Polynomial ergodicity is rather generic

- Abstracting the structural property of the drift under no abandonment, we obtain the same results for general drifts.

Corollary (general drifts)

Consider the following structural hypotheses (for a positive symmetric matrix Q).

(H1) $\limsup_{|x|\to\infty} \frac{\langle b(x), Qx\rangle}{|x|^{1+\theta}} < 0$, *for some* $\theta \in [0,1)$, *and* $a(x)$ *has slower than polynomial growth of order* $1+\theta$.

(H2) *For some constant* $\gamma \in [0,1)$, *one of the following hold.*

(i) *There exists some* $x_0 \in \mathbb{R}^d$ *and a positive constant* C, *such that*

$$\langle z_0, b(x)\rangle \;\ge\; -C\bigl(1+\langle z_0, x\rangle^{\gamma}\bigr)\,, \qquad \langle z_0, x\rangle \ge 0\,.$$

(ii) *There exists a positive constant* C, *such that*

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Then

$$\tilde{C}_1\Bigl(\frac{t}{\epsilon} + |x|^{\alpha-\epsilon}\Bigr)^{\frac{1-\gamma-\alpha}{1-\gamma-\epsilon}} \;\le\; \bigl\|\delta_x P^X_t(\mathrm{d}y) - \pi(\mathrm{d}y)\bigr\|_{\mathrm{TV}} \;\le\; \tilde{C}_2(\epsilon)\,(t\vee 1)^{\frac{1+\epsilon-\alpha-\theta}{1-\theta}}\,|x|^{\alpha-\epsilon}\,.$$

Note that necessarily $\gamma \ge \theta$ *by hypothesis.*

Existence of moments

I. Systems with no abandonment

- Under any Markov control, $\int_{\mathbb{R}^d} |x|^{\theta-1}\,\pi(\mathrm{d}x) = \infty$ for $\theta \notin \Theta_c$.
- This implies that under heavy-tailed arrivals there exists no stabilizing control.

II. Systems with abandonment

- Under any Markov control, $\int_{\mathbb{R}^d} |x|^{\theta}\,\pi(\mathrm{d}x) = \infty$ for $\theta \notin \Theta_c$.
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A conjecture

Provided $\tilde{\varrho} > 0$ and $1 \in \Theta_c$, the process $\{X(t)\}_{t\geq 0}$ is polynomially ergodic under any Markov control.

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- For the Markovian "V" model, Gamarnik & Stolyar (2012) have established geometric ergodicity for the prelimit, under any work conserving scheduling policy.

Ergodic control of a related class of jump diffusions

- In the study of ergodic control problems there is a certain gap between the stochastic control and PDE communities:
- The PDE community is investigating equations of the 'ergodic type', sometimes without connecting them to a stochastic interpretation, while the stochastic control community is often restricting the class of admissible controls to guarantee that the martingale problem is well posed, thus restricting the breadth of optimality.

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- A way to bridge this gap is to address the primal optimization problem, over infinitesimal ergodic measures.
- This has been considered in Fleming & Vermes (1989) for the discounted cost, Stockbridge (1990), Bhatt & Borkar (1996).
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- Given the recent advances in PDE theory and techniques, we revisit the linear programming formulation.
- The goal is to state the ergodic control problem for the operator $\mathcal{A}$ as a convex optimization subject to an elliptic equation for measures.
- Then derive the Hamilton–Jacobi–Bellman (HJB) equation and establish verification of optimality results via analytical methods, without assuming that the martingale problem for $\mathcal{A}$ is well posed.

The controlled operator

$$\mathcal{A}_z u(x,z) \;:=\; \sum_{i,j} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_i b^i(x,z)\frac{\partial u}{\partial x_i}(x) + \int_{\mathbb{R}^d} \bigl(u(x+y) - u(x) - \mathbb{1}_{\{|y|\le 1\}}\langle y, \nabla u(x)\rangle\bigr)\,\nu_x(\mathrm{d}y)\,.$$

- z is a control parameter that lives in a compact metric space $\mathcal{Z}$.

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- $\nu_x(\mathrm{d}y)$ is a finite Borel measure on $\mathbb{R}^d$ for each x.
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- $d \ge 2$.
- The coefficients of $\mathcal{A}$ are assumed to satisfy the following.

Assumption

(a) The matrix $a = [a^{ij}]$ is symmetric, positive definite, and locally Lipschitz continuous. The drift $b\colon \mathbb{R}^d \times \mathcal{Z} \to \mathbb{R}^d$ is continuous.

(b) The map $x \mapsto \boldsymbol{\nu}(x) := \nu_x(\mathbb{R}^d)$ is locally bounded.

(c) the map $x \mapsto \nu_x(K - x)$ is bounded on $\mathbb{R}^d$ for any fixed compact set $K \subset \mathbb{R}^d$.

Some notation

- Let $\mathcal{B}(\mathbb{R}^d, \mathcal{Z})$ denote the set of Borel measurable maps $v\colon \mathbb{R}^d \to \mathcal{Z}$. Such a map v is called a *stationary Markov control*, and we use the symbol $\mathfrak{U}_{\mathrm{sm}}$ to denote this class of controls.
- For $v \in \mathfrak{U}_{\mathrm{sm}}$, we use the simplified notation $b_v(x) := b\bigl(x, v(x)\bigr)$, and define $\mathcal{A}_v$, $\mathfrak{R}_v$ and ϱ_v analogously.
- Relaxed stationary Markov control: $v \in \mathfrak{U}_{\mathrm{sm}}$ may be viewed as a kernel on $\mathcal{P}(\mathcal{Z}) \times \mathbb{R}^d$

$$b_v(x) \;:=\; \int_{\mathcal{Z}} b(x,z)\, v(\mathrm{d}z \,|\, x)\,,$$

  and analogously for $\mathfrak{R}_v$.

Past work

- The only treatment for the ergodic control problem with non-local operators in $\mathbb{R}^d$ we could find is Menaldi & Robin (1997), under very strong blanket stability hypotheses.
- Even though the Lévy measure here is finite, there is no regularity assumption on the kernel, and this makes the problem quite difficult.

Infinitesimally invariant measures

We fix a countable dense subset $\mathcal{C}$ of $\mathcal{C}^2_0(\mathbb{R}^d)$ consisting of functions with compact supports.

Definition

A probability measure $\mu_v \in \mathcal{P}(\mathbb{R}^d)$, $v \in \mathfrak{U}_{\mathsf{sm}}$, is called *infinitesimal invariant* under $\mathcal{A}_v$ if

$$\mathcal{A}^*_v \mu_v \;=\; 0 \iff \int_{\mathbb{R}^d} \mathcal{A}_v f(x)\, \mu_v(\mathrm{d}x) \;=\; 0 \quad \forall\, f \in \mathcal{C}\,.$$

If such a μ_v exists, then we say that v is a *stable* control, and define the *(infinitesimal) ergodic occupation measure* $\pi_v \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Z})$ by $\pi_v(\mathrm{d}x, \mathrm{d}z) := \mu_v(\mathrm{d}x)\, v(\mathrm{d}z \,|\, x)$.

- $\mathfrak{U}_{\mathsf{ssm}}$: the set of stable controls.
- $\mathcal{M}$: the set of infinitesimal invariant probability measures.
- $\mathcal{G}$: the set of ergodic occupation measures.

The ergodic control problem for the operator $\mathcal{A}$

- Let $\mathcal{R}\colon \mathbb{R}^d \times \mathcal{Z} \mapsto \mathbb{R}_+$ be a continuous *coercive* function, which we refer to as the *running cost* function.
- Recall the set of infinitesimal ergodic occupation measures (a closed and convex subset of $\mathcal{P}(\mathbb{R}^d \times \mathcal{Z})$)

$$\mathcal{G} \;=\; \biggl\{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathcal{Z}) : \int_{\mathbb{R}^d \times \mathcal{Z}} \mathcal{A}_z f(x)\, \pi(\mathrm{d}x, \mathrm{d}z) = 0 \text{ for all } f \in \mathcal{C}\biggr\}.$$

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The ergodic control problem

$$\varrho_* \;:=\; \inf_{\pi \in \mathcal{G}} \pi(\mathcal{R}) \;=\; \inf_{\pi \in \mathcal{G}} \int_{\mathbb{R}^d \times \mathcal{Z}} \mathcal{R}\, \mathrm{d}\pi\,.$$

For $v \in \mathfrak{V}_{\mathsf{ssm}}$, we let $\varrho_v := \pi_v(\mathcal{R})$, and we say that v is *optimal* if $\varrho_v = \varrho_*$.

Structural Hypotheses

- Note that there was no assumption on linear growth of coefficients to prevent finite time explosion of solutions.

Structural Hypotheses

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Two Hypotheses

(H1) For any $v \in \mathfrak{V}_{\mathsf{sm}}$, the equation $\mathcal{A}_v u - u = 0$ has no bounded positive solution $u \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$.

(H2) There exist $\hat{v} \in \mathfrak{V}_{\mathsf{sm}}$, a nonnegative $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$, an open ball $\widehat{\mathcal{B}}$, and a positive constant κ_0 such that

$$\mathcal{A}_{\hat{v}} \mathcal{V}(x) \;\le\; \kappa_0 \mathbb{1}_{\widehat{\mathcal{B}}}(x) - \mathcal{R}_{\hat{v}}(x) \quad \forall\, x \in \mathbb{R}^d\,.$$

The α-discounted HJB

Theorem (α-discounted HJB)

For any $\alpha \in (0,1)$, there exists a minimal nonnegative solution $V_\alpha \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > 1$, to the HJB equation

$$\min_{z\in\mathcal{Z}} \left[\mathcal{A}_z V_\alpha(x) + \mathcal{R}(x,z)\right] \;=\; \alpha V_\alpha(x)\,.$$

Moreover, $\inf_{\mathbb{R}^d} \alpha V_\alpha \le \varrho_*$*, and the infimum of* V_α *is attained in the set*

$$\mathit{\Gamma}_\circ \;:=\; \Bigl\{x \in \mathbb{R}^d : \sup_{z\in\mathcal{Z}} \mathcal{R}(x,z) \le \varrho_*\Bigr\}\,.$$

Concerning passage to the limit as $\alpha \searrow 0$

Recall the Harnack inequality: In it's simplest form it states that if u be a non-negative harmonic function in D (i.e., $\Delta u = 0$), then, for any bounded subdomain $D' \Subset D$,

$$\sup_{D'} u \;\le\; C(d, D, D') \inf_{D'} u\,.$$

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Example (Harnack fails)

Consider an operator $\mathcal{A}$ in $\mathbb{R}^2$, with a the identity matrix, $b = (3, 0)$, and $\nu = \nu_x$ a Dirac mass at $\tilde{x} = (3, 0)$. Let $f_\epsilon \in \mathcal{C}^2(\mathbb{R}^2)$, with $\epsilon \in (0, 1)$, be defined in polar coordinates by

$$f_\epsilon(r, \theta) \;:=\; -\log(r)\,\mathbb{1}_{\{r \ge \epsilon\}} + \left(\tfrac{3}{4} - \tfrac{r^2}{\epsilon^2} + \tfrac{r^4}{4\epsilon^4} - \log(\epsilon)\right)\mathbb{1}_{\{r < \epsilon\}} \,.$$

Let u_ϵ be a function taking values

$$\begin{cases} f_\epsilon(r, \theta)\,, & \text{on } B_1\,, \\ \left(\frac{4}{\epsilon^2} - \frac{4\tilde{r}^2}{\epsilon^4} + f_\epsilon(\tilde{r}, \tilde{\theta})\right)\mathbb{1}_{\{\tilde{r} < \epsilon\}}\,, & \text{on } B_1(\tilde{x})\,, \\ \text{nonnegative}\,, & \text{o.w.} \end{cases}$$

Then u_ϵ is nonnegative on $\mathbb{R}^2$ and satisfies $\mathcal{A}u_\epsilon = 0$ in B_1.
However, $\frac{u_\epsilon(0,\theta)}{u_\epsilon(\mathrm{e}^{-1},\theta)} = -\log(\epsilon)$.

The ergodic HJB equation

Theorem (ergodic HJB)

Let V_α, $\alpha \in (0,1)$, be the family of solutions in α–discounted HJB equations. Then, as $\alpha \searrow 0$, $V_\alpha - V_\alpha(0)$ converges in $\mathcal{C}^{1,r}(\overline{B}_R)$ for any $r \in (0,1)$ and $R > 0$, to a function $V \in \mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for any $p > 1$, which is bounded from below in $\mathbb{R}^d$ and solves

$$\min_{z \in \mathcal{Z}} \bigl[\mathcal{A}_z V(x) + \mathcal{R}(x,z)\bigr] \;=\; \varrho\,, \tag{1}$$

with $\varrho = \varrho_*$. Also $\alpha V_\alpha(x) \to \varrho_*$ uniformly on compact sets. In addition, the solution of (1) with $\varrho = \varrho_*$ is unique in the class of functions $V \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$, satisfying $V(0) = 0$, which are bounded from below in $\mathbb{R}^d$. For $\varrho < \varrho_*$, there is no such solution.

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Theorem (verification)

If $v \in \mathfrak{U}_{\mathrm{ssm}}$ is optimal, then it satisfies

$$b^i_v(x)\,\partial_i V(x) + \mathcal{R}_v(x) \;=\; \inf_{z \in \mathcal{Z}} \bigl[b^i(x,z)\partial_i V(x) + \mathcal{R}(x,z)\bigr] \qquad a.e.\ x \in \mathbb{R}^d\,. \tag{2}$$

In addition, provided V is inf-compact, any stable $v \in \mathfrak{U}_{\mathrm{ssm}}$ which satisfies (2) is necessarily optimal.

More regularity if we impose some structure on ν

Theorem (the Poisson equation)

We assume (H1) and one of the following:

(a) $\nu = \nu_x$ *is translation invariant and has compact support.*

(b) ν_x *has locally compact support and a density* $\psi_x \in L^p(\mathbb{R}^d)$ *for some* $p > \frac{d}{2}$*, such that* $x \mapsto \|\psi_x\|_{L^p(\mathbb{R}^d)}$ *is locally bounded*

Let $\hat{v} \in \mathfrak{V}_{\mathrm{ssm}}$ *be such that* $\mathcal{R}_{\hat{v}}$ *is coercive relative to* $\varrho_{\hat{v}}$*. Then, up to an additive constant, there exists a unique* $\widehat{V} \in \mathcal{W}^{2,d}_{\mathrm{loc}}(\mathbb{R}^d)$ *which is bounded from below in* $\mathbb{R}^d$*, and satisfies*

$$\mathcal{A}_{\hat{v}}\, \widehat{V}(x) + \mathcal{R}_{\hat{v}}(x) \;=\; \beta \qquad \forall\, x \in \mathbb{R}^d\,,$$

for some $\beta = \varrho_{\hat{v}}$*. For* $\beta < \varrho_{\hat{v}}$*, there is no such solution.*

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$$\mathcal{A}_{\hat{v}}\, \widehat{V}(x) + \mathcal{R}_{\hat{v}}(x) \;=\; \beta \qquad \forall\, x \in \mathbb{R}^d\,,$$

for some $\beta = \varrho_{\hat{v}}$. *For* $\beta < \varrho_{\hat{v}}$, *there is no such solution.*

Theorem (stronger results)

Grant the hypotheses of the preceding theorem. Then the results on the ergodic HJB hold without assuming (H2). Moreover, provided V *is inf-compact, a control* $v \in \mathfrak{U}_{\mathsf{sm}}$ *is optimal if and only if it it a selector from the minimizer of the HJB.*

Thank you!