Unbalanced random matching markets: the stark effect of competition

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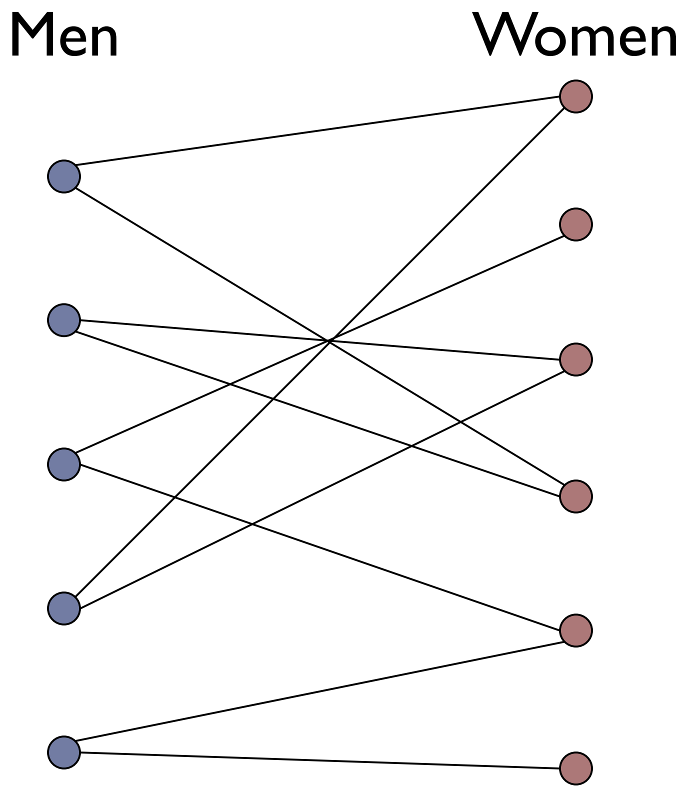
Matching markets

Markets characterized by:

- Indivisibilities
- Capacity constraints

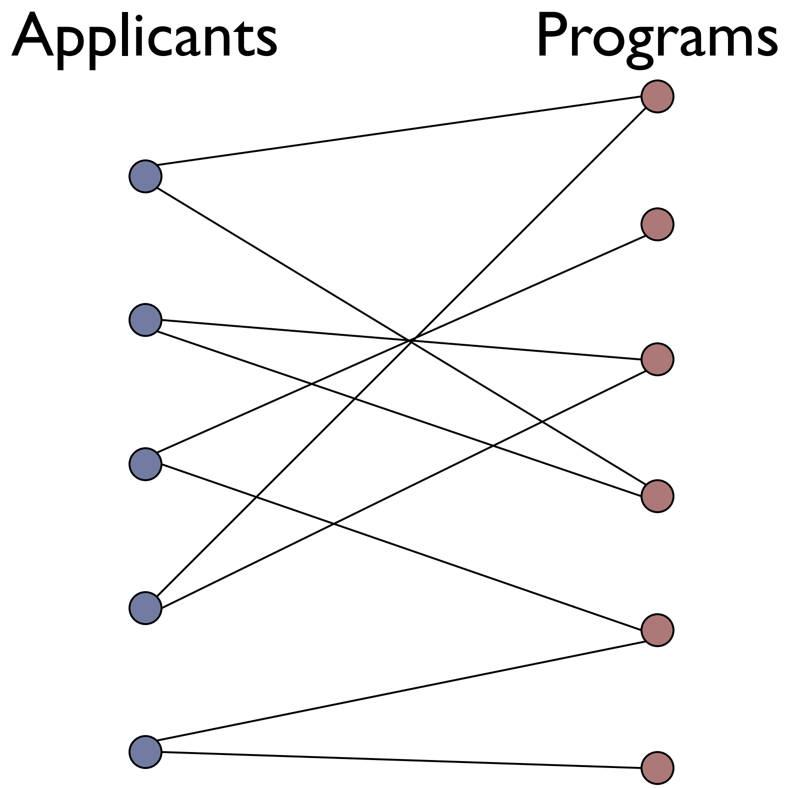
Often two-sided

Matching markets



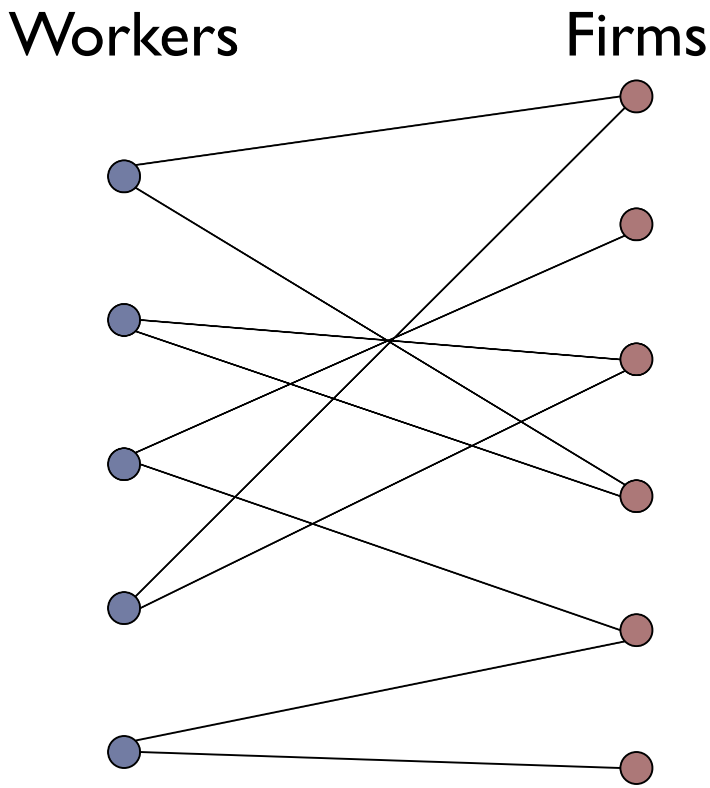
Dating/marriage market

Matching markets



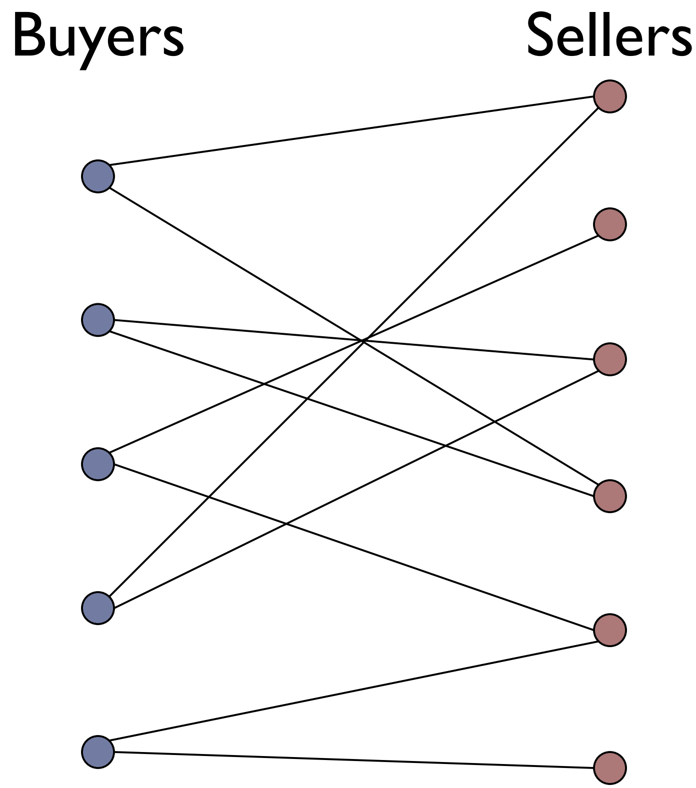
School/college admissions

Matching markets



Labor market

Matching markets



Housing market

Our goal

We study the effect of competition in matching markets

We study matching markets where

- Agents have heterogeneous preferences
  - Each agent has an ordered preference list over the other side
- There are no transfers.

[Gale & Shapley 1962]

Successful solution concept of stable matchings:

- Market designs
  - e.g. school admissions, residency match, engineering college admissions in India
- Verified in empirical studies of decentralized markets

Stable Matchings = Core Allocations

- A matching μ is stable if there is no blocking pair: man and woman (m, w) who both prefers each other over their current match.

- The set of stable matchings is a non-empty lattice, whose extreme points are the Men Optimal Stable Match (MOSM) and the Women Optimal Stable Match (WOSM)
  - Men are matched to their most preferred stable woman under the MOSM and their least preferred stable woman under the WOSM

Example

- The preferences of 3 men and 4 women:

A	B	C
1	2	4
2	4	1
3	3	2
4	1	3

1	2	3	4
A	C	A	B
B	A	C	C
C	B	B	A

Example

- A non stable matching (A and 2 would block)

A	B	C
1	2	4
2	4	1
3	3	2
4	1	3

1	2	3	4
A	C	A	B
B	A	C	C
C	B	B	A

Example

- The MOSM:

A	B	C
1	2	4
2	4	1
3	3	2
4	1	3

1	2	3	4
A	C	A	B
B	A	C	C
C	B	B	A

- (3 is unmatched)

Example

- The WOSM:

A	B	C
1	2	4
2	4	1
3	3	2
4	1	3

1	2	3	4
A	C	A	B
B	A	C	C
C	B	B	A

- (3 is unmatched)

Our model: random matching markets

Matching markets with heterogeneous preferences and no transfers.

Random Matching Market:

A set of n men $\mathcal{M}$ and a set of $n+k$ women $\mathcal{W}$

- Man m has complete preferences over women, drawn i.i.d. uniformly at random.
- Woman w has complete preferences over men, drawn i.i.d. uniformly at random.

Questions

- Average rank, or who gets to choose?
  - Define men's average rank of their wives under μ

$$R_{\text{Men}}(\mu) := \frac{1}{|\mu^{-1}(\mathcal{W})|} \sum_{m \in \mu^{-1}(\mathcal{W})} \text{Rank}_m(\mu(m))$$

  excluding unmatched men from the average
  - Average rank is 1 if all men got their most preferred wife, higher rank is worse.

- How many agents have multiple stable assignments?
  - Agents can manipulate iff they have multiple stable partners

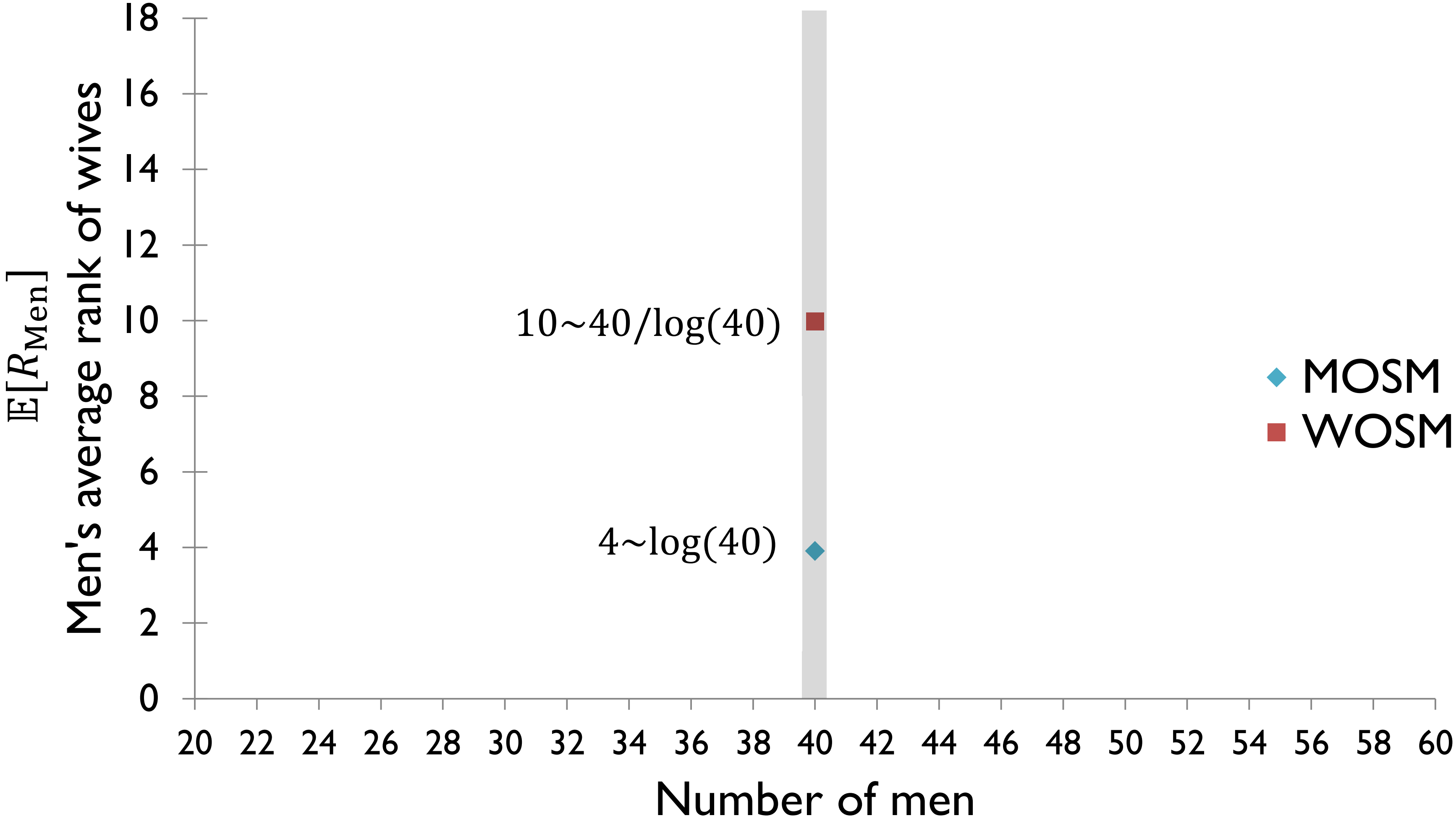
Previous Literature

- Pittel (1989), Knuth, Motwani, Pittel (1990), Roth, Peranson (1999) – when there are equal number of men and women
  - Under MOSM men's average rank of wives is $\log n$, but it is $n/\log n$ under the WOSM
  - The core is large – most agents have multiple stable partners
- Roth, Peranson (1999) document that in the NRMP the MOSM and WOSM are almost the same: suggest short lists as reason
- Immorlica, Mahdian (2005) and Kojima, Pathak (2009) show that if one side has short random preference lists the core is small
- Holzman and Samet (2013) show if preferences are highly correlated the core is small.

Is the small observed core driven by features like short lists & correlation in preferences?

We consider random matching markets with unequal number of men and women.

Men's average rank of wives, $|\mathcal{W}| = 40$



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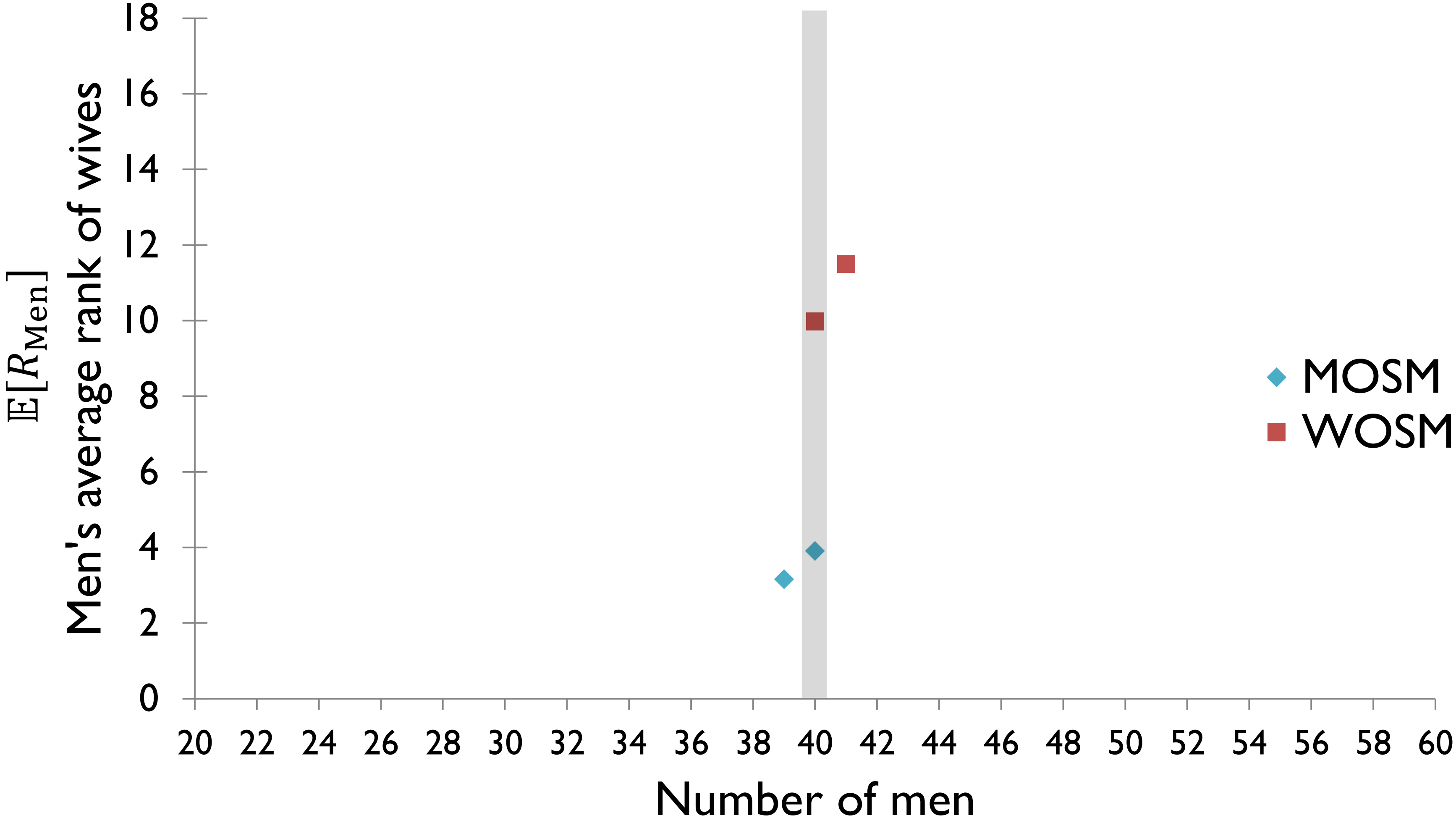

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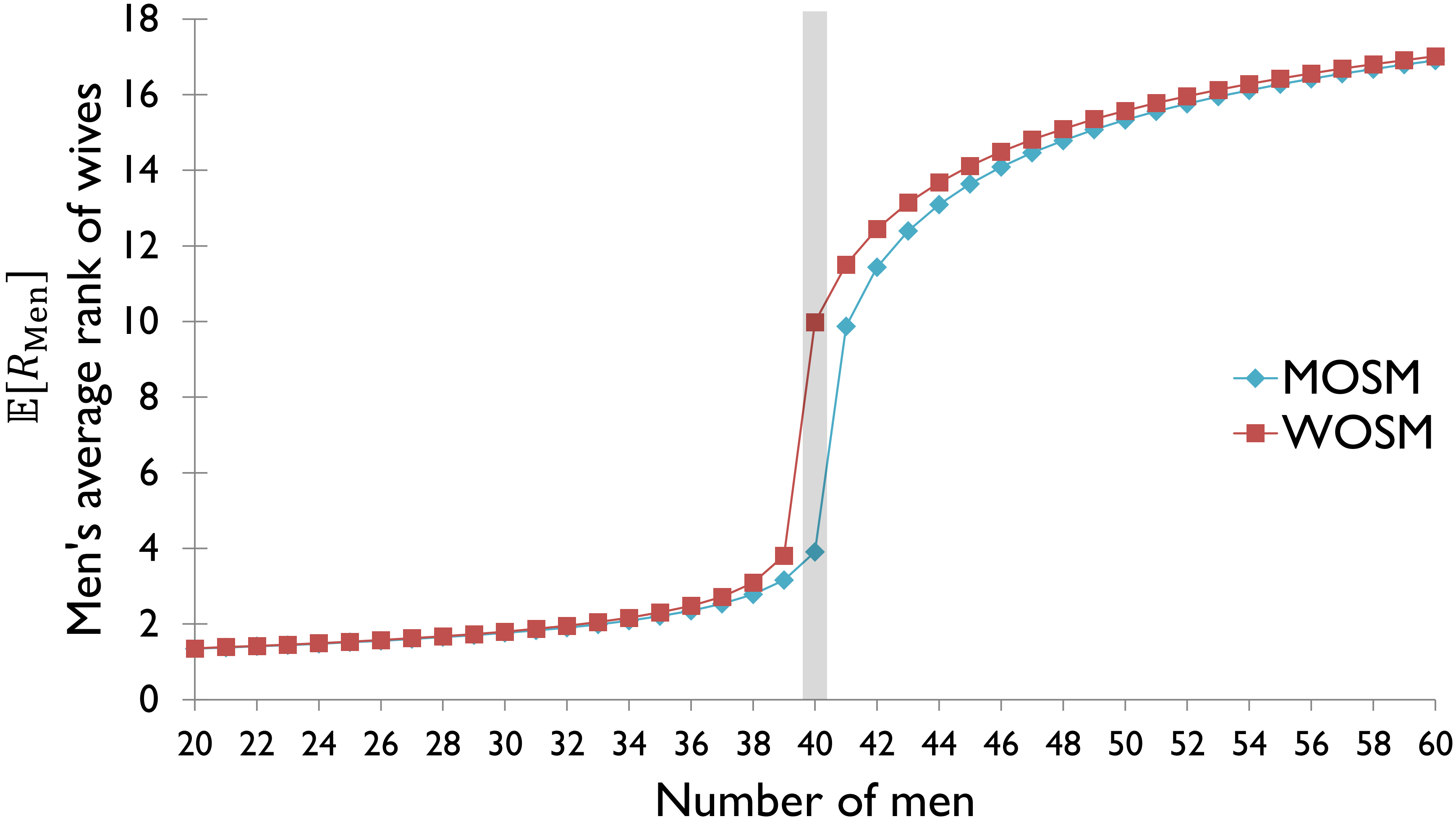
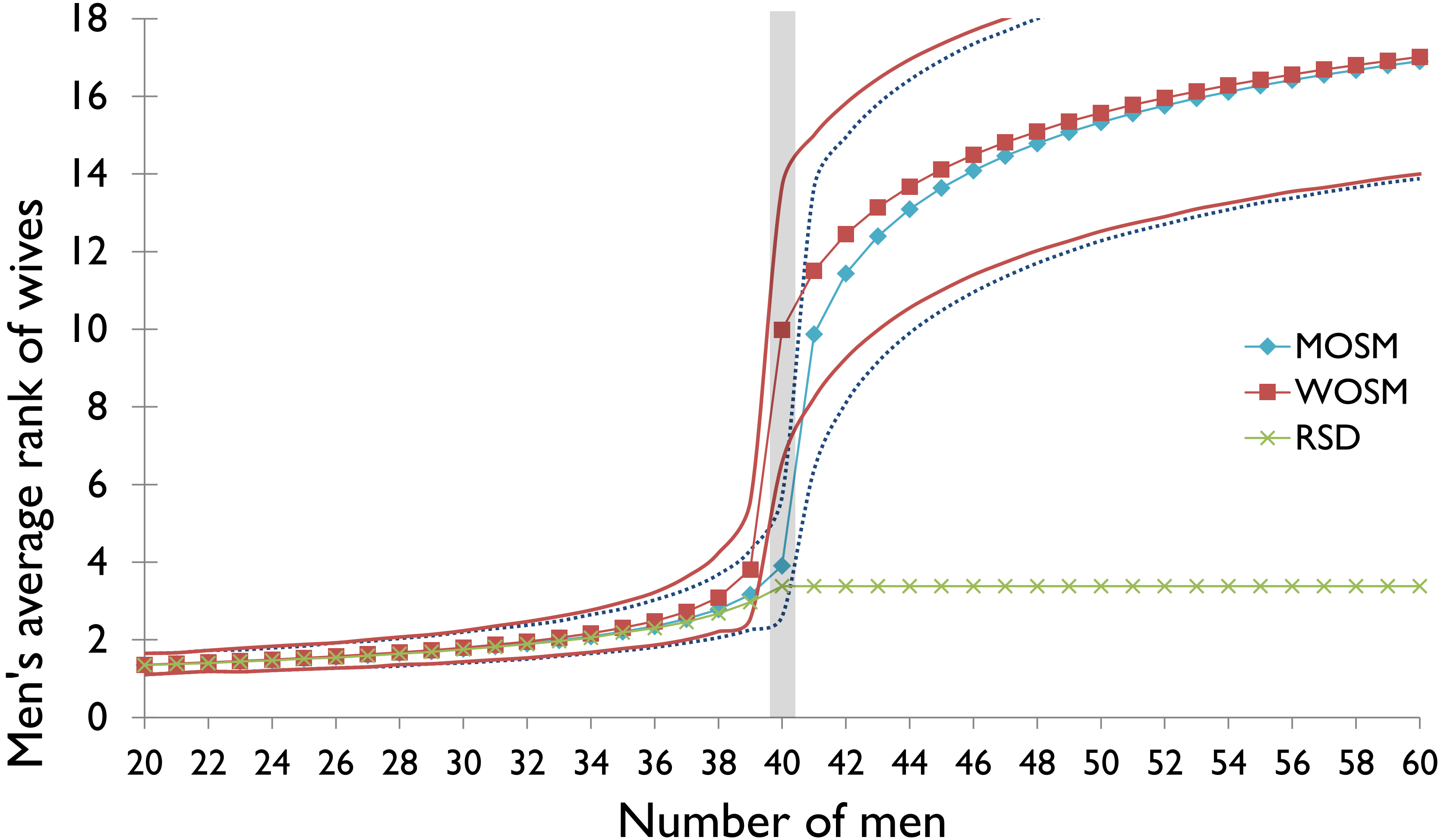
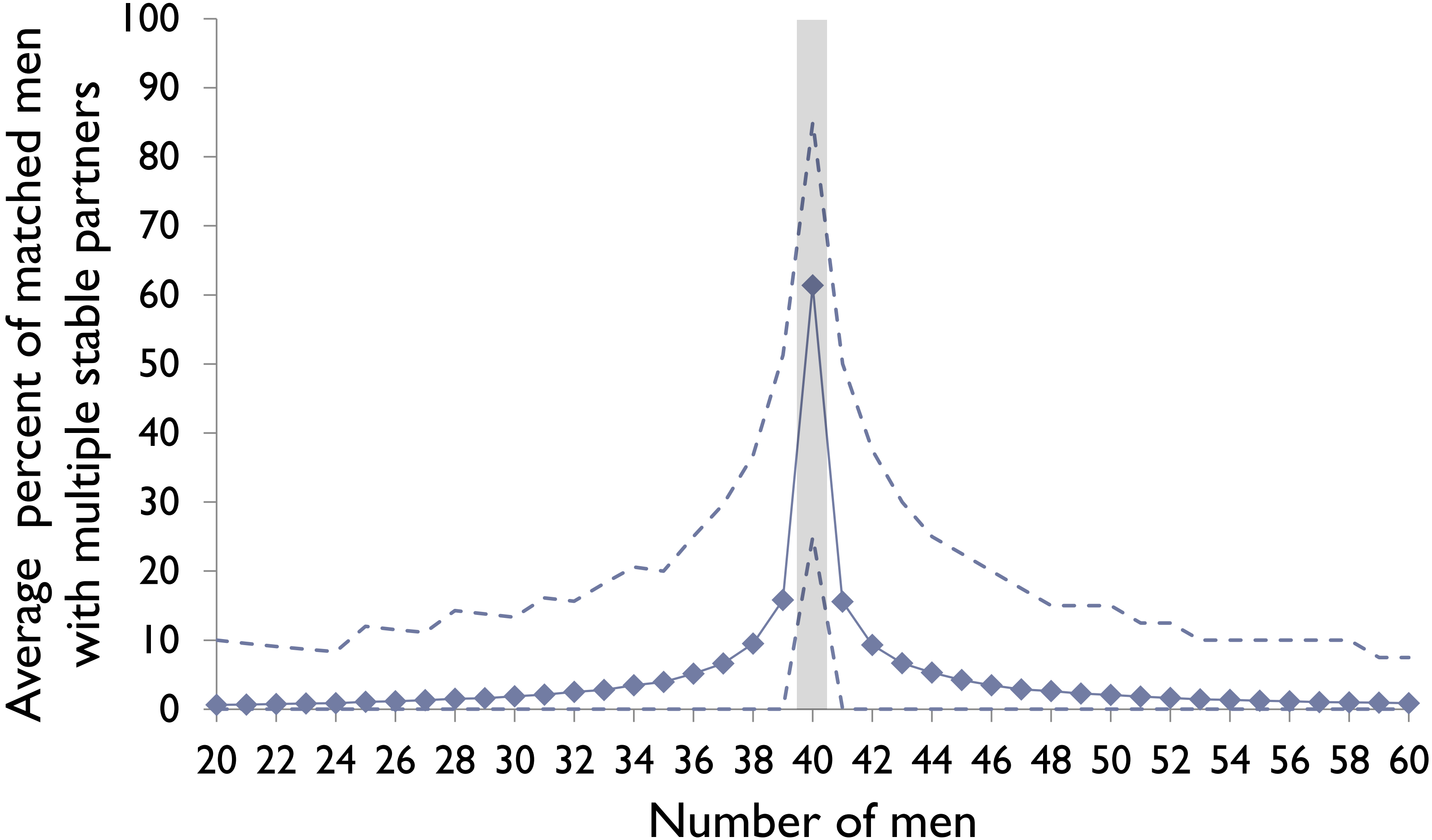


Men's average rank of wives, $|\mathcal{W}| = 40$



Percent of matched men with multiple stable partners $|\mathcal{W}| = 40$



Main Theorem

Theorem: Consider a random market with n men and $n+k$ women for $k = k(n) \geq 1$. With high probability in any stable matching,

$$R_{\text{Men}} \leq 1.01 \left(\frac{n+k}{n}\right) \log\left(\frac{n+k}{k}\right)$$

and

$$R_{\text{Women}} \geq \frac{n}{\left[1 + 1.01 \left(\frac{n+k}{n}\right) \log\left(\frac{n+k}{k}\right)\right]}$$

Moreover,

$$R_{\text{Men}}(\text{WOSM}) \leq (1 + o(1)) R_{\text{Men}}(\text{MOSM})$$

$$R_{\text{Women}}(\text{WOSM}) \geq (1 - o(1)) R_{\text{Women}}(\text{MOSM})$$

And $o(n)$ of men and $o(n)$ of women have multiple stable matches.

Main Theorem

- That is, under all stable matchings
  - men do almost as well as they would if they chose, ignoring women's preferences.
  - Women are either unmatched or roughly getting a randomly assigned man.

- The core is small
  - Limited choice for centralized mechanisms
  - Implies limited scope for manipulation of stable mechanisms
  - Facilitates comparative statics etc.

Corollary 1: One women makes a difference

Corollary: In a random market with n men and $n+1$ women, with high probability

$$R_{\text{Men}} \leq 1.01 \log n$$

and

$$R_{\text{Women}} \geq \frac{n}{1.01 \log n}$$

in all stable matches, and a vanishing fraction of agents have multiple stable partners.

Corollary 2: Large Unbalance

Corollary: Consider a random market with n men and $(1+\lambda)n$ women for $\lambda > 0$. Let $\kappa = 1.01\,(1+\lambda)\log(1+1/\lambda)$. With high probability, in all stable matchings

$$R_{\text{Men}} \le \kappa$$

and

$$R_{\text{Women}} \ge \frac{n}{1+\kappa}$$

Ashlagi, Braverman, Hassidim (2011) showed that the core is small in this setting.

Intuition

- In a competitive assignment market with n homogenous buyers and n homogenous sellers the core is large, but the core shrinks when there is one extra seller.
- In a matching market the addition of an extra woman makes all the men better off
  - Every man has the option of matching with the single woman
  - But only some men like the single woman
  - Changing the allocation of some men requires changing the allocation of many men: If some men are made better, and some women are made worse off, creating more options for men. All men benefit, and the core is small.

Proof overview

Calculate the WOSM using:

- Algorithm 1: Men-proposing Deferred Acceptance gives MOSM
- Algorithm 2: MOSM→WOSM

Both algorithms use a sequence of proposals by men

Stochastic analysis by sequential revelation of preferences

Algorithm 1: Men-proposing DA (Gale & Shapley)

Everyone starts unmatched. We add the men one at a time, running a 'chain' for each man.

Chain for adding m_k:

- Set m_k to be the proposer.
- The proposer proposes to his (next) most preferred woman w.
  - If w is unmatched, end chain and continue to add m_{k+1}
  - Otherwise, w rejects her less preferred man between m_k and her current partner. Repeat, with rejected man proposing.

Algorithm 2: MOSM $\rightarrow$ WOSM

We look for stable improvement cycles for women.

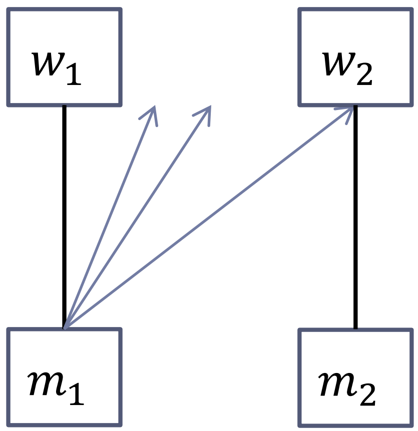
We iterate:

Phase: For candidate woman $\hat{w}$, reject her match m, starting a chain.

Two possibilities for how the chain ends:

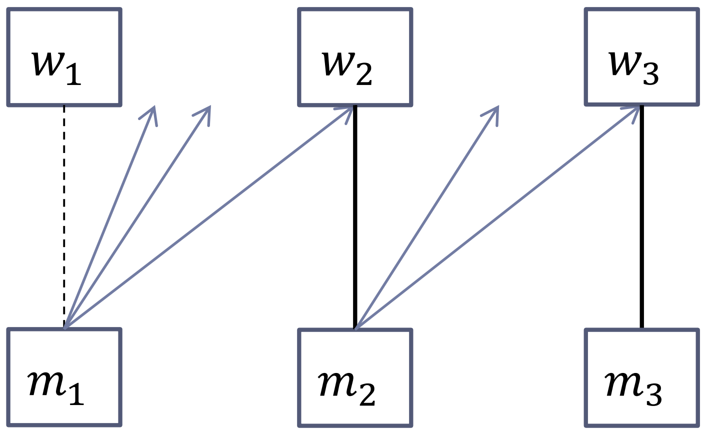
- (Improvement phase) Chain reaches $\hat{w} \Rightarrow$ new stable match.
- (Terminal phase) Chain ends with unmatched woman $\Rightarrow m$ is $\hat{w}$'s best stable match. $\hat{w}$ is no longer a candidate.

Illustration of Algorithm 2: MOSM $\rightarrow$ WOSM



w_1 is matched to m_1 under MOSM
w_2 prefers m_1 to m_2
rejects m_1 to start a chain

Illustration of Algorithm 2: MOSM → WOSM



w_2 prefers m_1 to m_2

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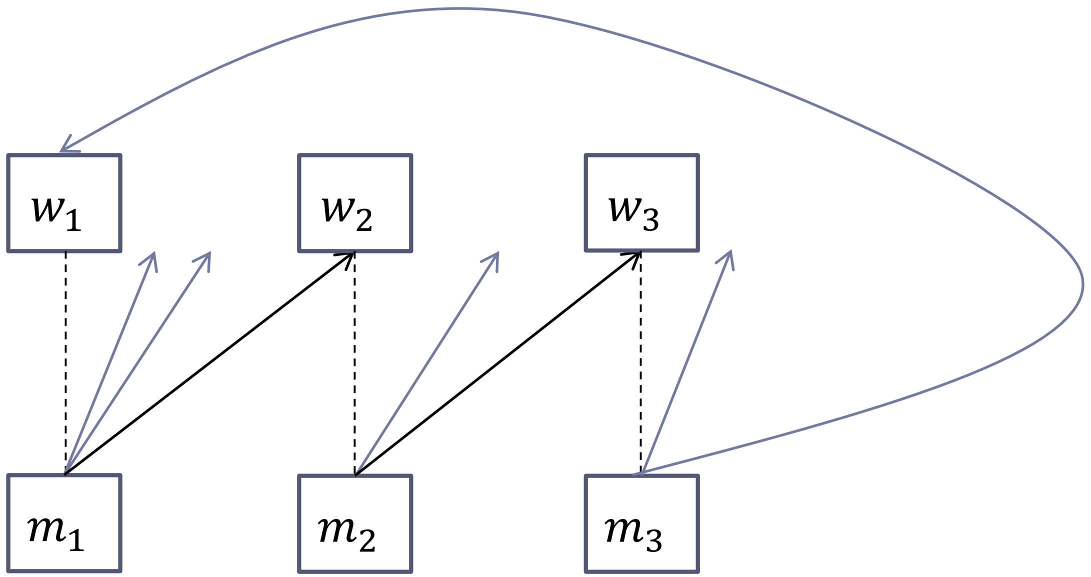
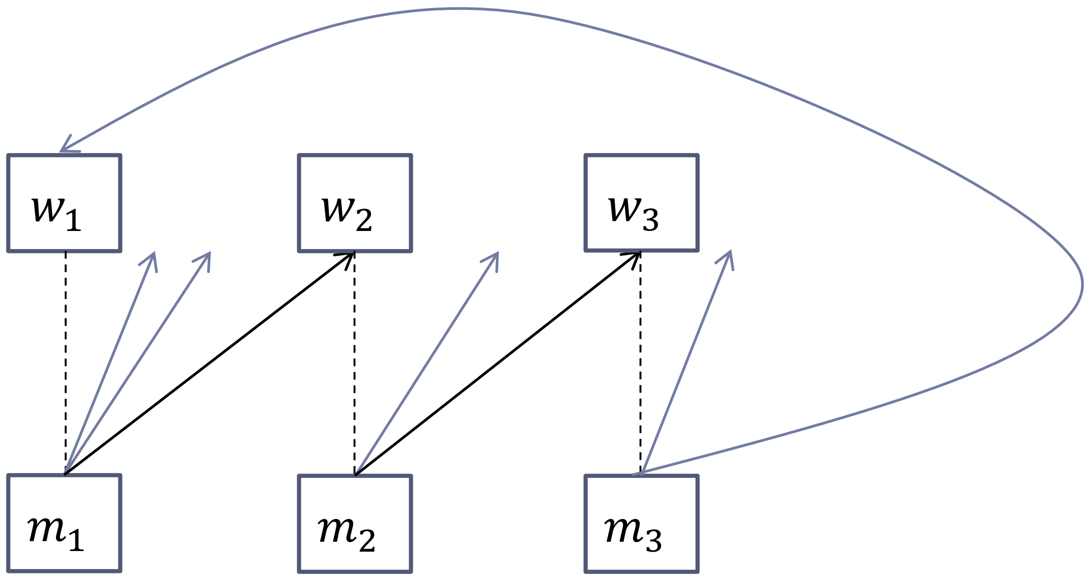
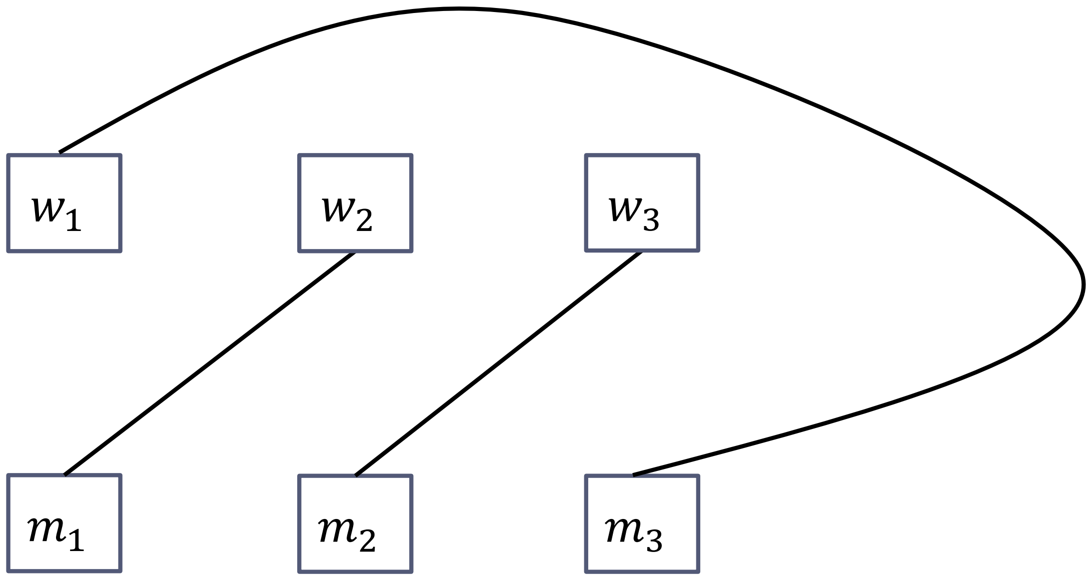


Illustration of Algorithm 2: MOSM → WOSM



w_1 prefers m_3 to m_1

Illustration of Algorithm 2: MOSM → WOSM



New stable match found. Update match and continue.

Illustration of Algorithm 2: MOSM → WOSM

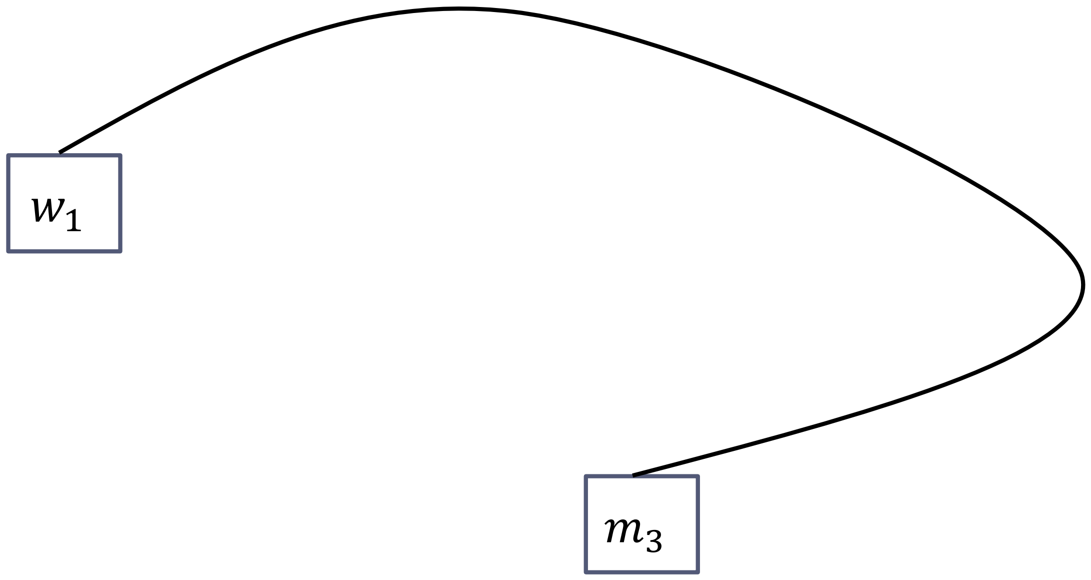


Illustration of Algorithm 2: MOSM → WOSM

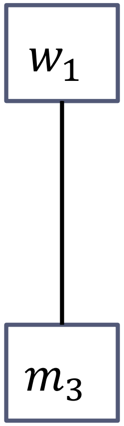
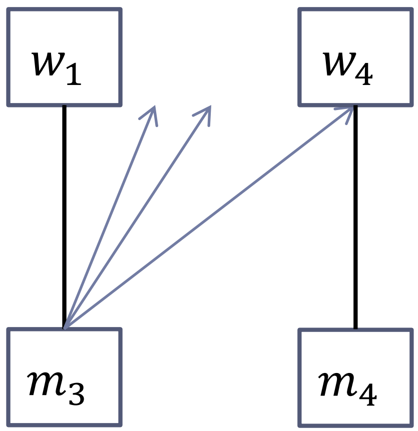


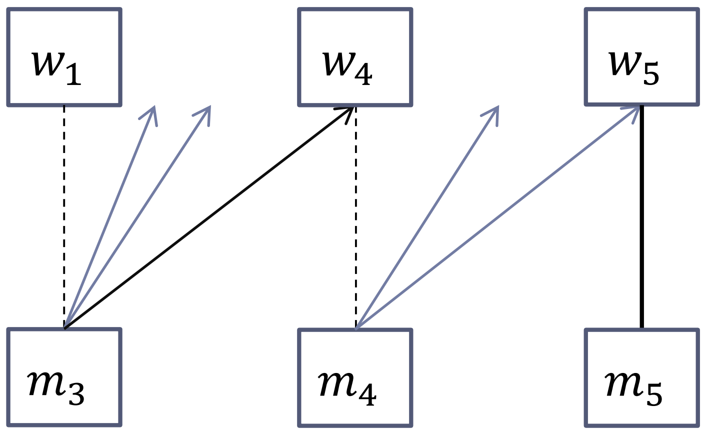
Illustration of Algorithm 2: MOSM → WOSM



w_1 rejects m_3 to start a chain

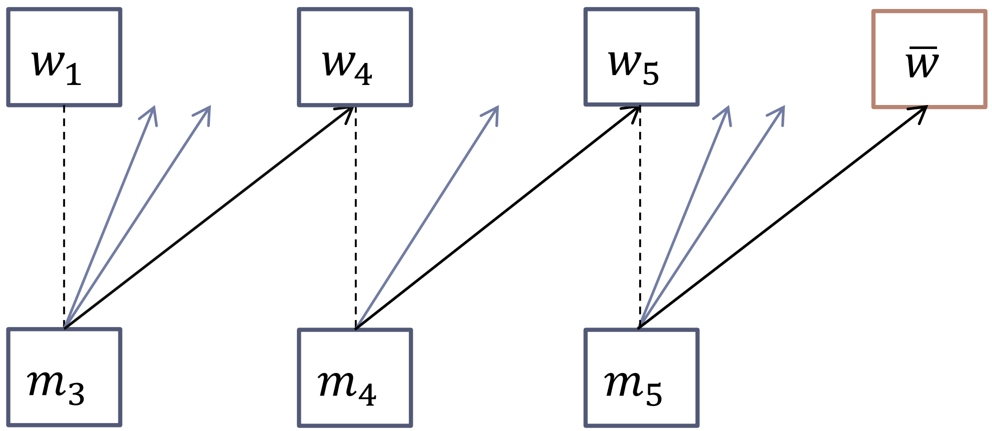
w_4 prefers m_3 to m_4

Illustration of Algorithm 2: MOSM → WOSM



w_5 prefers m_4 to m_5

Illustration of Algorithm 2: MOSM → WOSM

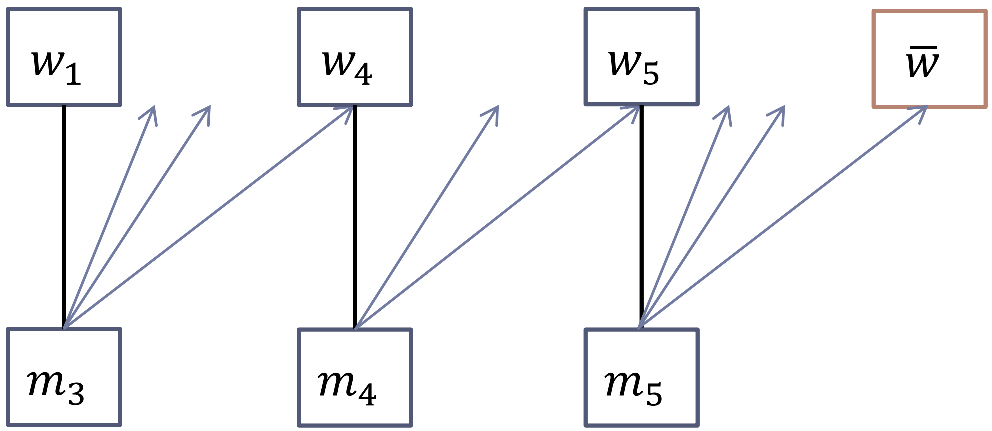


Chain ends with a proposal to unmatched woman $\overline{w}$

$\Rightarrow m_3$ is w_1's best stable partner

and similarly w_4 and w_5 already had their best stable partner

Illustration of Algorithm 2: MOSM → WOSM



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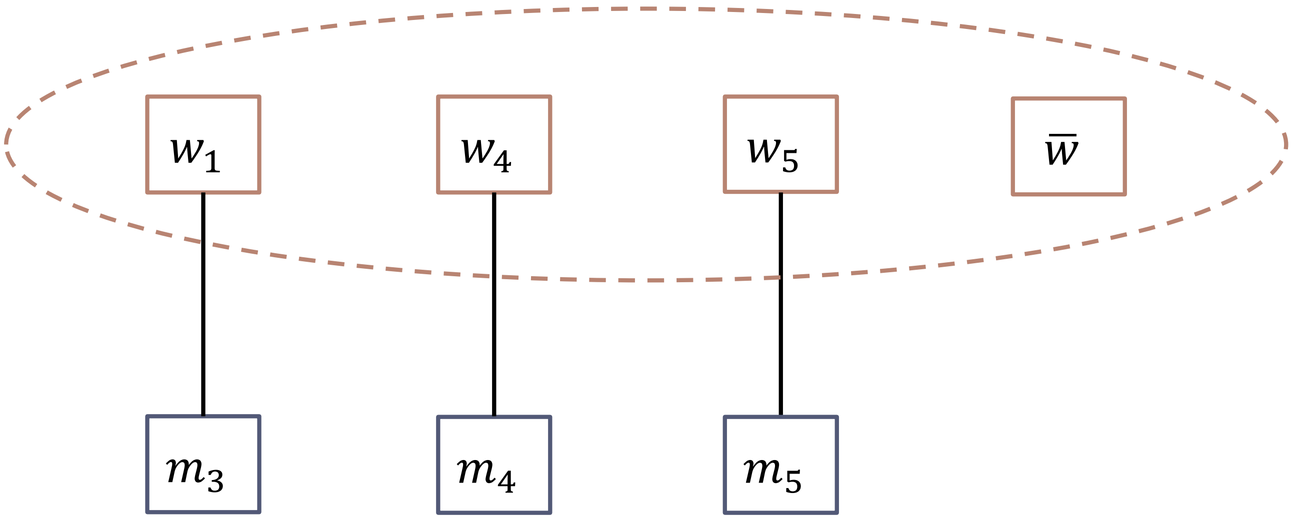

Chain ends with a proposal to unmatched woman $\overline{w}$

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and similarly w_4 and w_5 already had their best stable partner

Illustration of Algorithm 2: MOSM → WOSM

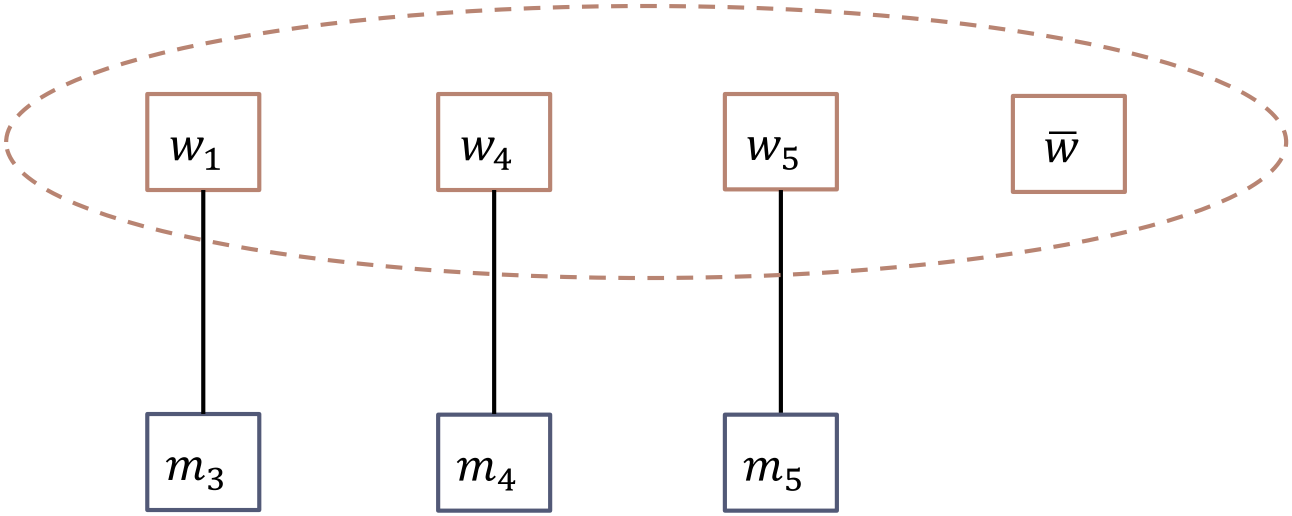
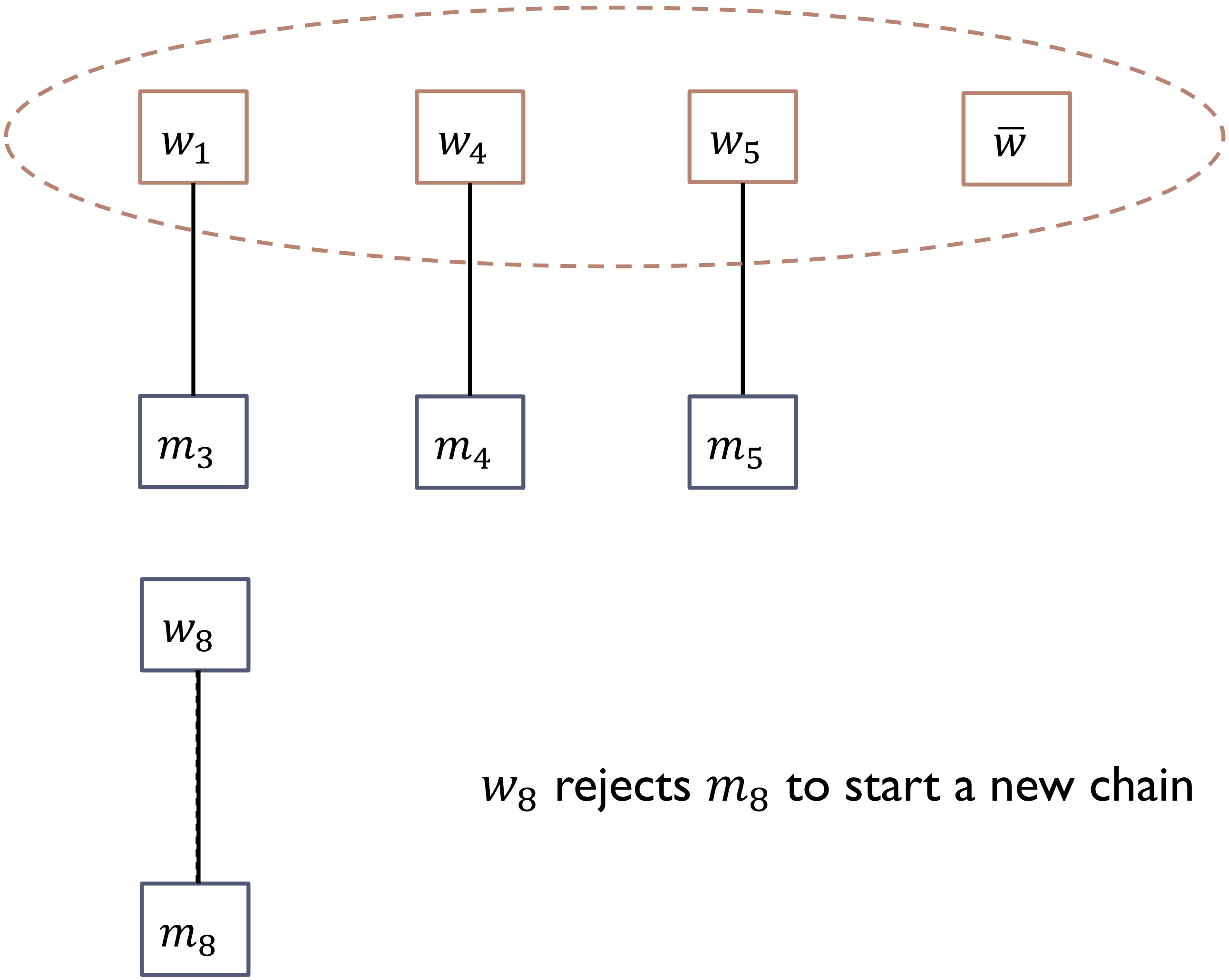
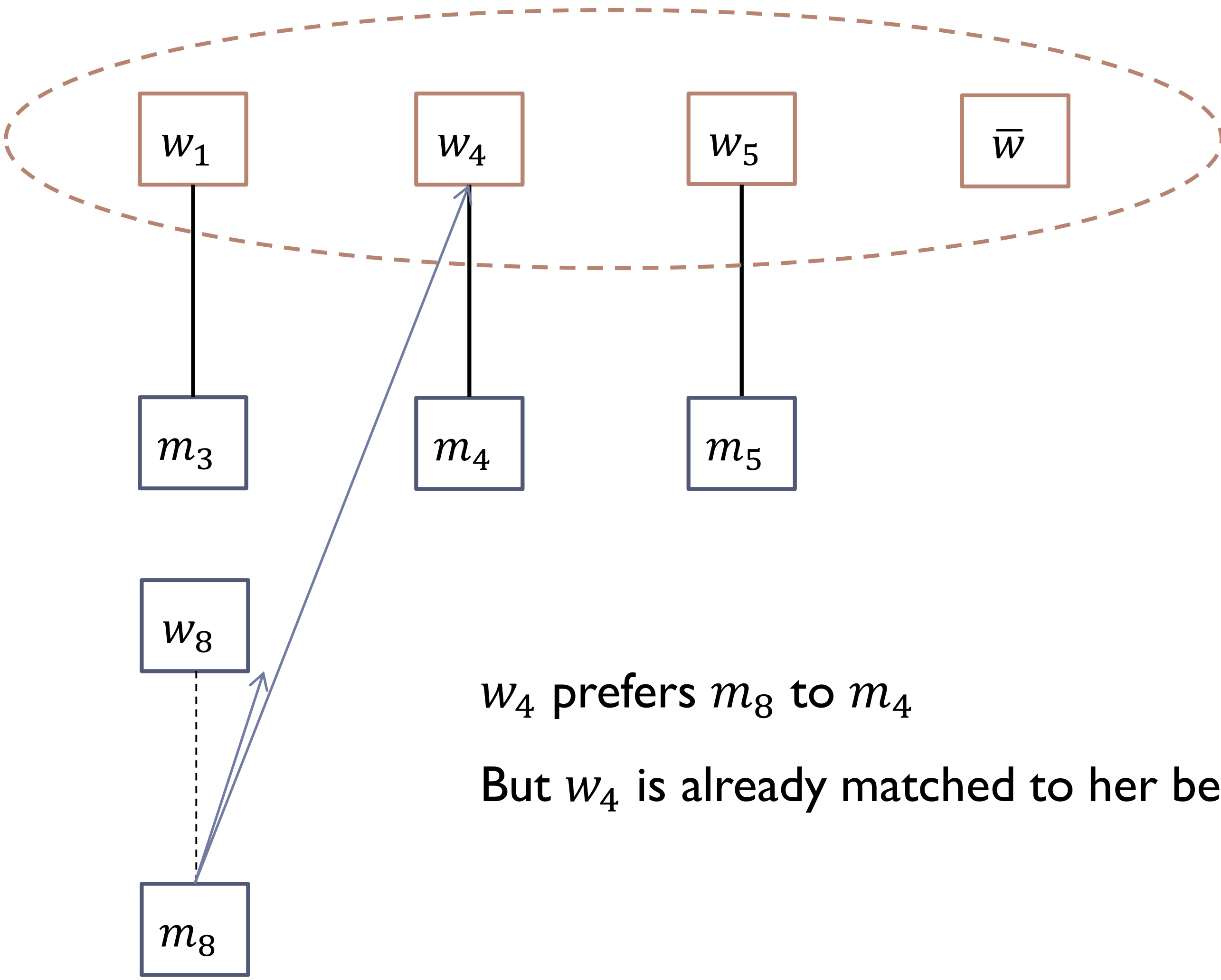


Illustration of Algorithm 2: MOSM → WOSM



w_8 rejects m_8 to start a new chain

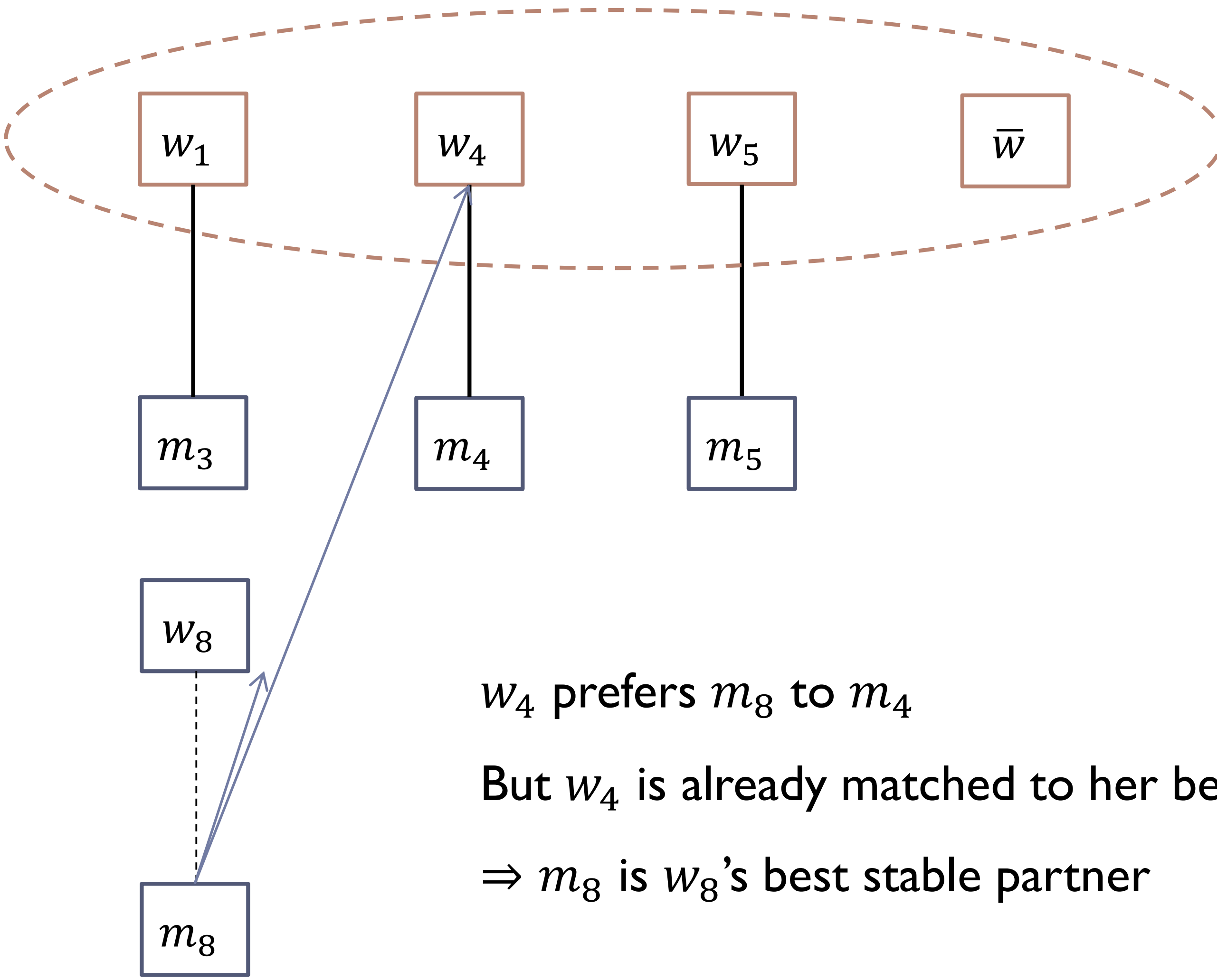
Illustration of Algorithm 2: MOSM → WOSM



w_4 prefers m_8 to m_4

But w_4 is already matched to her best stable partner

Illustration of Algorithm 2: MOSM → WOSM

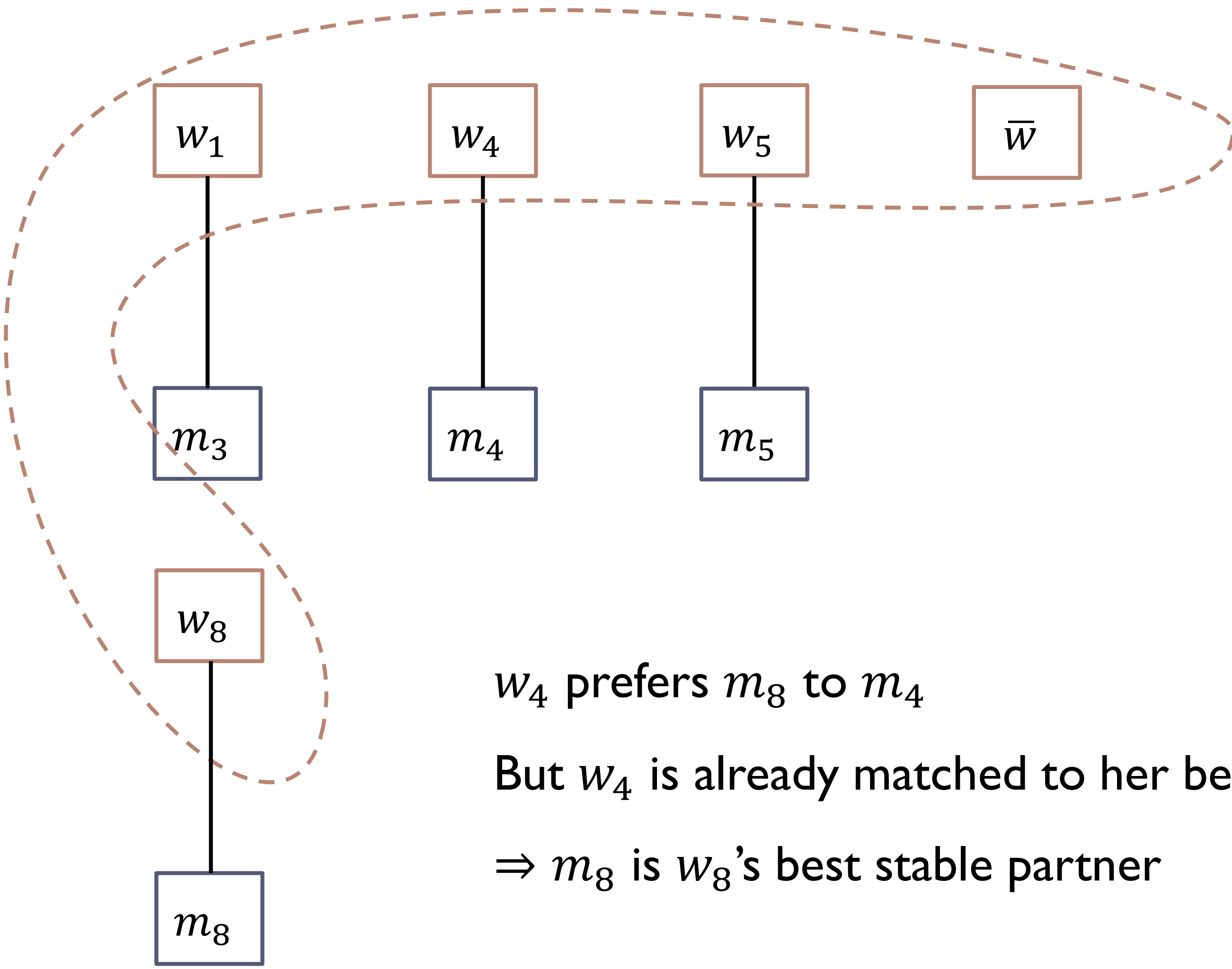


w_4 prefers m_8 to m_4

But w_4 is already matched to her best stable partner

$\Rightarrow$ m_8 is w_8's best stable partner

Illustration of Algorithm 2: MOSM $\rightarrow$ WOSM



w_4 prefers m_8 to m_4

But w_4 is already matched to her best stable partner

$\Rightarrow m_8$ is w_8's best stable partner

Algorithm 2: MOSM $\rightarrow$ WOSM

Initialize $S = \mathcal{W}_{\text{unmatched}}$

1. Choose $\widehat{w}$ in $\mathcal{W} \backslash S$ if non-empty.
2. Phase: Record the current match as $\hat{\mu}$. Woman $\widehat{w}$ rejects her partner, man m, starting a chain where $\widehat{w}$ accepts a proposal only if the proposal is preferred to m.
3. Two possibilities for how the chain ends:
   - (Improvement phase) If the chain ends with acceptance by $\widehat{w}$, we have found a new stable match. Return to Step 2.
   - (Terminal phase) Else the chain ends with acceptance by $\mathcal{W}_{\text{unmatched}}$. Woman $\widehat{w}$ has found her best stable partner. Roll the match back to $\hat{\mu}$. Add $\widehat{w}$ to S and return to Step 1.

Overview of stochastic analysis

- Analysis of MPDA is similar to that of Pittel (1989)

- Analysis of Algorithm 2: MOSM $\rightarrow$ WOSM more involved.

  Key finding: In a typical market, very few agents participate in improvement cycles.

Proof idea:

- Analysis of MPDA is similar to that of Pittel (1989)
  - Coupon collectors problem
- Analysis of Algorithm 2: MOSM $\rightarrow$ WOSM more involved.
  - S grows quickly
  - Once S is large improvement phases are rare
  - Together, in a typical market, very few agents participate in improvement cycles.

Strategic implications

- Men proposing DA (MPDA) is strategyproof for men, but no stable mechanism is strategyproof for all agents.
- A woman can manipulate MPDA only if she has multiple stable husbands
  - Misreport truncated preferences.

- In unbalanced matching market a diminishing number of women have multiple stable husbands
  - Mechanism is approximately strategyproof

Further questions

- Can we allow correlation in preferences?
  - Perfect correlation leads to a unique core.
  - But "short side" depends on more than N, K
    - Tiered market: 30 men, 40 women: 20 top, 20 mid
  - What is a general balance condition? Who chooses in more general settings?

- Are there any real/natural matching markets with large cores?
  - Extensive simulations suggest the answer is "no"

Men's average rank of wives

n	diff	-10	-1	0	+1	+5	+10
100	MOSM	29.5	20.3	5.0	4.1	3.0	2.6
	WOSM	30.1	23.6	20.3	4.9	3.2	2.6
200	MOSM	53.6	35.3	5.7	4.8	3.7	3.1
	WOSM	54.7	41.0	35.5	5.7	3.8	3.2
500	MOSM	115.8	75.9	6.7	5.7	4.5	3.9
	WOSM	118.0	86.6	76.2	6.7	4.7	4.0
1000	MOSM	203.8	136.2	7.4	6.4	5.2	4.6
	WOSM	207.5	155.1	137.3	7.4	5.4	4.7
2000	MOSM	364.5	249.6	8.1	7.1	5.9	5.3
	WOSM	370.8	280.7	249.1	8.1	6.1	5.4
5000	MOSM	793.1	560.0	9.1	8.1	6.8	6.2
	WOSM	804.7	622.5	560.2	9.1	7.0	6.3

Percent of men with multiple stable matches

n Diff:	-10	-1	0	+1	+5	+10
100	2.1	15.1	75.3	15.4	4.5	2.3
200	2.2	14.6	83.6	14.6	4.1	2.1
500	2.0	12.6	91.0	13.1	3.6	2.0
1000	1.9	12.3	94.5	12.2	3.4	2.0
2000	1.8	11.1	96.7	11.1	2.9	1.7
5000	1.5	10.1	98.4	10.2	2.8	1.5

Many-to-one simulations

- Similar features observed when capacities are small
- See the paper for details

Conclusion

Random unbalanced matching markets are surprisingly competitive:

- The short side chooses in all stable matchings
- The core is small – most agents have a single stable partner.

Our results suggest that matching markets generically have small cores

(See also Kanoria, Saban & Sethuraman (2015) for a similar result on matching markets with transfers)

Thank you!