Asymptotically Optimal Inventory Control For Assemble-to-Order Systems

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Outline

- The Assemble-to-Order (ATO) inventory system
- The ATO inventory control problem
- The one period ATO problem
- The stochastic programming (SP) based approach
- An SP lower bound for identical lead times
- An SP lower bound for general deterministic lead times
- Translation of SP solution into control policy
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Assemble-to-Order (ATO) Inventory Systems

- m products assembled from n components

  Bill of Materials (BOM): $A = \{a_{ji}\}$
  (Product i requires a_{ji} units of component j.)
  $\mathbf{A}_j$: j^{th} row of A = usage of component j by all products

- Deterministic component replenishment lead times:

$$0 = L_0 < L_1 < L_2 < ..... < L_K.$$

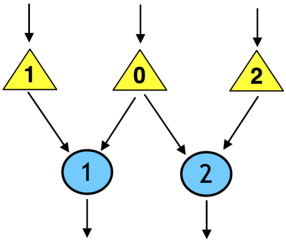
  Components with lead time L_k indexed by $n_{k-1} + 1, ..., n_k$
  (Suppliers are uncapacitated.)
  A^k: rows of A for components with lead time L_k (submatrix)
  ($n_0 = 0$, $n_K = n$, $k = 1, ..., K$).

- Compound Poisson demand for products.
- Negligible assembly time, so inventories kept at component level.
- Unsatisfied demands are backlogged, not lost.
- Unit inventory holding cost: $\mathbf{h}^k = (h_{n_{k-1}+1}, ..., h_{n_k}),\ 1 \le k \le K$.
- Unit backlog cost: $\mathbf{b} = (b_1, ..., b_m)$.

Goal: Find replenishment and allocation policies to minimize the long run average expected cost.

An Example: The 'W' Model

The W model has 3 components ($n = 3$) used to make 2 products ($m = 2$). This is the simplest model that involves 'component commonality', and has received a lot of attention in the literature.



Demand and Control Processes

- Demand process:

$$\mathcal{D}(t) = (\mathcal{D}_1(t), ..., \mathcal{D}_m(t)),\ \ t \geq 0,\ \text{demand during } [0, t]$$

Assumed to be a compound Poisson process, with an order (batch) size that has a finite moment of order 6.

$$\mathbf{D}(t_1, t_2) = \mathcal{D}(t_2) - \mathcal{D}(t_1), \quad t_2 > t_1 \geq 0.$$

Let $\mathbf{d}(t)$ denote the demand arriving at time t.

- Replenishment process (controlled):

$$\mathcal{R}^k(t) = (\mathcal{R}_{n_{k-1}+1}(t), ..., \mathcal{R}_{n_k}(t)),$$

components with lead time L_k ordered by time t.

$$\mathbf{R}^k(t) = \mathcal{R}^k(t) - \mathcal{R}^k(t - L_k),\ \ t \geq 0,\ \ 1 \leq k \leq K,\ \text{'pipeline'}.$$

- Allocation process (controlled):

$$\mathcal{Z}(t) = (\mathcal{Z}_1(t), ..., \mathcal{Z}_m(t))$$

demand served by time t

The ATO Inventory Control Problem

Result of inventory control:

- demand backlogs

$$\mathbf{B}(t) = \mathbf{B}(0) + \mathcal{D}(t) - \mathcal{Z}(t)$$

- inventory on-hand

$$\mathbf{I}^k(t) = \mathbf{I}^k(0) + \mathcal{R}^k(t - L_k) - A^k \mathcal{Z}(t), \ 1 \le k \le K$$

- inventory cost rate

$$C(t) = \sum_{k=1}^{K} \mathbf{h}^k \cdot \mathbf{I}^k(t) + \mathbf{b} \cdot \mathbf{B}(t).$$

Objective: Choose a feasible policy p that controls replenishment $(\mathcal{R}(t))$ and allocation $(\mathcal{Z}(t))$ to minimize the long-run average expected inventory cost

$$\mathcal{C}^p \equiv \limsup_{T \to \infty} \frac{1}{T} \int_{L_K}^{T+L_K} \mathbf{E}\left[C(t)\right] dt$$

Feasible policy:

- $\mathbf{B}(t) \ge 0, \quad \mathbf{I}(t) \ge 0$
- Decisions cannot be based on future demand information

One Period ATO (Song and Zipkin, 2003)

At the beginning of the period we choose $\mathbf{y} = (y_1, \ldots, y_n)$, the number of each component to order.

At the end of the period the components all arrive, we observe the demand for each product $\mathbf{D} = (D_1, \ldots, D_m)$, and choose $\mathbf{z} = (z_1, \ldots, z_m)$, the amount of demand to satisfy to minimize the total cost

$$\mathbf{b} \cdot (\mathbf{D} - \mathbf{z}) + \mathbf{h} \cdot (\mathbf{y} - A\mathbf{z})$$

subject to $0 \leq \mathbf{z} \leq \mathbf{D}, \quad A\mathbf{z} \leq \mathbf{y}$.
Note that

$$\mathbf{b} \cdot (\mathbf{D} - \mathbf{z}) + \mathbf{h} \cdot (\mathbf{y} - A\mathbf{z}) = \mathbf{b} \cdot \mathbf{D} + \mathbf{h} \cdot \mathbf{y} - \mathbf{c} \cdot \mathbf{z},$$

where $\mathbf{c} = \mathbf{b} + \mathbf{h}A$.

The above can be restated as a 2 stage stochastic linear program:

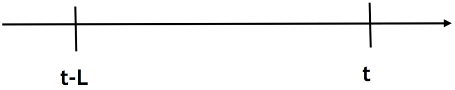
$$\begin{aligned} C_+^* &= \mathbf{b} \cdot E[\mathbf{D}] + \Psi \\ \Psi &= \min_{\mathbf{y} \in \mathbf{R}_+^n} \{\mathbf{h} \cdot \mathbf{y} + E[\Phi^0(\mathbf{y}, \mathbf{D})]\}, \\ \Phi^0(\mathbf{y}, \mathbf{x}) &= -\max_{\mathbf{z} \in \mathbf{R}_+^m} \{\mathbf{c} \cdot \mathbf{z} | \mathbf{z} \leq \mathbf{x}, A\mathbf{z} \leq \mathbf{y}\}. \end{aligned}$$

The Stochastic Programming Based Approach

1. Introduce a stochastic (linear) program (with complete recourse) whose solution provides a lower bound on the achievable cost in the inventory control problem
2. Solve the stochastic program (SP)
3. Translate the SP solution into a control policy for the inventory system
4. Prove that the SP based control policy is asymptotically optimal as the lead times grow large. (This is equivalent to holding lead times fixed and letting the arrival rates grow large.)

This is similar to the approach introduced by Harrison (1988) for 'stochastic processing networks' in heavy traffic. (But: There are no capacitated resources in this model, so there is no notion of heavy traffic.)

The Idea Behind the Stochastic Program Lower Bound: Identical Lead Times



- Assume that $L_i = L$ for all i
- Focus (myopically) on the cost rate at time t. This is affected by replenishment decisions made up to time $t - L$, and allocation decisions made up to time t.
- For the lower bound:
  - Assume no inventory and an empty pipeline at time $t - L$. (This is without loss of optimality.)
  - Assume backlogs $B_i(t - L) = \alpha_i \geq 0,\ 1 \leq i \leq m$.
  - Place order for components at time $t - L$ to show up at time t.
  - After observing $\mathbf{D}(t) = \mathcal{D}(t) - \mathcal{D}(t - L)$ make allocations at time t to (myopically) minimize the cost rate at time t

This is precisely the one period ATO problem, but with initial backlogs.

The Lower Bound SP for Identical Lead Times

Let $\mathbf{D} \stackrel{d}{=} \mathbf{D}(t) = \mathcal{D}(t) - \mathcal{D}(t-L)$ denote the demand over a lead time.

Let $c_i \equiv b_i + \sum_{j=1}^{n} a_{ji} h_j, \ 1 \le i \le m$.

We define the following 2 stage stochastic linear program:

$$\begin{aligned} \underline{C} &= \inf_{\alpha \in \mathbf{R}_+^m} \left\{ \mathbf{b} \cdot (\mathbf{E}[\mathbf{D}] + \alpha) + \Phi^1(\alpha) \right\}. && (1) \\ \Phi^1(\alpha) &\equiv \inf_{\mathbf{y} \in \mathbf{R}_+^n} \{ \mathbf{h} \cdot \mathbf{y} + E[\Phi^0(\mathbf{y}, \alpha + \mathbf{D})] \}, \\ \Phi^0(\mathbf{y}, \mathbf{x}) &= -\max_{\mathbf{z} \in \mathbf{R}_+^m} \{ \mathbf{c} \cdot \mathbf{z} | \mathbf{z} \le \mathbf{x}, A\mathbf{z} \le \mathbf{y} \}. && (2) \end{aligned}$$

Theorem (Dogru, R., Wang, 2010): Let p be any feasible policy, and $\mathcal{C}^p$ be the corresponding cost. Then

$$\mathcal{C}^p \ge \underline{C}.$$

Note: The infimum over α is typically not attained at a finite value.

An Equivalent Lower Bound Stochastic Program

For $\mathbf{y} \in \mathbf{R}^n$ and $\mathbf{x} \in \mathbf{R}^m_+$, let

$$\varphi^0(\mathbf{y}, \mathbf{x}) = -\max_{\mathbf{z} \in \mathbf{R}^m} \{\mathbf{c} \cdot \mathbf{z} | \mathbf{z} \le \mathbf{x},\ A\mathbf{z} \le \mathbf{y}\},$$

and

$$\varphi^1 = \inf_{\mathbf{y} \in \mathbf{R}^n} \{\mathbf{h} \cdot \mathbf{y} + \mathbf{E}[\varphi^0(\mathbf{y}, \mathbf{D})]\}.$$

Let $C^* = \mathbf{b} \cdot (\mathbf{E}[\mathbf{D}]) + \varphi^1$.

Theorem (R. and Wang, 2015). There exists $\mathbf{y}^* \in \mathbf{R}^n$ such that

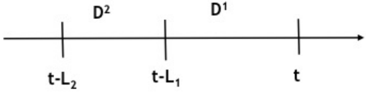
$$C^* = \mathbf{b} \cdot (\mathbf{E}[\mathbf{D}]) + \mathbf{h} \cdot \mathbf{y}^* + \mathbf{E}[\varphi^0(\mathbf{y}^*, \mathbf{D})],\ \text{and}\ C^* = \underline{C}.$$

The following LP is equivalent to the recourse LP $\varphi^0(\mathbf{y}, \mathbf{x})$:

$$\min\{\mathbf{c} \cdot \mathbf{B}\, | \mathbf{B} \ge \mathbf{0}, A\mathbf{B} \ge \mathbf{Q}\,\} - \mathbf{c} \cdot \mathbf{x},$$

where $\mathbf{Q} = A\mathbf{x} - \mathbf{y}$ and $\mathbf{B} = \mathbf{x} - \mathbf{z}$.

The Idea Behind the Stochastic Program Lower Bound: 2 Distinct Lead Times



- Focus (myopically) on the cost rate at time t. This is affected by replenishment decisions for components with lead time L_2 made up to time $t - L_2$, by replenishment decisions for components with lead time L_1 made up to time $t - L_1$, and allocation decisions made up to time t.
- For the lower bound:
  - Assume no inventory and an empty pipeline at time $t - L_2$. (This is without loss of optimality.)
  - Assume backlogs $B_i(t - L_2) = \alpha_i \geq 0,\ 1 \leq i \leq m$.
  - Place order for long lead time components at time $t - L_2$ to show up at time t.
  - After observing $\mathbf{D}^2(t) = \mathcal{D}(t - L_1) - \mathcal{D}(t - L_2)$, place order for short lead time components at time $t - L_1$to show up at time t.
  - After observing $\mathbf{D}(t) = \mathcal{D}(t) - \mathcal{D}(t - L_2)$ make allocations at time t to (myopically) minimize the cost rate at time t

A General SP-Based Lower Bound

In R. and Wang (2012) a particular $K+1$ stage stochastic program is shown to provide a lower bound on achievable cost.

In R., Wan and Wang (2018) this bound is transformed into:

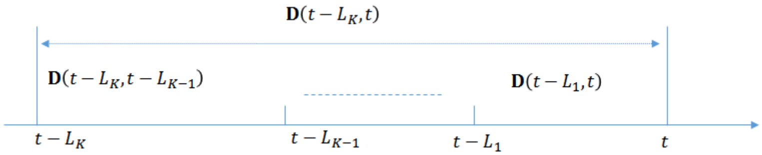
$$\underline{C} = \mathbf{b} \cdot E[\bar{\mathbf{D}}] + \varphi^K,$$

where

$$\begin{aligned}
\varphi^K &= \inf_{\mathbf{y}^K} \left\{ \mathbf{h}^K \cdot \mathbf{y}^K + \mathbf{E}[\varphi^{K-1}(\mathbf{y}^K, \mathbf{D}^K)] \right\}, \\
\varphi^k(\mathbf{y}^{k+1}, ..., \mathbf{y}^K, \mathbf{x}) &= \inf_{\mathbf{y}^k} \left\{ \mathbf{h}^k \cdot \mathbf{y}^k + \mathbf{E}[\varphi^{k-1}(\mathbf{y}^k, ..., \mathbf{y}^K, \mathbf{x} + \mathbf{D}^k)] \right\}, \; 1 \le k < K, \\
\varphi^0(\mathbf{y}^1, ..., \mathbf{y}^K, \mathbf{x}) &= -\max_{\mathbf{z}} \left\{ \mathbf{c} \cdot \mathbf{z} | \mathbf{z} \le \mathbf{x}, A^k \mathbf{z} \le \mathbf{y}^k, 1 \le k \le K \right\},
\end{aligned}$$

with $\mathbf{D}^k \stackrel{d}{=} \mathbf{D}(t - L_k, t - L_{k-1})$ $(1 \le k \le K)$, $\quad \bar{\mathbf{D}} = \mathbf{D}^1 + ... + \mathbf{D}^K$.

The optimal solution $\mathbf{y}^{*k}$ $(1 \le k \le K)$ are finite quantities.



Translation of SP Solution into a Control Policy: Allocation

- At any time t, solve for

$$\mathbb{B}(t) = \text{argmin}\{\mathbf{c}\cdot\mathbf{B}\,|\mathbf{B}\geq\mathbf{0}, A\mathbf{B}\geq\mathbf{Q}(t)\},\quad t\geq 0,$$

  where the component balance process $\mathbf{Q}$ is defined by

$$\mathbf{Q}(t)\equiv A\mathbf{B}(t)-\mathbf{I}(t).$$

- Use $\mathbb{B}(t)$ as target backlog levels: Allocate components to products so that (with $a_{ji}=0$ or 1)

$$(B_i(t)-\mathbb{B}_i(t))^+\cdot\min_{j:a_{ji}>0}\{I_j(t)\}=0,\quad 1\leq i\leq m.$$

Translation of SP Solution with Identical Lead Times into a Control Policy: Replenishment

- The inventory position of component j is defined as

$$I_j(t) + R_j(t) - \sum_{i=1}^{m} a_{ji} B_i(t),\ 1 \leq j \leq n$$

- A base-stock policy keeps the inventory position of each component at a given fixed level
- These fixed levels, $y_1, \ldots, y_n$, are known as base-stock levels
- After some (possibly empty) initial period of not ordering, during which the inventory positions drop to their base-stock levels, a base stock policy involves order-for-order replenishment.

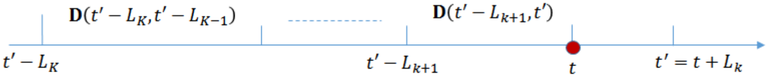
The SP based replenishment policy is a base-stock policy with $\mathbf{y}^*$, an optimal solution of φ^1 as base-stock levels.

Translation of SP Solution into a Control Policy: Replenishment

For components with lead time L_K, solve ϕ^K and use solution as base-stock levels.

At each time t and for each $k, 1 \leq k < K$, for components with lead time L_k,

- Solve $\varphi^k(\mathbf{y}^{k+1}, ..., \mathbf{y}^K, \mathbf{x})$ using
$$\mathbf{x} = \mathbf{D}(t + L_k - L_K, t),$$
with $\mathbf{y}^{k+1}, ..., \mathbf{y}^K$ obtained from $\varphi^{k'}(\mathbf{y}^{k'+1}, ..., \mathbf{y}^K, \mathbf{x})$, $k' = K, \ldots, k+1$, where
$$\mathbf{x} = \mathbf{D}(t + L_k - L_K, t + L_k - L_{k'}).$$
- Set target inventory positions $\mathbb{IP}^k(t) = \mathbf{y}^{*k}(t) - \mathbf{D}(t + L_k - L_K, t)$.
- Set inventory positions at
$$\mathbf{IP}^k(t) = \mathbf{IP}^{k-}(t) \vee \mathbb{IP}^k(t),$$
where $\mathbf{IP}^{k-}(t)$ are inventory positions before ordering, and $\mathbf{IP}^k(t)$ are inventory positions after placing the order.



The SP Based Policy is Typically Not Optimal

It may not be possible to have actual inventory positions and backlog levels obtained under any feasible policy match the ideal values of the SP:

- An inventory position can be raised (by ordering new units), but not reduced at will.
- Backlog levels cannot be raised at will to redistribute component shortage to demands.
- The target backlog level $\mathbb{B}(t)$ may not be equal to the ideal quantity obtained from the SP, $\mathbb{B}^*(t)$, because the actual inventory positions may not be equal to the target inventory positions.

There are some special cases of ATO systems where the actual inventory positions and backlogs can track the ideal values, yielding an optimal policy.

We show that, although the SP based policy is typically not optimal, it is asymptotically optimal as the lead times grow large.

Asymptotic Optimality

We introduce a family of systems indexed by L, with

$$L_1^{(L)} < L_2^{(L)} < \ldots.. < L_K^{(L)} = L$$

$$\text{and} \qquad \frac{L_k^{(L)}}{L} \to \text{constant as } L \to \infty \ (k = 1, \ldots, K).$$

$\mathcal{C}^{(L)}$: long-run average expected inventory cost of the L^{th} system,
$\underline{C}^{(L)}$: cost lower bound.

Theorem (R., Wan and Wang, 2018): Under any policy in the class that we propose,

$$\lim_{L \to \infty} \frac{\mathcal{C}^{(L)} - \underline{C}^{(L)}}{\sqrt{L}} = 0.$$

We show that $\underline{C}^{(L)}/\sqrt{L} \to \underline{C}^*$, with $0 < \underline{C}^* < \infty$, so this is equivalent to

$$\frac{\mathcal{C}^{(L)}}{\underline{C}^{(L)}} \to 1 \quad \text{as} \quad L \to \infty.$$

Although we state our limit theorem in terms of 'long lead times', we could, equivalently, allow the demand arrival rate to grow large.

FCLT Scaling

The limit theorem utilizes the standard scaling/normalization of the functional central limit theorem.

Let

$$\widehat{\mathbb{IP}}_j^{(L)}(t) \equiv \frac{\mathbb{IP}_j^{(L)}(Lt)}{\sqrt{L}}, \text{ and } \widehat{IP}_j^{(L)}(t) \equiv \frac{IP_j^{(L)}(Lt)}{\sqrt{L}}, \; 1 \le j \le n,$$

and let

$$\hat{\mathbb{B}}_i^{(L)}(t) \equiv \frac{\mathbb{B}_i^{(L)}(Lt)}{\sqrt{L}}, \; \hat{\mathbb{B}}_i^{*(L)}(t) \equiv \frac{\mathbb{B}_i^{*(L)}(Lt)}{\sqrt{L}}, \text{ and } \hat{B}_i^{(L)}(t) \equiv \frac{B_i^{(L)}(Lt)}{\sqrt{L}}, \; 1 \le i \le m.$$

Cost Difference

Let

$$\underline{\hat{C}}^{(L)} \equiv \frac{\underline{C}^{(L)}}{\sqrt{L}},\ \hat{\mathcal{C}}^{(L)}(t) \equiv \frac{E[C^{(L)}(Lt)]}{\sqrt{L}},\ \text{and } \hat{\mathcal{C}}^{(L)} = \limsup_{T\to\infty} \frac{1}{T}\int_1^{T+1} \hat{\mathcal{C}}^{(L)}(t)dt.$$

The difference between the actual and lower bound costs can be written as

$$\begin{aligned}
& \hat{\mathcal{C}}^{(L)} - \underline{\hat{C}}^{(L)} \\
= \ & \limsup_{T\to\infty} \frac{1}{T}\int_1^{T+1} \left(\hat{\mathcal{C}}^{(L)}(t) - \underline{\hat{C}}^{(L)}\right) dt \\
= \ & \limsup_{T\to\infty} \frac{1}{T}\int_1^{T+1} \sum_{j=1}^{n} h_j \left(\mathbf{E}\left[\widehat{IP}_j^{(L)}(t-\hat{L}_{k_j}^{(L)})\right] - \mathbf{E}\left[\widehat{\mathbb{IP}}_j^{(L)}(t-\hat{L}_{k_j}^{(L)})\right]\right) dt \\
& + \limsup_{T\to\infty} \frac{1}{T}\int_1^{T+1} \sum_{i=1}^{m} b_i \left(\mathbf{E}\left[\hat{B}_i^{(L)}(t)\right] - \mathbf{E}\left[\hat{\mathbb{B}}_i^{*(L)}(t)\right]\right) dt \\
\leq \ & \sum_{j=1} h_j \sup_{t\geq 0}\left|\mathbf{E}\left[\widehat{IP}_j^{(L)}(t) - \widehat{\mathbb{IP}}_j^{(L)}(t)\right]\right| + \sum_{i=1}^{m} b_i \sup_{t\geq 1}\left|\mathbf{E}\left[\hat{B}_i^{(L)}(t) - \hat{\mathbb{B}}_i^{*(L)}(t)\right]\right|
\end{aligned}$$

We develop a unified asymptotic framework to address the differences in both inventory positions and backlogs.

Stochastic Tracking Problem

- $\mathcal{T}_j(t)$: a family of *target processes* that satisfy the following Lipschitz condition:

  $$|\mathcal{T}_j(t_2) - \mathcal{T}_j(t_1)| \leq g_j \sum_{l \in \mathcal{K}_j} ||\mathbf{D}(t_2 - s_l, t_2) - \mathbf{D}(t_1 - s_l, t_1)||_1, \; 1 \leq j \leq n,$$

  where s_l $(l \in \mathcal{K}_j)$ and g_j $(1 \leq j \leq n)$ are finite positive constants.

- $\mathcal{W}_j(t)$: a family of *state-constrained target processes* defined by

  $$\mathcal{W}_j(t) = \mathcal{W}_j^-(t) \vee \mathcal{T}_j(t) \quad \text{and} \quad \mathcal{W}_j^-(t) = \mathcal{W}_j(t^-) - \mathcal{A}_j \cdot \mathbf{d}(t),$$

  where $\mathcal{A}_j$ is a constant vector $(1 \leq j \leq n)$

- Initial states of $\mathcal{T}_j(t)$ and $\mathcal{W}_j(t)$ $(1 \leq j \leq n)$ properly defined.

The identification of $\mathcal{T}_j(t)$ with $\mathbb{IP}_j(t)$, and $\mathcal{W}_j(t)$ with $IP_j(t)$ is immediate. The application of the stochastic tracking problem to the backlog requires some transformations.

Convergence in Stochastic Tracking Problem

We consider a family of systems indexed by L, and let

$$\hat{\mathcal{T}}_j^{(L)}(t) = \frac{\mathcal{T}_j^{(L)}(Lt)}{\sqrt{L}} \quad \text{and} \quad \hat{\mathcal{W}}_j^{(L)}(t) = \frac{\mathcal{W}_j^{(L)}(Lt)}{\sqrt{L}}, \ 1 \le j \le n.$$

Theorem (R., Wan and Wang, 2018): For $1 \le j \le n$, if $\lim_{L\to\infty} \mathbf{E}\left[\hat{\mathcal{W}}_j^{(L)}(0) - \hat{\mathcal{T}}_j^{(L)}(0)\right] = 0$, then

$$\lim_{L\to\infty} \mathbf{E}\left[\sup_{t\ge 0}\left(\hat{\mathcal{W}}_j^{(L)}(t) - \hat{\mathcal{T}}_j^{(L)}(t)\right)\right] = 0.$$

Intuitively, when $\hat{\mathcal{W}}_j^{(L)}(t) > \hat{\mathcal{T}}_j^{(L)}(t)$, $\hat{\mathcal{W}}_j^{(L)}(t)$ is decreasing at a rate that is $O(\sqrt{L})$, so $\hat{\mathcal{W}}_j^{(L)}(t)$ cannot get much larger than $\hat{\mathcal{T}}_j^{(L)}(t)$.

This is similar to a state-space collapse result.

Lipschitz Continuity

Note that an assumption of the stochastic tracking problem is that the processes $\mathcal{T}_j(t)$ are Lipschitz continuous in particular (sums of) differences of the arrival processes over certain intervals.

For the target backlog process $\mathbb{B}(t)$, which is obtained as an optimal solution of a linear program, this Lipschitz continuity follows from Hoffman's Lemma.

There are not (to the best of our knowledge) equivalent results for multistage stochastic programs.

Note that the quantity of demand arriving in an interval has a discrete distribution, so the multistage stochastic program can be written as an infinite linear program.

This infinite linear program can be made finite by truncating the demand.

Lipschitz continuity for the multistage SP is proved in R., Wan and Wang (2018) by applying a variant of Hoffman's Lemma, and showing that the associated Lipschitz constant does not grow as the truncation level grows.

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