Turán numbers of expanded forests

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Main results are joint with
Tao JIANG, Miami University, Oxford, OH, USA

Turán type problems

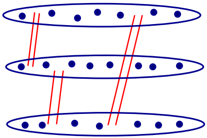
Given a k-uniform hypergraph $\mathcal{A}$.
What is the max number of edges of $\mathcal{F}$, $\mathcal{F} \subset \binom{[n]}{k}$, if $\mathcal{A} \not\subset \mathcal{F}$?
Notation of this threshold: $\mathbf{ex}^{(k)}(n, \mathcal{A}) := \max |\mathcal{F}|$.

E.g., the classical Turán theorem on max triangle free graphs:

$$\mathbf{ex}(n, K_3) = \lfloor \frac{1}{4} n^2 \rfloor.$$

The complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ is extremal.

In general: (Turán 1941) $e(G_n) > e(T_{n,p}) \implies K_{p+1} \subseteq G_n$
and here $T_{n,p}$:= the Turán graph, is the unique extremal graph.



Theorem (Erdős, Stone, Simonovits, Kővári, T. Sós, Turán)

If G is any graph and $n \to \infty$ then one of the three cases holds

— G is a forest and $\mathbf{ex}(n, G) = O(n)$,

— G has cycles, but it is bipartite, then for some $c := c(G) > 0$ one has $\Omega(n^{1+c}) < \mathbf{ex}(n, G) < O(n^{2-c})$,

— $\chi(G) = p + 1 \geq 3$, then

$$\mathbf{ex}(n, G) = (1 + o(1)) \left(1 - \frac{1}{p}\right) \binom{n}{2}.$$

QUESTION:

What is the situation for k-uniform hypergraphs?

A modest partial answer:

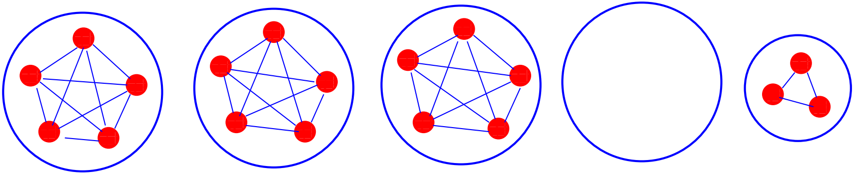
We have asymptotic for some 'tree-like' hypergraphs.

Fact (If $\mathbb{T}$ is a tree (forest) of v vertices then)

$$\mathrm{ex}(n, \mathbb{T}) \leq (v-2)n.$$

Proof. Leave out degrees $\leq (v-2)$.
If an H is left with $\delta(H) \geq k-1$, then $\mathbb{T}$ can be embedded into H.

Lower bound: disjoint complete graphs on $v-1$ vertices. No P_6:



$$\mathrm{ex}(n, \mathbb{T}_v) \geq \frac{v-2}{2}n - O(v^2).$$

For every forest F with $e(F) > 1$ we have $\mathrm{ex}(n, F) = O(n)$.

Trees in graphs; the Erdős-Sós conjecture

Theorem (Ajtai, Komlós, Simonovits, Szemerédi 2014+)

If $\mathbb{T}$ is a tree of v vertices, $v \geq v_0$, then

$$\mathbf{ex}(n, \mathbb{T}) \leq \frac{1}{2}(v-2)n.$$

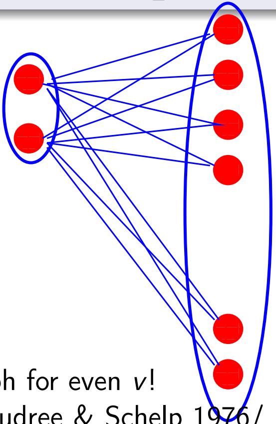
Infinitely difficult (over 180 pages).

Other versions by
Ajtai, Hladky, Komlós, Piquet, Simonovits, Szemerédi 2014+

Why is it so difficult?
Probably because there are many different (almost) extremal constructions.

Theorem (Erdős-Gallai, 1959. If P_v is a path of v vertices then)

$$\mathrm{ex}(n, P_v) \leq \frac{1}{2}(v-2)n.$$



No P_6.
Another extremal graph for even v!
Exact $\mathrm{ex}(n, P_v)$ by Faudree & Schelp 1976/ Kopylov 1977.

Forests in hypergraphs

What is the situation for k-uniform hypergraphs?

A recursive definition: A single k-set is a k-forest.
Suppose that $\mathbb{T} = \{E_1, E_2, \dots, E_u\} \subseteq \binom{V}{k}$ is a k-forest and
$A := A_{u+1} \subset E_i$ for some $1 \leq i \leq u$, and $B \cap V = \emptyset$, $|A| + |B| = k$.
Then $\{E_1, E_2, \dots, E_u, E_{u+1}\}$ is a k-forest with $E_{u+1} := A \cup B$.

$k = 2:$ graphs (the usual forests and trees).

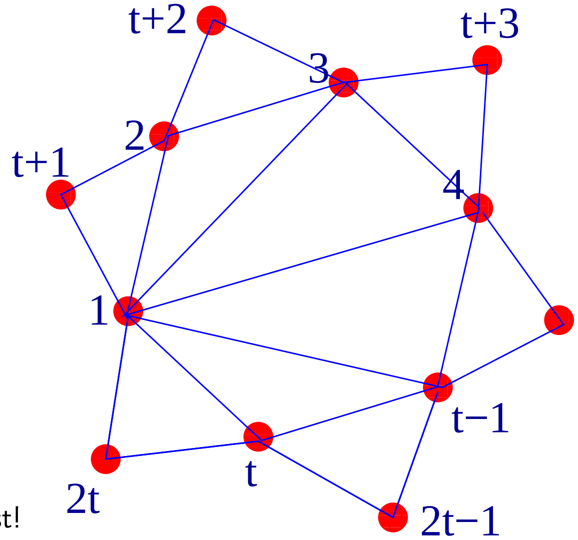
Theorem (easy greedy algorithm)

Let $\mathbb{T}$ be a (partial) k-forest of v vertices. Then

$$\mathbf{ex}^{(k)}(n, \mathbb{T}) \leq (v - k)\binom{n}{k-1}.$$

Gives the **correct order of magnitude**. If $\cap\mathbb{T} = \emptyset$, $\mathbf{ex} \geq \binom{n-1}{k-1}$.

A partial forest is not necessarily a forest



$\mathbb{C}_t^{(k)}$ is not a forest!
It is a partial k-tree for $k \geq 3$.

A starting point: The Erdős–Ko–Rado theorem

The simplest forest possible: disjoint hyperedges.

$\mathcal{F} \subset \binom{[n]}{k}$, a k-uniform set-system
on the n-element underlying set $[n] = \{1, 2, \ldots, n\}$, $\quad n \geq k \geq 2$.

Theorem (Erdős–Ko–Rado 1961)

If hyperedges pairwise intersect each other and $n \geq 2k$, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

EQUALITY only for centered systems ($n > 2k$).
Construction: all k-element subsets containing a given element

$$\mathcal{F}_1 := \{F : |F| = k,\ 1 \in F \subset [n]\}.$$

Erdős' Matching Conjecture

matching number: $\nu(\mathcal{F}) :=$
the maximum number of pairwise disjoint members (edges) of $\mathcal{F}$.
$\mathbb{M}_\nu^{(k)} :=$ ν pairwise disjoint k-sets.

$$\mathcal{A}^{(k)}(\nu) := \binom{[k\nu-1]}{k}, \qquad |\mathcal{A}| = \binom{k\nu-1}{k}.$$
$$\mathcal{B}_n^{(k)}(\nu) := \{B \in \binom{[n]}{k} : B \cap [\nu-1] \neq \emptyset\}, \qquad |\mathcal{B}| = \binom{n}{k} - \binom{n-\nu+1}{k}$$

Conjecture (The Matching Conjecture, Erdős 1965)

If $\mathcal{F} \subset \binom{[n]}{k}$, $\nu(\mathcal{F}) < \nu$ *and* $n \geq k\nu - 1$ *then*

$$|\mathcal{F}| \leq \mathbf{ex}^{(k)}(n, \mathbb{M}_\nu) = \max\left\{\binom{k\nu-1}{k}, \binom{n}{k} - \binom{n-\nu+1}{k}\right\}$$

The right hand side is $(\nu - 1 + o(1))\binom{n-1}{k-1}$.

The status of Erdős Matching Conjecture

- Obvious for $k = 1$, (singletons)
- True for graphs $k = 2$ (Erdős, Gallai, 1959)
- The case $\nu = 2$ is the classical Erdős, Ko, Rado
- $\mathbf{ex}^{(k)}(n, \mathbb{M}_\nu) \le (\nu - 1)\binom{n-1}{k-1}$ $(\forall n, k, \nu)$ by Frankl
- True for $n > n_0(k, \nu)$. Erdős 1965.
- $n_0(k, \nu) < 2k^3\nu$ (Bollobás, Daykin, Erdős, 1976)
- $n_0(k, \nu) < O(k\nu^2)$ (Frankl & ZF, unpublished)

More recently (2011-2012)

Alon, Frankl, Huang, Rödl, Ruciński, and Sudakov
Alon, Huang and Sudakov/ Frankl, Rödl and Ruciński:
connections to other problems

- $n_0(k, \nu) < 3k^2\nu$ (Huang, Loh and Sudakov, 2011)
- $k = 3$ SOLVED Łuczak and Mieczkowska for large ν, Frankl $\forall \nu$
- Best bound (Frankl 2012): True for $n \ge (2\nu - 1)k - \nu$.

Tight k-uniform trees

Definition (tight k-tree)

A sequence of k-element sets $\mathbb{T} := \{E_1, E_2, \dots, E_u\}$ *such that* $\forall\ E_i$ $(2 \le i \le u)$ *there exists an earlier edge* $|E_\alpha \cap E_i| = k-1$, $\alpha = \alpha(i) < i$ *and* E_i *has a new vertex:* $|E_i \setminus \cup_{j<i} E_j| = 1$.

For $k = 2$ (i.e., for graph) this is the usual tree.

$$\mathbf{ex}^{(k)}(n, \mathbb{T}) = ?$$

when $\mathbb{T}$ (and k) are fixed and $n \to \infty$.

The Turán number of tight k-trees

Let $\mathbb{T}$ be a tight k-tree of v vertices.
One can obtain a $\mathbb{T}$-free k-graph by packing $(v-1)$-vertex sets:
Two sets overlap in $\leq k-2$. Replace them by complete k-graphs.
Rödl, 1985, gives

$$\mathbf{ex}^{(k)}(n,\mathbb{T}) \geq (1-o(1))\frac{\binom{n}{k-1}}{\binom{v-1}{k-1}} \times \binom{v-1}{k} = (1+o(1))\frac{v-k}{k}\binom{n}{k-1}.$$

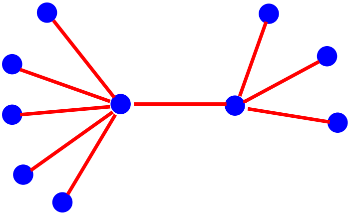
By a recent result of Keevash 2014:
Given $\mathbb{T}$, for ∞ many n's there is no need for $(1+o(1))$.
The error term is only $O(n^{k-2})$.

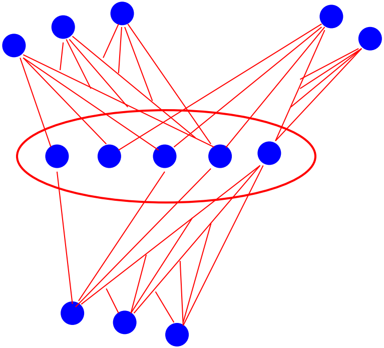
Conjecture (Erdős and Sós for graphs, Kalai 1984 for all k)

$$\forall \mathbb{T} \qquad \mathbf{ex}^{(k)}(n,\mathbb{T}) \leq \frac{v-k}{k}\binom{n}{k-1}.$$

Starlike trees

The Kalai conjecture has been proved (Frankl and ZF, 1987) for **star-like** tight trees, i.e.,
when $\mathbb{T}$ has a *central edge* meeting all others in $k-1$ vertices.
For $k=2$ these are the diameter 3 trees.





Other forests?

$\mathbf{ex}(n, \mathbb{T}) = \Theta(n^{k-1})$ was simple. (For $\cap\mathbb{T} = \emptyset$.)
We are looking for exact formulas for

$$\lim_{n\to\infty} \mathbf{ex}^{(k)}(n, \mathbb{T}) / \binom{n}{k-1}.$$

We have seen examples where the extremal family was spreading out evenly as a design (starlike tight trees).
For some other forests the extremal families were concentrated (Erdos' thm for ν disjoint k-sets).

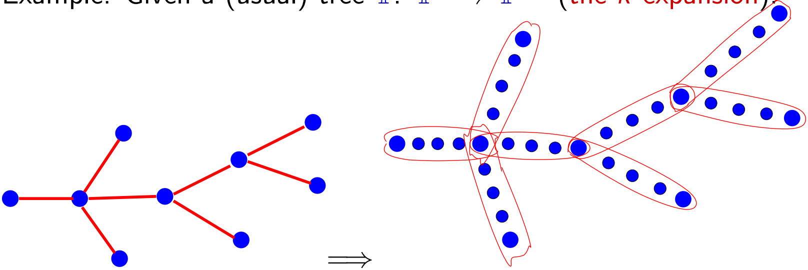
Aim of this lecture:
New exact asymptotic for $\mathbf{ex}(n, \mathbb{T})$ for a large class of trees with Erdős type, **concentrated** extremum.

Hypergraph expansions

Definition (k-expansion)

Given a (hyper)graph G the k-expansion of G is the k-graph $G^{(k)}$ obtained from G by enlarging each edge E of G with a set of $k - |E|$ vertices disjoint from $V(G)$ such that distinct edges are enlarged by disjoint $(k - |E|)$-sets.

Example: Given a (usual) tree $\mathbb{T}$. $\mathbb{T} \Longrightarrow \mathbb{T}^{(k)}$ (the k-expansion).



Expansions of a matching = Erdős' theorem.

Theorem (Frankl 1977, a path of two edges. Conj'd by Erdős-Sós)

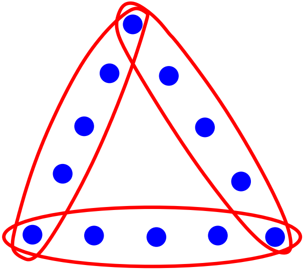
Fix $k \geq 4$. Then $ex_k(n, P_2^{(k)}) = \binom{n-2}{k-2}$ for $n > n_0(k)$, with equality only for a 2-star (two points are contained in all k-sets).

The case $k = 4$ was completed for all n by Keevash, Mubayi, Wilson '06.

The case $\chi(G) > k$ by Pikhurko / and by Mubayi 2006.
Then $ex_k(n, G^{(k)}) = \Theta(n^k)$.

Expanded triangles

$\Delta^{(k)} := \{A, B, C\}$, $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \emptyset$.



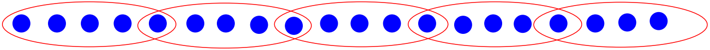
Theorem (Frankl&ZF 1987 Chvátal's 1972 triangle conjecture)

If $\mathcal{F} \subset \binom{[n]}{k}$ does not contain a $\Delta^{(k)}$ (and $n > n_k$, $k \geq 3$), then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

Equality only for centered families.

Does not imply EKR.



A 5-edge linear (*loose*, 1-*tight*) path, $\mathbf{P}_5^{(k)}$.

Cannot be covered by two vertices.

$$\mathbf{ex}(n, \mathbf{P}_5^{(k)}) \geq |\mathcal{F}_2| = \binom{n-1}{k-1} + \binom{n-2}{k-1}.$$

$\mathcal{F}_t := \{F \in \binom{[n]}{k} : F \cap \{1, 2, \dots, t\} \neq \emptyset\}$.
$\mathcal{F}_t$ does not contain a linear path of length $2t+1$, $\mathbf{P}_{2t+1}$.

The Turán number of a linear path

Theorem (ZF, T. Jiang, R. Seiver 2011, $t = 1$ Frankl, ZF 1987)

If $\mathcal{F} \subset \binom{[n]}{k}$ does not contain a linear path of length $2t+1$, $k \geq 4$ and $n > n_{k,t}$ then $|\mathcal{F}| \leq |\mathcal{F}_t|$.

$$\mathbf{ex}(n, \mathbf{P}_{2t+1}) = \binom{n-1}{k-1} + \cdots + \binom{n-t}{k-1}.$$

Equality only for $\mathcal{F}_t$.

For large n improves Győri, Katona, Lemons 2010 by a factor of k, and Mubayi, Verstraëte 2007 by a factor of 2.

Case $k = 3$: same true. By Kostochka, Mubayi, Verstraëte 2014+.

The Turán number of a linear path, even case

For even length: one can add a 2-intersecting system to $\mathcal{F}_t$.

Theorem (same authors 2011, $t = 0$ Frankl 1977)

If $n > n_{k,t}$ (we have $kt < O(\log\log n)$) and $k \geq 4$, then

$$\mathbf{ex}(n, \mathbf{P}_{2t+2}) = \binom{n-1}{k-1} + \cdots + \binom{n-t}{k-1} + \binom{n-t-2}{k-2}.$$

The only extremal system:
$\mathcal{F}_t \cup \{F : |F| = k, \{t+1, t+2\} \subset F \subset \{t+1, \dots, n-1, n\}\}$.

Proof: By **Delta-system method!**

Again, the case $k = 3$ was completed by Kostochka, Mubayi, Verstraëte 2014+.
Different method, for very large n.

The Turán number of a linear cycles

Theorem (ZF, T. Jiang, 2013, $t = 1$ Frankl, ZF 1987)

If $\mathcal{F} \subset \binom{[n]}{k}$ does not contain a linear CYCLE of length $2t+1$ $k \geq 5$ and $n > n_{k,t}$ then $|\mathcal{F}| \leq |\mathcal{F}_t|$.

$$\mathbf{ex}(n, \mathbf{C}_{2t+1}) = \binom{n-1}{k-1} + \cdots + \binom{n-t}{k-1}.$$

Equality only for $\mathcal{F}_t$.

We also have an asymptotic for $k = 4$. $\mathbf{ex} = (t + o(1))\binom{n-1}{k-1}$.
Kostochka, Mubayi and Verstraëte 2014+ for $k = 3$.

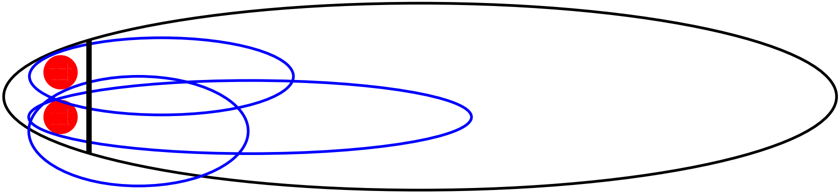
Same bound as for PATH, $\mathbf{P}_{2t+1}$, but more difficult to prove.
Neither result implies the other.

The case of $\mathbf{ex}(n, \mathbf{C}_{2t+2})$ is a slightly more complicated.

Crosscuts in hypergraphs

Definition (Frankl and ZF 1987)

Given an k-graph $\mathcal{H}$, a crosscut of $\mathcal{H}$ is subset X of $V(\mathcal{H})$ such that $\mathcal{H}$ meets all members of $\mathcal{H}$ in a singleton.
$\sigma(\mathcal{H}) := \min |X|$ *(it it exists).*

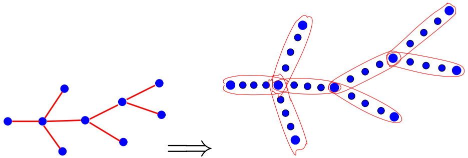


$$\mathcal{F}_t^0 := \{F \in \binom{[n]}{k} : |F \cap \{1, 2, \dots, t\}| = 1\}.$$

$\mathcal{F}_{\sigma-1}^0$ does not contain $\mathcal{H}$.

$$\mathbf{ex}^{(k)}(n, \mathcal{H}) \geq |\mathcal{F}_{\sigma-1}| = (\sigma(\mathcal{H}) - 1 + o(1))\binom{n-1}{k-1}.$$

The Turán number of expanded forests



Theorem (Asymptotic for blown up linear forests (ZF 2011))

For $k \geq 4$ we have

$$(\sigma - 1)\binom{n-\sigma+1}{k-1} \leq \mathbf{ex}(n, \mathbb{T}^{(k)}) = (\sigma - 1 + o(1))\binom{n-1}{k-1}.$$

Kostochka-Mubayi-Verstraëte 2014:
previous results for expansions of paths, cycles and trees hold for all $k \geq 3$ (esp for triple systems, $k = 3$).

Main result: expansions of $(k-2)$-forests

Theorem (ZF & T. Jiang 2014+)

Suppose that $\mathcal{H}$ is a k-forest such that each edge has at least two degree 1 vertices, $\sigma := \sigma(\mathcal{H})$. Then

$$(\sigma-1)\binom{n-\sigma+1}{k-1} \le \mathbf{ex}(n,\mathcal{H}) = (\sigma-1+o(1))\binom{n-1}{k-1}.$$

Error term is only $O(n^{k-2})$.
Stability of extremum: yes.
From this: exact results.
Same for $\mathcal{H}' \subset \mathcal{H}$ (partial forests like the cycle $\mathbf{C}_\ell^{(k)}$), $\sigma := \sigma(\mathcal{H}')$.

It does not hold with only one degree 1 vertex on each edge.
Except for $k=3$ (Kostochka et al.)

Some tools of the proof

1. Delta system, consider kernels.
2. The structure of typical hyperedges (Type I).
3. Separating large degree vertices, L.
4. Typical k-sets meet L in a singleton.
5. The number of typical edges $\leq (\sigma - 1)\binom{n-\ell}{k-1}$.
6. To prove a Kruskal-Katona type thm for the shadow of $\mathcal{H}$-free hypergraphs.
7. A **weak stability** of the extremum.

Stability of ext(n, K_{p+1})

Theorem (Large p-chromatic subgraphs (ZF 2010))

Suppose $K_{p+1} \not\subset G$, $|V(G)| = n$ and $e(G) \geq e(T_{n,p}) - t$. Then there exists a p-chromatic subgraph H_0, $E(H_0) \subset E(G)$ such that

$$e(H_0) \geq e(G) - t.$$

There are other (more exact) results, (Hanson, Toft 1991, Győri 1987, 1991, Alon 1996, Tuza et al., Bollobás & Scott, ...)
But! here there is no $\varepsilon, \delta, n_0, \ldots$. It is true for every n and t.

Corollary (ZF 2010, a special case of Simonovits' Stability thm.)

Suppose G is K_{p+1}-free with $e(G) \geq e(T_{n,p}) - t$. Then $\exists$ a complete p-chromatic graph H, $V(H) = V(G)$, such that

$$|E(G) \triangle E(H)| \leq 3t.$$

The superstability of Erdős theorem

Theorem (Recall Erdős' Matching theorem)

If $\mathcal{F} \subset \binom{[n]}{k}$, $\nu(\mathcal{F}) < \nu$ *and* $n > n_0(k,\nu)$, *then*

$$|\mathcal{F}| \leq \mathbf{ex}^{(k)}(n, \mathbb{M}_\nu) = \binom{n}{k} - \binom{n-\nu+1}{k} = (\nu-1)\binom{n-1}{k-1} + O(n^{k-2}).$$

Much more is true. It is easy to prove a concentration theorem. $\forall \mathcal{F}$: $\nu(\mathcal{F}) < \nu$ implies that $\exists$ a set L of size $\nu - 1$ such that

$$|\{F \in \mathcal{F} : F \cap L = \emptyset\}| = O(n^{k-2}).$$

All but a small number of hyperedges meet L.

No superstability for most trees

Construction of $\mathcal{H}(m, r, t)$. $r \geq t \geq 2$, $\alpha \in \mathbf{Z}$, $V(\mathcal{H}) := [r] \times [m]$.
$F = \{(1, y_1), (2, y_2), \ldots, (r, y_r)\} \in \mathcal{H}$ if and only if

$$\sum_{1 \leq i \leq t} y_i \equiv \alpha \mod m.$$

$|\mathcal{H}| = (n/r)^{r-1}$ and $\forall$ $|L| < n/(2r)$ misses at least $\frac{1}{2}(\frac{n}{r})^{r-1}$ edges.
So if $\mathbb{T}$ has a superstability property (like $\mathbb{M}_\nu$ does), then it must be embedded into $\mathcal{H}(m, r, t)$ for all $t \geq 2$ for all large m.

Theorem (Main step in the proof of our results)

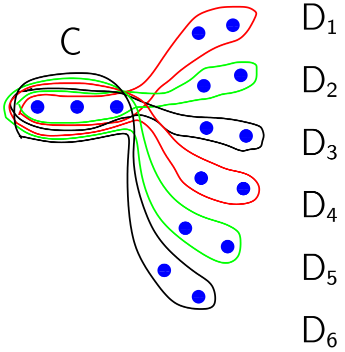
If $\mathbb{T}$ can be embedded into all $\mathcal{H}(m, r, t)$, and every $E \in \mathbb{T}$ has a degree 1 vertex, then $\exists$ L, $|L| = n^\varepsilon$, such that

$$|\{F \in \mathcal{F} : F \cap L = \emptyset\}| = O(n^{k-1-\varepsilon}).$$

Almost all but a small number of hyperedges meet L.

A powerful tool: delta systems

Def: The sets $\{D_1, D_2, \ldots, D_\ell\}$ are forming a *delta-system* of size ℓ with center (kernel) C if
$D_i \cap D_j = C \quad (\forall\ 1 \le i < j \le \ell)$.



The Erdős-Rado bound for Δ-systems

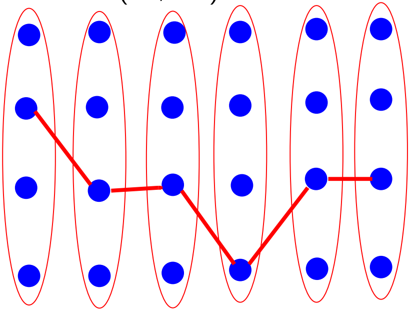
Theorem (Erdős, Rado 1960, large Δ-systems in k-unif families)

$\exists\ \varphi(k,\ell)$: *If $|\mathcal{F}| > \varphi(k,\ell)$, then it contains Δ-system of size ℓ.*

$(\ell-1)^k \leq \varphi(k,\ell) < k!(\ell-1)^k$.

(Kostochka 1997): $\varphi(k,3) < Ck!\left(\frac{\log\log\log k}{\log\log k}\right)^k$

Erdős: $\varphi(k,3) < C^k$? (\$1,000).



The *intersection structure* of $F \in \mathcal{F}$ with respect to the family $\mathcal{F}$

$$\mathcal{I}(F,\mathcal{F}) := \{F \cap F' : F' \in \mathcal{F}, F \neq F'\}.$$

Def: a k-uniform family $\mathcal{F} \subset \binom{[n]}{k}$ is *k-partite* if $\exists$ a k-partition $[n] = X_1 \cup \cdots \cup X_k$ such that $|F \cap X_i| = 1$ for $\forall i$ and $\forall\ F \in \mathcal{F}$.

Then $\exists$ a natural projection $\pi : 2^{[n]} \to [k]$.

$$\pi(S) := \{i : S \cap X_i \neq \emptyset\} \ \& \ \ \pi(\mathcal{I}(F,\mathcal{F})) := \{\pi(S) : S \in \mathcal{I}(F,\mathcal{F})\}.$$

The Δ-system method

Algebraic methods frequently do not yield sufficient information on local structure. We seek new methods, building blocks.

The intersection semilattice lemma

The next statement is a kind of compactness.
Instead of the whole $\mathcal{F} \subset \binom{[n]}{k}$ one should consider only a $\mathcal{J} \subset 2^{[k]}$, where the **kernel of kernels is again a kernel**.

Theorem (ZF 1983)

There exists a positive bound $c(k,\ell) > 0$ as follows:
Every k-uniform family $\mathcal{F}$ contains a subsystem $\mathcal{F}^* \subset \mathcal{F}$ such that
(1) $|\mathcal{F}^*| \geq c(k,\ell)|\mathcal{F}|$,
(2) $\mathcal{F}^*$ **k-partite**,
(3) *the intersection structure is* **homogeneous**, $\exists\, \mathcal{J} \subset 2^{\{1,2,\dots,k\}}$ *such that* $\pi(\mathcal{I}(F,\mathcal{F}^*)) = \mathcal{J}$ *for all edge* $F \in \mathcal{F}^*$,
(4) $\mathcal{J}$ **closed under intersection**, $(A, B \in \mathcal{J} \implies A \cap B \in \mathcal{J})$,
(5) *every* **pairwise intersection is a kernel**, *i.e.,* $\forall F_1, F_2 \in \mathcal{F}^*$ $\exists\, F_3, \dots, F_\ell \in \mathcal{F}^*$ *such that* $\{F_1, F_2, \dots, F_\ell\}$ *is a* Δ*-system.*

The End